

**LC HELIX ON HYPERSURFACES IN
MINKOWSKI SPACE \mathbb{E}_1^{n+1}**

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Abstract: In this paper we give the definition of a generalized LC helix for a non-null curve lying on a hypersurface in \mathbb{E}_1^{n+1} by using the Levi Civita's notion of parallel vector field. Also we give some basic properties and characterization of generalized LC helices.

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1. Introduction

Helix is one of the most fascinating curves in science and nature. Scientists have

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long held a fascination, sometimes bordering on mystical obsession for helical structures in nature. Helices arise in nanosprings, carbon nanotubes, α -helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA.

A curve of constant slope or general helix in Euclidean 3-space \mathbb{E}^3 , is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 [14] is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. If both of k_1 and k_2 are non-zero constants it is, of course, a general helix. It is known that straight line and circle are degenerate helix examples. ($k_1 = 0$, if the curve is straight line and $k_2 = 0$, if the curve is a circle) [9].

The notion of a generalized helix can be generalized to higher dimensions in many ways. In [12] the same definition is proposed but in \mathbb{E}^n . In [7] the definition which was defined by Hayden is more restrictive: the fixed direction makes a constant angle with all the vectors of the Frenet frame. This definition only works in the odd dimensional case. Moreover, in the same reference, it is proved that the definition is equivalent to the fact that the ratios $\frac{k_{n-1}}{k_{n-2}}, \frac{k_{n-3}}{k_{n-4}}, \dots, \frac{k_2}{k_1}$ being the curvatures, are constant. This statement is related with the Lancret Theorem for generalized helices in \mathbb{E}^3 (the ratio of torsion to curvature is constant). In [11] the curves in \mathbb{E}^n for which all the ratios $\frac{k_{n-1}}{k_{n-2}}, \frac{k_{n-3}}{k_{n-4}}, \dots, \frac{k_2}{k_1}$ are constant which was called *ccr* curves. In the same reference, it is shown that in the even dimensional case, a curve has constant curvature ratios if and only if its tangent indicatrix is a geodesic in the flat torus.

As a matter of fact, circular helix is the simplest three-dimensional spirals. One of the most interesting spiral example is k -Fibonacci spirals. These curves appear naturally from studying the k -Fibonacci numbers $\{F_{k,n}\}_{n=0}$ and the related hyperbolic k -Fibonacci function. Fibonacci numbers and the related Golden Mean or Golden section appear very often in theoretical physics and physics of the high energy particles [3, 4]. Three-dimensional, k -Fibonacci spirals was studied from a geometric point of view in [5].

Helices have been studied intensively by geometers and other scientists. In [2] some characterizations for a non-degenerate curve to be a generalized helix by using harmonic curvatures of the curve in n -dimensional Euclidean space \mathbb{E}^n . Moreover, in [10] curvature conditions are given for $AW(k)$ -type curves in Lorentzian space and shown that under what conditions $AW(k)$ -type curves are helix.

The Lancret theorem was revisited and solved by M. Barros in 3-dimensional

real space-forms by using Killing vector fields along curves. Also, in [1] M. Barros showed that a relation such as $\tau = b\kappa + a$ for a general helices in 3-dimensional real space-form, where a and b are constant.

A helical curve is a curve in 3-dimensional space form $M^3(c)$ of constant curvature c whose both curvature and torsion are non zero constants. It reduces to a Riemannian a circle or a geodesic [15].

Levi Civita's notion of parallellism (or parallel motion of a direction along a curve) is defined for the vector field along a curve located on a hypersurface. In this paper we give the definition of a generalized LC helix for a non-null curve lying on a hypersurface in \mathbb{E}_1^{n+1} by using the Levi Civita's notion of parallel vector field. Also we give some basic properties and characterization of generalized LC helices. In particular, if the hypersurface which the non- null curve lying on is replaced by \mathbb{E}_1^n , then the definition of a helix on \mathbb{E}_1^n coincides with the definition of generalized LC helix.

2. Preliminaries

Let E_1^{n+1} be the $(n + 1)$ dimensional pseudo-Euclidean space with index 1 endowed with the indefinite inner product given by

$$g(x, y) = -x_1y_1 + \sum_{i=2}^{n+1} x_iy_i,$$

where $x = (x_1, x_2, \dots, x_{n+1})$, $y = (y_1, y_2, \dots, y_{n+1})$ is the usual coordinate system. Let M be a hypersurface in E_1^{n+1} and p be a point on M and $v \in T_pM$ a tangent vector. Then v is said to be spacelike, timelike or null according to $g(v, v) > 0$, $g(v, v) < 0$, or $g(v, v) = 0$ and $v \neq 0$, respectively. Notice that the vector $v = 0$ is spacelike. The category into which a given tangent vector falls is called its causal character. These definitions can be generalized for curves as follows. A curve α on M is said to be spacelike if all of its velocity vectors α are spacelike, similarly for timelike and null [1].

Let us recall from [13, 8] the definition of the Frenet frame and curvatures.

Let M be a hypersurface in E_1^{n+1} and $\alpha : I \subset \mathbb{R} \rightarrow M$ be non-null curve on M . A non-null curve $\alpha(s)$ is said to be a unit speed curve if $g(\alpha(s), \alpha(s)) = \varepsilon_0$, (ε_0 being $+1$ or -1 according to α is spacelike or timelike respectively). Let $\{V_1, V_2, \dots, V_n\}$ be the moving Frenet frame along the unit speed curve α , where V_i ($i = 1, 2, \dots, n$) denote i th Frenet vector fields and k_i be i th curvature functions of the curve ($i = 1, 2, \dots, n - 1$). Then the Frenet formulas are given

by

$$\begin{aligned}
 \nabla_{V_1} V_1 &= k_1 V_2, \\
 \nabla_{V_1} V_i &= -\varepsilon_{i-2} \varepsilon_{i-1} k_{i-1} V_{i-1} + k_i V_{i+1}, \quad 1 < i < n \\
 \nabla_{V_1} V_n &= -\varepsilon_{n-2} \varepsilon_{n-1} k_{n-1} V_{n-1}
 \end{aligned}
 \tag{1}$$

where $g(V_i, V_i) = \varepsilon_{i-1}$, and ∇ is the Levi-Civita connection of M .

Let M be a hypersurface in E_1^{n+1} with the Levi-Civita connection ∇ and suppose $\alpha : I \subset \mathbb{R} \rightarrow M$ is non-null curve on M . For any tangent vector X at $\alpha(s)$ is said to be a Levi Civita's notion of parallel vector field on M of a direction along a curve if $\nabla_{V_1} X = 0$. Also, the Levi Civita's notion of parallel vector field has constant length [6].

3. LC Helix on Hypersurfaces in Minkowski Space

In this section we define LC helices on hypersurfaces in Minkowski space and we give some characterizations for LC helices on hypersurfaces in the same space.

Definition 1. Let M be a hypersurface in E_1^{n+1} and $\alpha : I \subset \mathbb{R} \rightarrow M$ be non-null curve on M . A non-null curve $\alpha(s)$ is said to be a generalized LC helix if there exists a Levi Civita's notion of parallel vector field X on M such that $g(V_1, X)$ is a constant function. Any line parallel this direction X is called the axis of the LC generalized helix.

Definition 2. Let M be a hypersurface in E_1^{n+1} and non-null curve $\alpha : I \subset \mathbb{R} \rightarrow M$ be a unit speed generalized LC helix. Harmonic curvatures of α is defined by $H_i : I \subset \mathbb{R} \rightarrow \mathbb{R}$,

$$\begin{aligned}
 H_1 &= \varepsilon_0 \varepsilon_1 \frac{k_1}{k_2}, \\
 H_i &= (\varepsilon_{i-1} \varepsilon_i k_i H_{i-2} + \nabla_{V_1} H_{i-1}) \frac{1}{k_{i+1}}, \quad 2 \leq i \leq n-2,
 \end{aligned}
 \tag{2}$$

where k_1, k_2, \dots, k_{n-1} are curvatures functions of the curve α which are not necessarily constant.

Theorem 3. Let M be a hypersurface in E_1^{n+1} and $\alpha : I \subset \mathbb{R} \rightarrow M$ be a unit speed generalized LC helix. Let $\{V_1, V_2, \dots, V_n\}$, $\{H_1, H_2, \dots, H_{n-2}\}$ be denote the Frenet frame and the higher ordered harmonic curvatures of the curve, respectively. Then the following equations is holds

$$g(V_{i+2}, X) = H_i g(V_1, X), \quad 1 \leq i \leq n-2,
 \tag{3}$$

where X is axis of the LC generalized helix.

Proof. We use mathematical induction on i . Since X is axis of the generalized LC helix α , we get

$$X = \lambda_1 V_1 + \lambda_2 V_2 + \dots + \lambda_n V_n.$$

From the definition of generalized LC helix we have

$$g(V_1, X) = \varepsilon_0 \lambda_1 \tag{4}$$

By taking the derivative of (4) and applying the Frenet formulas twice, we obtain

$$g(V_2, X) = 0,$$

$$g(V_3, X) = H_1 g(V_1, X).$$

respectively. Hence it is shown that (3) is true for $i = 1$. We now assume (3) is true for the first $i - 1$. Then we have

$$g(V_{i+1}, X) = H_{i-1} g(V_1, X). \tag{5}$$

By taking the derivative of (5) and applying the Frenet formulas, we get

$$-\varepsilon_{i-1} \varepsilon_i k_i g(V_i, X) + k_{i+1} g(V_{i+1}, X) = \nabla_{V_1} H_{i-1} g(V_1, X).$$

By using the our induction hypothesis, $g(V_i, X) = H_{i-2} g(V_1, X)$, we have

$$\frac{1}{k_{i+1}} (\varepsilon_{i-1} \varepsilon_i k_i H_{i-2} + \nabla_{V_1} H_{i-1}) g(V_1, X) = g(V_{i+2}, X),$$

it follows that

$$g(V_{i+2}, X) = H_i g(V_1, X).$$

□

Theorem 4. Let M be a hypersurface in E_1^{n+1} and non-null curve $\alpha : I \subset \mathbb{R} \rightarrow M$ be a unit speed generalized LC helix with Frenet vector fields $\{V_1, V_2, \dots, V_n\}$, and harmonic curvatures $\{H_1, H_2, \dots, H_{n-2}\}$. If α is generalized LC helix, then $\sum_{i=1}^{n-2} \varepsilon_{i+1} H_i^2 = c$, where c is any constant.

Proof. Let α be generalized LC helix with the arc length parameter s . Then by the definition we can assume that

$$g(V_1, X) = \varepsilon_0 \lambda_1, \quad (6)$$

for some non zero constant λ_1 . By taking the derivative of (6) with respect to s and applying the Frenet formulas we have

$$g(k_1 V_2, X) = 0.$$

Since $k_1 \neq 0$, then we have

$$g(V_2, X) = 0.$$

On the other hand, the fact that X is axis of the LC generalized helix α , we get

$$\begin{aligned} X &= \sum_{i=1}^n \lambda_i V_i, \\ X &= \varepsilon_0 g(V_1, X) V_1 + \varepsilon_2 g(V_3, X) V_3 + \dots + \varepsilon_{n-1} g(V_n, X) V_n. \end{aligned}$$

From Theorem(3) we can write

$$X = \varepsilon_0 g(V_1, X) V_1 + g(V_1, X) \sum_{j=1}^{n-2} \varepsilon_{j+1} H_j V_{j+2},$$

and notice that X is a unit Levi Civita's notion of parallel vector field, we obtain

$$\varepsilon_0 (g(V_1, X))^2 + (g(V_1, X))^2 \sum_{j=1}^{n-2} \varepsilon_{j+1} H_j^2 = 1.$$

Thus we get

$$\sum_{j=1}^{n-2} \varepsilon_{j+1} H_j^2 = \frac{1 - \varepsilon_0 \lambda_1^2}{\lambda_1^2}.$$

This completes the proof. □

Then we have the following corollary.

Corollary 5. *If X is axis of the LC generalized helix α on M , then we can write*

$$X = \sum_{i=1}^n \lambda_i V_i.$$

By using the Theorem (3), we get

$$X = g(V_1, X) \left(\varepsilon_0 V_1 + \sum_{j=1}^{n-2} \varepsilon_{j+1} H_j V_{j+2} \right)$$

where $g(V_1, X) = \varepsilon_0 \lambda_1 = \text{constant}$. Also

$$\frac{W}{\|W\|} = \frac{\varepsilon_0 V_1 + \varepsilon_2 H_1 V_3 + \dots + \varepsilon_{n-1} H_{n-2} V_n}{\sqrt{\varepsilon_0 + \varepsilon_2 H_1^2 + \dots + \varepsilon_{n-1} H_{n-2}^2}}$$

is a unit axis of the LC generalized helix α .

Theorem 6. Let M be a hypersurface in E_1^{n+1} and non-null curve $\alpha : I \subset \mathbb{R} \rightarrow M$ be a unit speed curve with Frenet vector fields $\{V_1, V_2, \dots, V_n\}$, and harmonic curvatures $\{H_1, H_2, \dots, H_{n-2}\}$. Then α is a generalized LC helix if and only if $W = \varepsilon_0 V_1 + \varepsilon_2 H_1 V_3 + \dots + \varepsilon_{n-1} H_{n-2} V_n$ is Levi Civita's notion of parallel vector field.

Proof. Suppose that α is a generalized LC helix on M and X is axis of α . From Corollary(5), we get

$$X = \varepsilon_0 \lambda_1 (\varepsilon_0 V_1 + \varepsilon_2 H_1 V_3 + \dots + \varepsilon_{n-1} H_{n-2} V_n). \tag{7}$$

By taking the derivative of (7), and using the definition of the Levi Civita's notion of parallel vector field we can easily obtain

$$\nabla_{V_1} W = 0.$$

Thus W is Levi Civita's notion of parallel vector field.

Conversely, since W is Levi Civita's notion of parallel vector field then $\|W\| = \text{constant}$. We consider the normalisation of the Levi Civita's notion of parallel vector field as follows

$$X = \frac{1}{\|W\|} W.$$

Therefore we have

$$g(V_1, X) = \frac{1}{\|W\|} = \text{constant}.$$

Thus α is a generalized LC helix. □

Theorem 7. Let M be a hypersurface in E_1^{n+1} , and α be a geodesic on M . Then α is a generalized LC helix.

Proof. Let α be a geodesic on M with the arc length parameter s and suppose that $X = V_1$. From the Gauss formula we have

$$\begin{aligned} \nabla_{V_1} X &= \nabla_{V_1}^0 X - \langle \nabla_{V_1}^0 X, N \rangle N \\ &= \frac{d^2\alpha}{ds^2} - \langle \frac{d^2\alpha}{ds^2}, N \rangle N \end{aligned}$$

where ∇^0 is the Levi-Civita connection of E_1^{n+1} and N is normal unit vector field on M . By the definition of geodesic we have $\frac{d^2\alpha}{ds^2} = \lambda N$ for some constant λ . Hence

$$\nabla_{V_1} X = 0,$$

and clearly $g(X, V_1) = g(V_1, V_1) = \varepsilon_0$ is constant. Thus α is a generalized LC helix. □

From now, let us consider M is a hypersurface in E_1^4 and $\alpha : I \subset \mathbb{R} \rightarrow M$ be a non-null curve with Frenet vector fields $\{V_1, V_2, V_3\}$ and curvatures $\{k_1, k_2\}$. Then the Frenet formulas are given by

$$\begin{aligned} \nabla_{V_1} V_1 &= k_1 V_2, \\ \nabla_{V_1} V_2 &= -\varepsilon_0 \varepsilon_1 k_1 V_1 + k_2 V_3, \\ \nabla_{V_1} V_3 &= -\varepsilon_1 \varepsilon_2 k_2 V_2. \end{aligned}$$

Theorem 8. *Let M be a hypersurface in E_1^4 , and α be a regular curve on M . Then α satisfies the following equation*

$$\begin{aligned} \nabla_{V_1}^3 V_1 - \left(2\frac{k_1}{k_1} + \frac{k_2}{k_2}\right) \nabla_{V_1}^2 V_1 \\ + \left[-\frac{k_1}{k_1} + \frac{k_1 k_2}{k_1 k_2} + 2\left(\frac{k_1}{k_1}\right)^2 + \varepsilon_0 \varepsilon_1 k_1^2 + \varepsilon_1 \varepsilon_2 k_2^2\right] \\ \nabla_{V_1} V_1 + \varepsilon_0 \varepsilon_1 k_1 k_2 \left(\frac{k_1}{k_2}\right)' V_1 = 0. \end{aligned} \tag{8}$$

Theorem 9. *Let M be a hypersurface in E_1^4 , and α be a regular curve on M . Then α is a generalized LC helix if and only if $\frac{k_1}{k_2}$ is constant.*

Proof. Let M be a hypersurface in E_1^4 , and α is a generalized LC helix on M . Without loss of generality, assume α has unit speed. By the definition of LC helix, there exists a Levi Civita’s notion of parallel vector field X such that

$$g(V_1, X) = c_1 \tag{9}$$

for some non zero constant c_1 . By taking the derivative of (9) and applying the Frenet formulas

$$\begin{aligned} g(\nabla_{V_1} V_1, X) &= 0, \\ g(k_1 V_2, X) &= 0. \end{aligned}$$

Since $k_1 \neq 0$, then we get

$$g(V_2, X) = 0.$$

Now, X is perpendicular to V_2 , so

$$X = aV_1 + bV_3, \tag{10}$$

for some non zero function a, b . Because X is a unit Levi Civita's notion of parallel vector field, $\nabla_{V_1} X = 0$. By taking the derivative of (10), and applying the Frenet formulas we have

$$0 = a V_1 + (ak_1 - \varepsilon_1 \varepsilon_2 k_2 b) V_2 + b V_3.$$

Since $\{V_1, V_2, V_3\}$ are linearly independent we have

$$a = 0,$$

$$ak_1 - \varepsilon_1 \varepsilon_2 k_2 b = 0,$$

$$b = 0.$$

Hence

$$\frac{k_1}{k_2} = \varepsilon_1 \varepsilon_2 \frac{b}{a} = \text{constant}.$$

Now suppose that $\frac{k_1}{k_2}$ is constant. We can choose $\frac{k_1}{k_2} = \varepsilon_1 \varepsilon_2 c$, for some non zero constant c and define

$$X = \frac{1}{c} V_1 + V_3$$

to get

$$\nabla_{V_1} X = \frac{1}{c} k_1 V_2 - \varepsilon_1 \varepsilon_2 k_2 V_2 = 0.$$

Hence, X is a Levi Civita's notion of parallel vector field and clearly $g(X, V_1) = \frac{1}{c}$ is constant. Thus α is a generalized LC helix. □

Example 10. Timelike LC Helix on S_1^3 .

$$\alpha(t) = \left(\frac{2}{3} \cos 3t, \frac{2}{3} \sin 3t, \frac{\sqrt{5}}{3} \cosh 3t, \frac{\sqrt{5}}{3} \sinh 3t \right).$$

is a unit speed curve on S_1^3 . The Frenet vector fields $\{V_1, V_2, V_3\}$ and the curvatures k_1 and k_2 , are obtained as follows:

$$\begin{aligned} V_1 &= \left(-2 \sin 3t, 2 \cos 3t, \sqrt{5} \sinh 3t, \sqrt{5} \cosh 3t \right), \\ V_2 &= \left(-\frac{\sqrt{5}}{3} \cos 3t, -\frac{\sqrt{5}}{3} \sin 3t, \frac{2}{3} \cosh 3t, \frac{2}{3} \sinh 3t \right), \\ V_3 &= \left(\sqrt{5} \sin 3t, -\sqrt{5} \cos 3t, -2 \sinh 3t, -2 \cosh 3t \right), \\ k_1 &= 4\sqrt{5}, \quad k_2 = 9. \end{aligned}$$

Then $\alpha(t)$ is a timelike LC helix with the axis

$$X = \left(\frac{\sqrt{5}}{10} \sin 3t, -\frac{\sqrt{5}}{10} \cos 3t, \frac{1}{4} \sinh 3t, \frac{1}{4} \cosh 3t \right).$$

Theorem 11. Let M be a hypersurface in E_1^4 , and α be a regular curve on M . Then α is a generalized LC helix if and only if the equation

$$\begin{aligned} \nabla_{V_1}^3 V_1 - \left(3 \frac{k_1}{k_2} \right) \nabla_{V_1}^2 V_1 \\ + \left[-\frac{k_1''}{k_1} + 3 \left(\frac{k_1}{k_1} \right)^2 + \varepsilon_0 \varepsilon_1 k_1^2 + \varepsilon_1 \varepsilon_2 k_2^2 \right] \nabla_{V_1} V_1 = 0. \end{aligned} \quad (11)$$

holds for α .

Proof. Let α be a generalized LC helix. Since α is a regular curve, it satisfies the equation in (11). Moreover, from Theorem(9) we have $\frac{k_1}{k_2}$ is constant. Thus

$$-\left(2 \frac{k_1}{k_1} + \frac{k_2}{k_2} \right) = -3 \frac{k_1}{k_1},$$

$$\left[-\frac{k_1''}{k_1} + \frac{k_1}{k_1} \frac{k_2}{k_2} + 2 \left(\frac{k_1}{k_1} \right)^2 + \varepsilon_0 \varepsilon_1 k_1^2 + \varepsilon_1 \varepsilon_2 k_2^2 \right]$$

$$= -\frac{k_1''}{k_1} + 3\left(\frac{k_1}{k_1}\right)^2 + \varepsilon_0\varepsilon_1k_1^2 + \varepsilon_1\varepsilon_2k_2^2,$$

$$\varepsilon_0\varepsilon_1k_1k_2\left(\frac{k_1}{k_2}\right)' = 0,$$

then setting these equalities in (8) we obtain equation (11).

Conversely assume that the regular curve α satisfies the equality in . Since α is a regular curve it also satisfies the equality in (11). Substracting (11) from (8), we get

$$X\nabla_{V_1}^2V_1 + Y\nabla_{V_1}V_1 + \varepsilon_0\varepsilon_1ZV_1 = 0, \tag{12}$$

where

$$X = \frac{k_1}{k_1} - \frac{k_2}{k_2},$$

$$Y = \frac{k_1}{k_1} \frac{k_2}{k_2} - \left(\frac{k_1}{k_1}\right)^2,$$

$$Z = k_1k_2\left(\frac{k_1}{k_2}\right)'.$$

Setting the equations

$$\nabla_{V_1}V_1 = k_1V_2,$$

$$\nabla_{V_1}^2V_1 = (-\varepsilon_0\varepsilon_1k_1^2)V_1 + k_1V_2 + k_1k_2V_3,$$

in (12), we obtain

$$X(-\varepsilon_0\varepsilon_1k_1^2V_1 + k_1V_2 + k_1k_2V_3) + Yk_1V_2 + \varepsilon_0\varepsilon_1ZV_1 = 0$$

and

$$(-\varepsilon_0\varepsilon_1k_1^2X + \varepsilon_0\varepsilon_1Z)V_1 + (Xk_1 + Yk_1)V_2 + (k_1k_2X)V_3 = 0.$$

Since $\{V_1, V_2, V_3\}$ are linearly independent we have

$$-\varepsilon_0\varepsilon_1k_1^2X + \varepsilon_0\varepsilon_1Z = 0,$$

$$Xk_1 + Yk_1 = 0,$$

$$k_1k_2X = 0.$$

Hence

$$\left(\frac{k_1}{k_2}\right)' = 0.$$

Solution of this equation leads to

$$\frac{k_1}{k_2} = \text{constant}$$

Thus the regular curve α is a generalized LC helix. □

Theorem 12. *Let M be a hypersurface in E_1^4 , and α be a regular curve on M . If α is a cyclic LC helix (i.e. $k_1 = \text{constant}$, $k_2 = \text{constant}$) then*

$$\nabla_{V_1}^3 V_1 + (\varepsilon_0 \varepsilon_1 k_1^2 + \varepsilon_1 \varepsilon_2 k_2^2) \nabla_{V_1} V_1 = 0.$$

Proof. The proof is straight forward from Theorem(11). □

Theorem 13. *Let M be a hypersurface in E_1^4 and $\alpha : I \subset \mathbb{R} \rightarrow M$ be a unit speed curve with Frenet vector fields $\{V_1, V_2, V_3\}$. Then α is a generalized LC helix if and only if $\det(\nabla_{V_1} V_3, \nabla_{V_1}^2 V_3, \nabla_{V_1}^3 V_3) = 0$.*

Proof. Let α be a generalized LC helix. From the Frenet formulas we have

$$\nabla_{V_1} V_3 = -\varepsilon_1 \varepsilon_2 k_2 V_2. \tag{13}$$

By taking the derivative of (13) and applying the Frenet formulas twice, we get

$$\nabla_{V_1}^2 V_3 = -\varepsilon_1 k_1 k_2 V_1 - \varepsilon_1 \varepsilon_2 k_2 V_2 - \varepsilon_1 \varepsilon_2 k_2^2 V_3,$$

$\nabla_{V_1}^3 V_3 = (-\varepsilon_1 k_1 k_2 - 2\varepsilon_1 k_1 k_2) V_1 + (-\varepsilon_1 k_1^2 - \varepsilon_1 \varepsilon_2 k_2 + k_2^3) V_2 + (-3\varepsilon_1 \varepsilon_2 k_2 k_2),$ respectively.

A straightforward computation leads to the following

$$\det(\nabla_{V_1} V_3, \nabla_{V_1}^2 V_3, \nabla_{V_1}^3 V_3) = \varepsilon_1 k_2^5 \left(\frac{k_1}{k_2}\right).$$

Since α is a general LC helix, then $\frac{k_1}{k_2}$ is constant, from which we have

$$\det(\nabla_{V_1} V_3, \nabla_{V_1}^2 V_3, \nabla_{V_1}^3 V_3) = 0.$$

Conversely, we assume that the equation $\det(\nabla_{V_1} V_3, \nabla_{V_1}^2 V_3, \nabla_{V_1}^3 V_3) = 0$ holds, we easily obtain that $\frac{k_1}{k_2}$ is a constant function, then from Theorem (9) we have α is a generalized LC helix, which concludes the proof. □

Theorem 14. *Let M be a hypersurface in E_1^4 and $\alpha : I \subset \mathbb{R} \rightarrow M$ be a unit speed curve with Frenet vector fields $\{V_1, V_2, V_3\}$. Then α is a generalized LC helix if and only if $\det(\nabla_{V_1}^2 \alpha, \nabla_{V_1}^3 \alpha, \nabla_{V_1}^4 \alpha) = 0$.*

Proof. Let α be a generalized LC helix. From the definition of Frenet frame

$$\nabla_{V_1}\alpha = V_1.$$

Thus

$$\nabla_{V_1}(\nabla_{V_1}\alpha) = \nabla_{V_1}V_1 = k_1V_2. \tag{14}$$

By taking the derivative of (14) and applying the Frenet formulas twice, we get

$$\nabla_{V_1}(\nabla_{V_1}^2\alpha) = \nabla_{V_1}^2V_1 = (-\varepsilon_0\varepsilon_1k_1^2)V_1 + k_1V_2 + k_1k_2V_3,$$

$$\nabla_{V_1}(\nabla_{V_1}^3\alpha) = \nabla_{V_1}^3V_1 = AV_1 + BV_2 + CV_3,$$

respectively, where $A = -3\varepsilon_0\varepsilon_1k_1k_1, B = -\varepsilon_0\varepsilon_1k_1^3 + k_1 - \varepsilon_1\varepsilon_2k_1k_2^2, C = 2k_1k_2 + k_1k_2$.

A straightforward computation leads to the following

$$\det(\nabla_{V_1}^2\alpha, \nabla_{V_1}^3\alpha, \nabla_{V_1}^4\alpha) = \varepsilon_0\varepsilon_1k_1^5 \left(\frac{k_2}{k_1}\right).$$

Since α is a general LC helix, then $\frac{k_1}{k_2}$ is constant, from which we have

$$\det(\nabla_{V_1}^2\alpha, \nabla_{V_1}^3\alpha, \nabla_{V_1}^4\alpha) = 0.$$

Conversely, we assume that the equation $\det(\nabla_{V_1}^2\alpha, \nabla_{V_1}^3\alpha, \nabla_{V_1}^4\alpha) = 0$ holds, we easily obtain that $\frac{k_1}{k_2}$ is a constant function, then from Theorem (9) we have a generalized LC helix, which concludes the proof. □

4. Conclusion

Helix is one of the most fascinating curves in science and nature. Helices arise in nanosprings, carbon nanotubes, α -helices, DNA double and collagen triple helix. Due to these properties, helix has played a very important role in the study of scientist.

Lorentzian geometry helps to bridge the gap between modern differential geometry and the mathematical physics of general relativity by giving an invariant treatment of Lorentzian geometry.

In this paper we give the definition of a generalized LC helix for a non-null curve lying on a hypersurface in \mathbb{E}_1^{n+1} by using the Levi Civita's notion of parallel vector field. In particular, if the hypersurface which the non-null curve lying on is replaced by \mathbb{E}_1^n , then the definition of a helix on \mathbb{E}_1^n coincides with the definition of generalized LC helix.

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