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On the Striction Curves of Involute and Bertrandian Frenet Ruled Surfaces in E^3

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Abstract

In this paper, we consider twelve special ruled surfaces associated to the evolute curve α, involute curve α^* and Bertrand curve α^{**} with $k_1 \neq 0$. They are called as Frenet ruled surface, Involute Frenet ruled surfaces, and Bertrandian Frenet ruled surface cause of their generators are the Frenet vector fields of evolute curve α. First we give all the parametrizations of all Frenet ruled surfaces. Further we give only three matrices of striction curves along the Frenet ruled surface, Involute Frenet ruled surfaces, and Bertrandian Frenet ruled surface.

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1 Introduction and Preliminaries

A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. A ruled surface is one which can be generated by the motion of a straight line in Euclidean $3-space$ [3], [4]. Choosing a directrix on the surface, i.e. a smooth unit speed curve $\alpha(s)$ orthogonal to the straight lines, and then choosing $v(s)$ to be unit vectors along the curve in the direction of the lines, the velocity vector α_s and v satisfy $\langle\alpha^{'}, v\rangle = 0$. The fundamental forms of the $B-scroll$ with null directrix and Cartan frame in the Minkowskian 3-space is examined in [12]. The properties of the $B-scroll$ are also examined in Euclidean $3-space$ and $n-space$ and in Lorentzian $3-space$ and $n-space$ with time-like directrix curve and null rulings ([6],[9], [10], [11], [13], [18]). The striction point on a ruled surface $\varphi(s, v) = \alpha(s) + ve(s)$ is the foot of the common normal between two consecutive generators (or ruling). To illustrate the current situation, we bring here the famous example of L. K. Graves [6], so called the $B-scroll$. The special ruled surfaces $B-scroll$ over null curves with null rulings in 3-dimensional Lorentzian space form has been introduced by L. K. Graves. The Gauss map of B-scrolls has been examined in [2]. Deriving curves based on the other curves is a subject in geometry. Involute-evolute curves, Bertrand curves are this kind of curves. The involute of a given curve is a well-known concept in Euclidean $3-space$. We can say that; evolute and involute is a method of deriving a new curve based on a given curve. The involute of the curve is called sometimes *the evolvent*. Involvents play a part in the construction of gears. The evolute is the locus of the centers of tangent circles of the given planar curve [16]. Let α and α^* be the curves in Euclidean $3-space$. The tangent lines to a curve α generate a surface called the tangent surface of α. If the curve α^* which lies on the tores intersect the tangent lines orthogonally is called an involute of α. If a curve α^* is an involute of α, then by definition α is an evolute of α^*. Hence given α, its evolutes are the curves whose tangent lines intersect α orthogonally. By using the similiar method we produce a new ruled surface based on the other ruled surface. $Involute\ B-scroll$ is defined in [13] . The differential geometric elements of the *involute* $\tilde{D}$ *scroll* are examined in [17].It is well-known that, if a curve is differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve α, is called Frenet-Serret apparatus of the curves. Let Frenet vector fields be $V_1(s), V_2(s), V_3(s)$ of α and let the first and second curvatures of the curve $\alpha(s)$ be $k_1(s)$ and $k_2(s)$, respectively. The quantities $\{V_1, V_2, V_3, D, k_1, k_2\}$ are collectively Frenet-Serret apparatus of the curves. Let a rigid object move along a regular curve described parametrically by $\alpha(s)$. This object has its own intrinsic coordinate system. Here Darboux vector D is the areal velocity vector of the Frenet frame of a space curve. It is named after Gaston Darboux who discovered it. The Darboux vector provides a concise way of interpreting curvature k_1 and torsion k_2 geometrically; curvature is the measure of the rotation of the Frenet frame about the binormal unit vector, and torsion is the measure of the rotation of the Frenet frame about the tangent unit vector. For any unit speed curve α,in terms of the Frenet-Serret apparatus, the Darboux vector can be expressed as $D(s) = k_2(s)V_1(s) + k_1(s)V_3(s)$ where curvature functions are defined by $k_1 = k_1(s) = \|V_1(s)\|$ *and* $k_2(s) = -\left\langle V_2, \dot{V}_3 \right\rangle$. The Darboux vector field of α and it has the following symmetrical properties [7], $D \times V_1 = \dot{V}_1, D \times V_2 = \dot{V}_2, D \times V_3 = \dot{V}_3$. Let a vector field be $\tilde{D}(s) = \frac{k_2}{k_1}(s)V_1(s) + V_3(s)$ along $\alpha(s)$ under the condition that $k_1(s) \neq 0$ and it is called the modified Darboux vector field of α [14]. Also it is trivial that $\tilde{D}(s)' = \left(\frac{k_2}{k_1}\right)' V_1$. The Frenet formulae are also well known as

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Let unit speed regular curve $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be given. For $\forall s \in I$, then the curve α^* is called the involute of the curve α, if the tangent at the point $\alpha(s)$ to the curve α passes through the tangent at the point $\alpha^*(s)$ to the curve α^* and $\langle V_1, V_1^* \rangle = 0$. The distance between corresponding points of the involute curve in $\mathbb{E}^3$ is $d\big(\alpha(s), \alpha^*(s)\big) = |\sigma - s|, \sigma = constant, \forall s \in I$,then we may write that

$$\alpha^*(s) = \alpha(s) + (\sigma - s)V_1(s).$$

The relations between the Frenet frames $\{V_1(s), V_2(s), V_3(s), D(s)\}$ and $\{V_1^*(s), V_2^*(s), V_3^*(s), \tilde{D}^*(s)\}$ are as follows:

$$\begin{cases} V_1^* = V_2 \\ V_2^* = \dfrac{k_2V_1 + k_1V_3}{(k_1^2 + k_2^2)^{\frac{1}{2}}} \\ V_3^* = \dfrac{k_2V_1 + k_1V_3}{(k_1^2 + k_2^2)^{\frac{1}{2}}} \\ \tilde{D}^* = \dfrac{k_2}{\sqrt{k_1^2 + k_2^2}}V_1 - \dfrac{k_1'k_2 - k_1k_2'}{(k_1^2 + k_2^2)^{\frac{3}{2}}}V_2 + \dfrac{k_1}{\sqrt{k_1^2 + k_2^2}}V_3. \end{cases} \tag{1.1}$$

Theorem 1.1 *Let (α, α^{*}) be a involute-evolute curves in $\mathbb{E}^3$. For the curvatures and the torsions of the involute-evolute curve (α, α^{*}) we have,*

$$\left\{ k_1^* = \frac{\sqrt{k_1^2 + k_2^2}}{(\sigma - s)\, k_1} \quad k_2^* = \frac{k_1 k_2' - k_1' k_2}{(\sigma - s)\, k_1 (k_1^2 + k_2^2)} \right. , ([8], [15]). \tag{1.2}$$

Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^{**} : I \to \mathbb{E}^3$ be the C^2- class differentiable unit speed two curves and let $V_1(s), V_2(s), V_3(s)$ and $V_1^{**}(s), V_2^{**}(s), V_3^{**}(s)$ be the Frenet frames of the curves α and α^{**}, respectively. If the principal normal vector V_2 of the curve α is linearly dependent on the principal normal vector V_2^{**} of the curve α^{**}, then the pair (α, α^{**}) are called Bertrand curve pair, [8],[15]. Also α^{**} is called Bertrand mate. If the curve α^{**} is Bertrand mate of α, then we may write that

$$\alpha^{**}(s) = \alpha(s) + \lambda V_2(s) \tag{1.3}$$

If the curve α^{**} is Bertrand mate $\alpha(s)$, then we have that $\langle V_1^{**}(s), V_1(s) \rangle = \cos\theta = constant$.

Theorem 1.2 *The distance between corresponding points of the Bertrand curve pair in $\mathbb{E}^3$ is constant, [8],[15].*

Theorem 1.3 *If $k_2(s) \neq 0$ along $\alpha(s)$, then $\alpha(s)$ is a Bertrand curve if and only if there exist nonzero real numbers λ and β such that constant $\lambda k_1 + \beta k_2 = 1$ for any $s \in I$. It follows from this fact that a circular helix is a Bertrand curve, [8].*

Theorem 1.4 *Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^{**} : I \to \mathbb{E}^3$ be the C^2- class differentiable unit speed two curves and the quantities $\left\{ V_1, V_2, V_3, \tilde{D}, k_1, k_2 \right\}$ and $\left\{ V_1^{**}, V_2^{**}, V_3^{**}, \tilde{D}^{**}, k_1^{**}, k_2^{**} \right\}$ are collectively Frenet-Serret apparatus of the curves α and the Bertrand mate α^{**}, respectively, then*

$$\begin{cases} V_1^{**} = \frac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2 + \beta^2}} \\ V_2^{**} = V_2 \\ V_3^{**} = \frac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2 + \beta^2}}; \ \lambda k_2 > 0 \\ \tilde{D}^{**} = \frac{k_1 \sqrt{\lambda^2 + \beta^2}}{(\beta k_1 - \lambda k_2)} \tilde{D}. \end{cases}$$

the first and second curvatures of the offset curve α^{**} are given by

$$k_1^{**} = \frac{\beta k_1 - \lambda k_2}{(\lambda^2 + \beta^2)\, k_2}, \quad k_2^{**} = \frac{1}{(\lambda^2 + \beta^2)\, k_2}, [8].$$

1.1 Striction curve of Frenet ruled surface

Frenet ruled surface is one which can be generated by the motion of a Frenet vector of any curve in Euclidean $3-space$. Tangent, Normal, Binormal, Darboux ruled surfaces of any curve are collectively named Frenet ruled surfaces. They have the following equations.

Definition 1.5 *In the Euclidean $3-space$, let $\alpha(s)$ be the arclengthed curve. The equations*

$$\begin{cases} \varphi_1(s,u_1) = \alpha(s) + u_1 V_1(s) \\ \varphi_2(s,u_2) = \alpha(s) + u_2 V_2(s) \\ \varphi_3(s,u_3) = \alpha(s) + u_3 V_3(s) \\ \varphi_4(s,u_4) = \alpha(s) + u_4 \tilde{D}(s) \end{cases}$$

are the parametrization of Frenet ruled surfaces which are called $V_1-scroll$ (tangent ruled surface), $V_2-scroll$ (normal ruled surface), $V_3-scroll$ (binormal ruled surface), Darboux ruled surface, respectively in [5].

Definition 1.6 *The striction point on a ruled surface $\varphi(s,v) = \alpha(s)+ve(s)$ is the foot of the common normal between two consecutive generators (or ruling). The set of striction points defines the striction curve given by*

$$c(s) = \alpha(s) - \frac{\langle \alpha_s, e_s \rangle}{\langle e_s, e_s \rangle} e(s).$$

For more detail see in, [3].

Theorem 1.7 *The striction curves of four Frenet ruled surfaces is given by the following matrix*

$$\begin{bmatrix} c_1-\alpha \\ c_2-\alpha \\ c_3-\alpha \\ c_4-\alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1}{k_2^2+k_2^2} & 0 \\ 0 & 0 & 0 \\ \frac{-k_2}{k_1\left(\frac{k_2}{k_1}\right)'} & 0 & \frac{-1}{\left(\frac{k_2}{k_1}\right)'} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Proof. Lets the striction curve of a tangent ruled surface φ_1 is $c_1(s)$, since

$$c_1(s) = \alpha(s) - \frac{\langle V_1, k_1 V_2 \rangle}{\langle V_1(s)_s, V_1(s)_s \rangle} V_1(s)$$

the striction curve of a tangent ruled surface is its directrix curve α,that is

$$c_1(s) = \alpha(s).$$

Similarly, the striction curves of normal ruled surface, binormal ruled surface and darboux ruled surface are given following this equations:

$$\begin{aligned} c_2(s) &= \alpha + \frac{k_1}{k_1^2 + k_2^2} V_2, \\ c_3(s) &= \alpha(s), \\ c_4(s) &= \alpha - \frac{k_2}{k_1 \left(\frac{k_2}{k_1}\right)'} V_1 - \frac{1}{\left(\frac{k_2}{k_1}\right)'} V_3 \end{aligned}$$

this completes the proof. It is easy to give the matrix, since we have already get the following equalities; this completes the proof. ■

2 Striction curves of Involute and Bertrandian Frenet ruled surface

In this section, first the tangent, normal, binormal, Darboux Frenet ruled surfaces of the *involute* α^* of the evolute α have been given as in the following definitions Further we write their parametric equations in terms of the Frenet apparatus of the evolute α with arclength s, they are called "*Involute tangent*, *normal*, *binormal*, *Darboux Frenet ruled surfaces* of curve α" as in the following way. In short way we use "*the Involute Frenet ruled surfaces* α."

Definition 2.1 *In the Euclidean* $3-$ *space, let* $\alpha^*(s)$ *be involute of* $\alpha(s)$ *with arcparameter* s. *The equations*

$$\begin{cases} \varphi_1^*(s, v_1) = \alpha^*(s) + v_1 V_1^*(s) \\ \varphi_2^*(s, v_2) = \alpha^*(s) + v_2 V_2^*(s) \\ \varphi_3^*(s, v_3) = \alpha^*(s) + v_3 V_3^*(s) \\ \varphi_4^*(s, v_4) = \alpha^*(s) + v_4 \tilde{D}^*(s) \end{cases}$$

are the parametrization of Frenet ruled surface of involute curve $\alpha^*(s)$

These surfaces express depending upon the evolute curve:

$$\begin{cases} \varphi_1^*(s, v_1) = \alpha(s) + (\sigma - s) V_1(s) + v_1 V_2(s), \\ \varphi_2^*(s, v_2) = \alpha(s) + (\sigma - s) V_1(s) + v_2 \left(\frac{-k_1 V_1 + k_2 V_3}{\left(k_1^2 + k_2^2\right)^{\frac{1}{2}}} \right), \\ \varphi_3^*(s, v_3) = \alpha(s) + (\sigma - s) V_1(s) + v_3 \left(\frac{k_2 V_1 + k_1 V_3}{\left(k_1^2 + k_2^2\right)^{\frac{1}{2}}} \right), \\ \varphi_4^*(s, v_4) = \alpha(s) + (\sigma - s) V_1(s) \\ \qquad + v_4 \left(\frac{k_2}{\sqrt{k_1^2 + k_2^2}} V_1 - \frac{k_1' k_2 - k_1 k_2'}{\left(k_1^2 + k_2^2\right)^{\frac{3}{2}}} V_2 + \frac{k_1 V_3}{\sqrt{k_1^2 + k_2^2}} \right) \end{cases}$$

are the parametrization of Frenet ruled surface of evolute curve $\alpha(s)$, which are called **Involute Normal ruled surface**, **Involute binormal ruled surface** and **Involute Darboux ruled surface**, respectively. They are collectively **Involute Frenet ruled surface**

Also we give the tangent, normal, binormal, Darboux Frenet ruled surfaces of the Bertran mate α^{**}. Similarly we write their parametric equations in terms of the Frenet apparatus of the Bertrand curve α. Hence they are called "*Bertrandian tangent, normal, binormal, Darboux Frenet ruled surfaces* of curve α as in the following way.

Definition 2.2 *In the Euclidean* $3-$ *space, let* $\alpha^{**}(s)$ *be Bertrand mate of* $\alpha(s)$ *with arcparameter* s. *The equations*

$$\begin{cases} \varphi_1^{**}(s,w_1) = \alpha^{**}(s) + w_1 V_1^{**}(s), \\ \varphi_2^{**}(s,w_2) = \alpha^{**}(s) + w_2 V_2^{**}(s), \\ \varphi_3^{**}(s,w_3) = \alpha^{**}(s) + w_3 V_3^{**}(s), \\ \varphi_4^{**}(s,w_4) = \alpha^{**}(s) + w_4 \tilde{D}^{**}(s) \end{cases}$$

are the parametrization of Frenet ruled surface of Bertrand mate $\alpha^{**}(s)$.

From theorem 1.4, we can write these surface equations

$$\begin{cases} \varphi_1^{**}(s,w_1) = \alpha + \lambda V_2 + w_1 \frac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2+\beta^2}}, \\ \varphi_2^{**}(s,w_2) = \alpha + (\lambda + w_2) V_2, \\ \varphi_3^{**}(s,w_3) = \alpha + \lambda V_2 + w_3 \left(\frac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2+\beta^2}} \right), \\ \varphi_4^{**}(s,w_4) = \alpha + \lambda V_2 + w_4 \frac{k_1 \sqrt{\lambda^2+\beta^2}}{(\beta k_1 - \lambda k_2)} \tilde{D}. \end{cases}$$

are the parametrization of Frenet ruled surface which are called **Bertrandian Tangent ruled surface**, **Bertrandian Normal ruled surface**, **Bertrandian Binormal ruled surface** and **Bertrandian Darboux ruled surface**, respectively. They are collectively **Bertrandian Frenet ruled surface**

Theorem 2.3 *The striction curves of four Frenet ruled surfaces along the involute curve* α^* *is given by the following matrix*

$$\begin{bmatrix} c_1^* - \alpha^* \\ c_2^* - \alpha^* \\ c_3^* - \alpha^* \\ c_4^* - \alpha^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1^*}{k_1^{*2}+k_2^{*2}} & 0 \\ 0 & 0 & 0 \\ -\frac{k_2^*}{k_1^* \left(\frac{k_2^*}{k_1^*}\right)'} & 0 & -\frac{1}{\left(\frac{k_2^*}{k_1^*}\right)'} \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}.$$

Proof. It is trivial that four the striction curves of four Frenet ruled surfaces along the *involute* curve α^* are

$$\begin{aligned}
c_1^*(s) &= \alpha^*(s) \\
c_2^*(s) &= \alpha^*(s) + \frac{k_1^*}{k_1^{*2}+k_2^{*2}} V_2^*(s) \\
c_3^*(s) &= \alpha^*(s) \\
c_4^*(s) &= \alpha^* - \frac{\frac{k_2^*}{k_1^*}}{\left(\frac{k_2^*}{k_1^*}\right)'} V_1^* - \frac{1}{\left(\frac{k_2^*}{k_1^*}\right)'} V_3^*.
\end{aligned}$$

Hence we have the proof. ■

Theorem 2.4 *The equatinos of the striction curves of four Involute Frenet ruled surfaces of the curve* α *in terms of Frenet apparatus of evolute curve* α

$$\begin{bmatrix} c_1^*-\alpha \\ c_2^*-\alpha \\ c_3^*-\alpha \\ c_4^*-\alpha \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ \lambda\left(1-\frac{k_1^2}{\left(k_1^2+k_2^2\right)(1+m)}\right) & 0 & \lambda\frac{k_1k_2}{\left(k_1^2+k_2^2\right)(1+m)} \\ \lambda & 0 & 0 \\ \lambda-\frac{k_2}{m'\ \eta^{\frac{1}{2}}} & -\frac{m}{m'} & \frac{k_1}{m'\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix},$$

here $(\sigma-s)=\lambda . k_1^2+k_2^2=\eta$ *and* $\left(\frac{k_2}{k_1}\right)'=\mu.$

Proof. The equatinos of the striction curves of four Frenet ruled surfaces along the involute curve α^* are

$$\begin{aligned}
c_1^*(s) &= \alpha+\lambda V_1 \\
c_2^*(s) &= \alpha^* + \frac{k_1^*}{k_1^{*2}+k_2^{*2}} V_2^* \\
c_3^*(s) &= \alpha+\lambda V_1 \\
c_4^*(s) &= \alpha^* - \frac{\frac{k_2^*}{k_1^*}}{\left(\frac{k_2^*}{k_1^*}\right)'} V_1^* - \frac{1}{\left(\frac{k_2^*}{k_1^*}\right)'} V_3^*.
\end{aligned}$$

Now substituing $k_1^* = \dfrac{\sqrt{k_1^2+k_2^2}}{(\sigma-s)\,k_1}$, $k_2^* = \dfrac{k_1k_2'-k_1'k_2}{(\sigma-s)\,k_1(k_1^2+k_2^2)}$ in the equatinos of the striction curves of four Frenet ruled surfaces along the involute curve α^* we have

$$\begin{aligned}
c_2^*(s) &= \alpha+\lambda\left(1-k_1\frac{k_1\left(k_1^2+k_2^2\right)^2}{\left(k_1^2+k_2^2\right)^3+\left(k_2'k_1-k_1'k_2\right)^2}\right)V_1 \\
&\quad +\lambda k_2\frac{k_1\left(k_1^2+k_2^2\right)^2}{\left(k_1^2+k_2^2\right)^3+\left(k_2'k_1-k_1'k_2\right)^2}V_3
\end{aligned}$$

$$\begin{aligned} c_4^*(s) &= \alpha(s) + \left(\lambda + \frac{k_2}{\sqrt{k_1^2+k_2^2}}\left(\frac{\left(k_1'k_2 - k_1k_2'\right)}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}\right)'\right)V_1 \\ &\quad - \frac{k_1'k_2 - k_1k_2'}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}\left(\frac{\left(k_1'k_2 - k_1k_2'\right)}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}\right)'V_2 + \frac{k_1}{\sqrt{k_1^2+k_2^2}}\left(\frac{\left(k_1'k_2 - k_1k_2'\right)}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}\right)'V_3. \end{aligned}$$

and in short way lets $m = \frac{\left(k_1'k_2 - k_1k_2'\right)}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}$. Hence we can give the following equations;

$$c_2^*(s) = \alpha + \lambda\left(1 - \frac{k_1^2}{\left(k_1^2+k_2^2\right)(1+m)}\right)V_1 + \lambda\frac{k_1k_2}{\left(k_1^2+k_2^2\right)(1+m)}V_3$$

and

$$c_4^*(s) = \alpha + \left(\lambda - \frac{k_2}{m'\left(k_1^2+k_2^2\right)^{\frac{1}{2}}}\right)V_1 - \frac{m}{m'}V_2 + \frac{k_1}{m'\left(k_1^2+k_2^2\right)^{\frac{1}{2}}}V_3$$

where $\lambda = \sigma - s$ .Further it is easy to write the matrix form in terms of Frenet apparatus of evolute curve α.This completes the proof. ∎

Corollary 2.1 *If* $m = \frac{\left(k_1'k_2 - k_1k_2'\right)}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}}$ *is a constant than there is not the striction curve of the Involute Darboux ruled surface.*

Corollary 2.2 *If* $m = \frac{\left(k_1'k_2 - k_1k_2'\right)}{\left(k_1^2+k_2^2\right)^{\frac{3}{2}}} = -1$ *than there is not the striction curve of the Involute normal ruled surface.*

Theorem 2.5 *The striction curves of four Frenet ruled surfaces along the Bertrand mate* α^{**} *is given by the following matrix*

$$\begin{bmatrix} c_1^{**} - \alpha^{**} \\ c_2^{**} - \alpha^{**} \\ c_3^{**} - \alpha^{**} \\ c_4^{**} - \alpha^{**} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1^{**}}{k_1^{**2}+k_2^{**2}} & 0 \\ 0 & 0 & 0 \\ -\frac{k_2^{**}}{k_1^{**}\left(\frac{k_2^{**}}{k_1^{**}}\right)'} & 0 & -\frac{1}{\left(\frac{k_2^{**}}{k_1^{**}}\right)'} \end{bmatrix} \begin{bmatrix} V_1^{**} \\ V_2^{**} \\ V_3^{**} \end{bmatrix}.$$

Proof. It is easy to give the following matrix for the striction curves of four Frenet ruled surfaces along along the Bertrand mate α^{**}

$$\begin{aligned} c_1^{**}(s) &= \alpha^{**}(s) \\ c_2^{**}(s) &= \alpha^{**}(s) + \frac{k_1^{**}}{k_1^{**2}+k_2^{**2}}V_2^{**}(s) \\ c_3^{**}(s) &= \alpha^{**}(s) \\ c_4^{**}(s) &= \alpha^{**} - \frac{\frac{k_2^{**}}{k_1^{**}}}{\left(\frac{k_2^{**}}{k_1^{**}}\right)'}V_1^{**} - \frac{1}{\left(\frac{k_2^{**}}{k_1^{**}}\right)'}V_3^{**}. \end{aligned}$$

■

Theorem 2.6 *The equatinos of the striction curves of four Bertrandian Frenet ruled surfaces along the evolute curve α in terms of Frenet apparatus of evolute curve α*

$$\begin{bmatrix} c_1^{**}-\alpha \\ c_2^{**}-\alpha \\ c_3^{**}-\alpha \\ c_4^{**}-\alpha \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & \left(\lambda+\frac{m\left(\lambda^2+\beta^2\right)k_2}{(m^2+1)}\right) & 0 \\ 0 & \lambda & 0 \\ \frac{m'\sqrt{\lambda^2+\beta^2}k_2}{m^3} & \lambda & \frac{m'\sqrt{\lambda^2+\beta^2}k_1}{m^3} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

here $m=\beta k_1-\lambda k_2$ and $(\sigma-s)=\lambda$.

Proof. Now substituing $k_1^{**}=\frac{\beta k_1-\lambda k_2}{(\lambda^2+\beta^2)k_2}$, $k_2^{**}=\frac{1}{(\lambda^2+\beta^2)k_2}$, $,k_2k_2^{**}=\frac{1}{(\lambda^2+\beta^2)}$ in the equatinos of the striction curves of four Frenet ruled surfaces along the Bertrand mate α^{**}

$$\begin{aligned} c_1^{**}(s) &= \alpha^{**}(s)=\alpha(s)+\lambda V_2 \\ c_2^{**}(s) &= \alpha^{**}(s)+\frac{k_1^{**}}{k_1^{**2}+k_2^{**2}}V_2^{**}(s) \\ c_3^{**}(s) &= \alpha^{**}(s)=\alpha(s)+\lambda V_2 \\ c_4^{**}(s) &= \alpha^{**}(s)-\frac{D^{**}(s)}{\left(\frac{k_2^{**}}{k_1^{**}}\right)'}. \end{aligned}$$

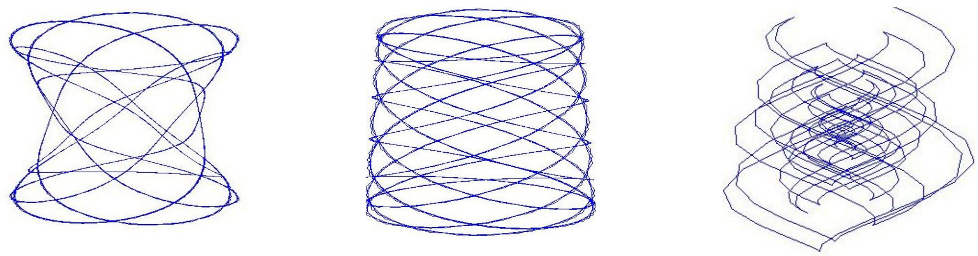
we have the equationc in terms of Frenet apparatus of bertrand curve α, where $m=\beta k_1-\lambda k_2$. ■

Example 1 *Let us consider the following Bertrand curve α, involute curve α^* and Bertrand mate α^{**}, respectively, [1].*

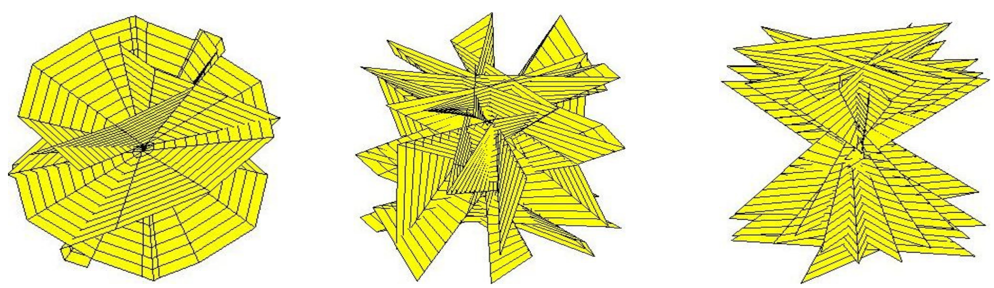
$$\alpha(s)=\Big(\frac{9}{208}\sin 16s-\frac{1}{117}\sin 36s,-\frac{9}{208}\cos 16s+\frac{1}{117}\cos 36s,\frac{6}{65}\sin 10s\Big)$$

$$\begin{aligned} \alpha^*(s) &= \Big(\frac{9}{208}\sin 16s-\frac{1}{117}\sin 36s+(\sigma-s)\big(\frac{9}{13}\cos 16s-\frac{4}{13}\cos 36s\big),-\frac{9}{208}\cos 16s \\ &\quad +\frac{1}{117}\cos 36s+(\sigma-s)\big(\frac{9}{13}\sin 16s-\frac{4}{13}\sin 36s\big),\frac{6}{65}\sin 10s+\frac{12}{13}(\sigma-s)\cos 10s\Big) \end{aligned}$$

$$\begin{aligned} \alpha^{**}(s) &= \Big(\frac{9}{208}\sin 16s-\frac{1}{117}\sin 36s+\frac{12}{13}\lambda\cos(26s),-\frac{9}{208}\cos 16s+\frac{1}{117}\cos 36s \\ &+ \frac{12}{13}\lambda\sin 26s,\frac{6}{65}\sin 10s-\frac{5}{13}\lambda\Big) \end{aligned}$$

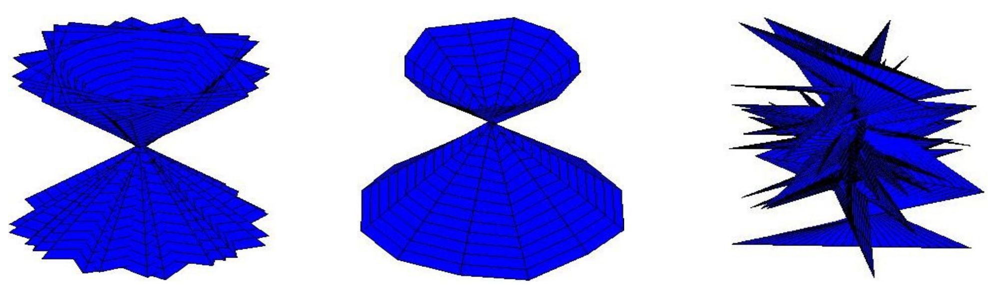


α Curve Bertrand mate of α Involute curve of α



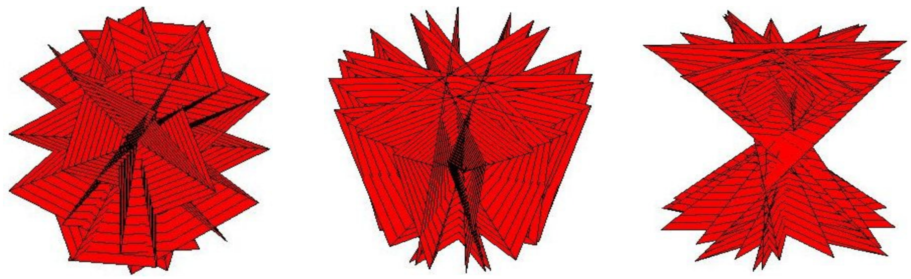
a b c

a: Tangent ruled surface
b: Bertrandian tangent ruled surface
c: Involute tangent ruled surface.



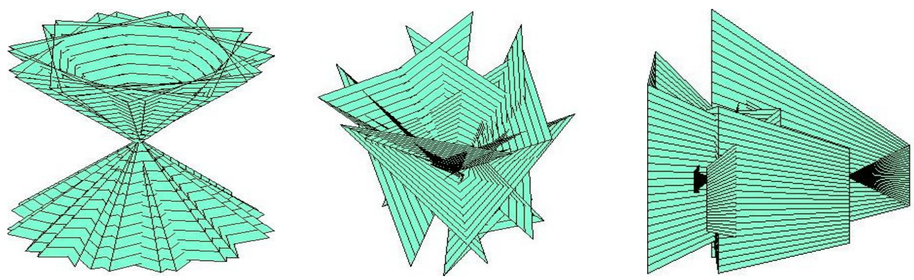
d e f

d: Normal ruled surface
e: Bertrandian normal ruled surface
f: Involute normal ruled surface.



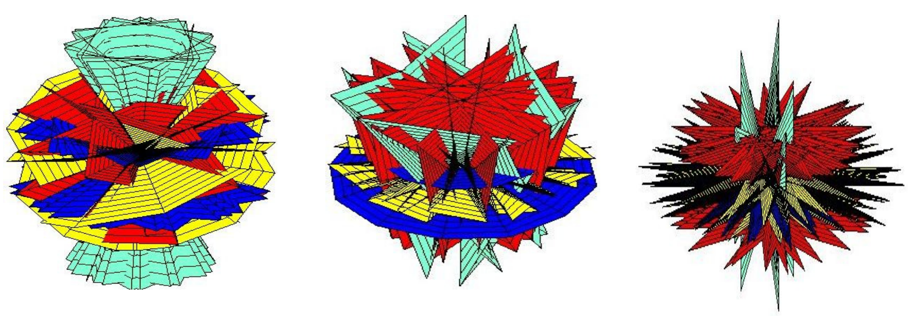
g h i

g: Binormal ruled surface
h: Bertrandian binormal ruled surface
i: Involute binormal ruled surface.



j k m

j: Darboux ruled surface
k: Bertrandian darboux ruled surface
m: Involute darboux ruled surface.



n o p

Figure 1: The figures n, o and p show the status of the surfaces relative to each other, respectively.

These figures are drawn with Mapple program for $\lambda = \sigma = 1$.

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