

Research Article

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Comparison between the (G'/G) - expansion method and the modified extended tanh method

DOI 10.1515/phys-2016-0006

Received August 23, 2015; accepted December 22, 2015

Abstract: In this paper, firstly, we give a connection between well known and commonly used methods called the $(\frac{G'}{G})$ -expansion method and the modified extended tanh method which are often used for finding exact solutions of nonlinear partial differential equations (NPDEs). We demonstrate that giving a convenient transformation and formula, all of the solutions obtained by using the $(\frac{G'}{G})$ -expansion method can be converted the solutions obtained by using the modified extended tanh method. Secondly, contrary to the assertion in some papers, the $(\frac{G'}{G})$ -expansion method gives neither all of the solutions obtained by using the other method nor new solutions for NPDEs. Namely, while the modified extended tanh method gives more solutions in a straightforward, concise and elegant manner without reproducing a lot of different forms of the same solution. On the other hand, the $(\frac{G'}{G})$ -expansion method provides less solutions in a rather cumbersome form. Lastly, we obtain new exact solutions for the Lonngren wave equation as an illustrative example by using these methods.

Keywords: The $(\frac{G'}{G})$ -expansion method; The modified extended tanh function method; The Lonngren wave equation; An exact solution

PACS: 02.30.Jr ; 02.70.Wz ; 02.60.Lj ; 03.65.Ge

1 Introduction

Over the past two decades several expansion methods for finding travelling-wave solutions to nonlinear evolution equations have been proposed, developed and extended owing to the fact that there are a lot of applications of non-

linear differential equations (NPDEs) describing different processes in many scientific areas. Particularly, because direct searching for exact solutions to NPDEs becomes more and more attractive partly due to symbolic computation in recent years, various methods for obtaining exact solutions to NPDEs have been proposed and used. Two of the methods used to solve NPDEs are the tanh function method and the $(\frac{G'}{G})$ -expansion method. The former is firstly introduced by Malfliet [1] and used in some papers [2 – 14] deriving different variations. The latter is firstly introduced by Wang and Zhang [15] and has been widely used in some papers [16 – 21].

The first aim of this paper is to demonstrate that there is a concrete connection between the $(\frac{G'}{G})$ - expansion method and the modified extended tanh method. In other words, giving a convenient transformation and formula, the solutions obtained by the $(\frac{G'}{G})$ -expansion method can be converted the solutions obtained by the modified extended tanh method.

The second aim of this paper is the $(\frac{G'}{G})$ -expansion method does not give new solutions unlike claims in some papers; however, the modified extended tanh method gives more solutions in a straightforward, concise and elegant manner without reproducing a lot of different forms of the same solution.

Although the merits and demerits of these two methods and their variations was showed and compared in some papers [22 – 28], our contribution in this paper is that we compare the most used form of the $(\frac{G'}{G})$ -expansion method and the last and powerful variation of the tanh function method called the modified extended tanh function method and present more concrete and general formula to compare these two methods. Thereby, the common and different solutions obtained by these methods can be realized easily.

In order to illustrate we solve the Lonngren wave equation

$$\frac{\partial^2}{\partial t^2} (u_{xx} - \alpha u + \beta u^2) + u_{xx} = 0, \quad (1.1)$$

which describes electric signals in telegraph lines on the basis of the tunnel diode [29 – 31]. Note that in the book

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[32], sufficient conditions of the blow-up for the corresponding initial-boundary-value problems were obtained.

This paper is organized as follows. In section 2, we have presented descriptions of the methods. In section 3, we have showed the connections between these two methods. In section 4, we have solved the Lonngren wave equation using these methods and obtained new solutions which could not be attained in the past.

2 Outline of the Methods

2.1 Description of the (G'/G)-Expansion Method

In this section, we describe the (G'/G)-expansion method for finding travelling wave solutions of nonlinear partial differential equations. Suppose that a nonlinear partial differential equation (NPDE), say in two independent variables x and t , is given by

$$P(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \dots) = 0 \tag{2.1}$$

where $u(x, t)$ is an unknown function, P is a polynomial in $u = u(x, t)$ and its various partial derivatives, in which highest order derivatives and nonlinear terms are involved.

The summary of the (G'/G)-expansion method, can be presented in the following six steps:

Step 1: To find the travelling wave solutions of Eq. (2.1) we introduce the wave variable

$$u(x, t) = u(\xi), \xi = x - ct, \tag{2.2}$$

where the constant c is generally termed the wave velocity. Substituting Eq. (2.2) into Eq. (2.1), we obtain the following ordinary differential equation (ODE) in ξ (which illustrates a principal advantage of a travelling wave solution, i.e., a PDE is reduced to an ODE).

$$P(U, cU', U', c^2U'', c^2U'', U'', \dots) = 0 \tag{2.3}$$

Step 2: If necessary we integrate Eq. (2.3) as many times as possible and set the constants of integration to be zero for simplicity.

Step 3: We suppose the solution of nonlinear partial differential equation can be expressed by a polynomial in (G'/G) as

$$u(\xi) = \sum_{i=0}^M a_i \left(\frac{G'}{G}\right)^i, \tag{2.4}$$

where $G = G(\xi)$ satisfies the second-order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{2.5}$$

where $G' = \frac{dG}{d\xi}$, $G'' = \frac{d^2G}{d\xi^2}$, and a_i, λ and μ are real constants with $a_M \neq 0$. Using the general solutions of Eq. (2.5), we have

$$\left(\frac{G'}{G}\right) = \begin{cases} -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + c_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)} \right), & \lambda^2 - 4\mu > 0 \\ -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \left(\frac{-c_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + c_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{c_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + c_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} \right), & \lambda^2 - 4\mu < 0 \\ -\frac{\lambda}{2} + \left(\frac{c_2}{c_1 + c_2 \xi}\right), & \lambda^2 - 4\mu = 0 \end{cases} \tag{2.6}$$

Moreover, it follows from Eq. (2.4) and (2.5) that

$$\begin{cases} U' = -\sum_{i=1}^M i a_i \left(\left(\frac{G'}{G}\right)^{i+1} + \lambda \left(\frac{G'}{G}\right)^i + \mu \left(\frac{G'}{G}\right)^{i-1}\right), \\ U'' = \sum_{i=1}^M i a_i \left((i+1) \left(\frac{G'}{G}\right)^{i+2} + (2i+1) \lambda \left(\frac{G'}{G}\right)^{i+1} \right. \\ \quad \left. + i(\lambda^2 + 2\mu) \left(\frac{G'}{G}\right)^i + (2i-1) \lambda \mu \left(\frac{G'}{G}\right)^{i-1} \right. \\ \quad \left. + (i-1) \mu^2 \left(\frac{G'}{G}\right)^{i-2} \right), \end{cases}$$

and so on. Here the prime denotes the derivative with respect to ξ .

Step 4: The positive integer M can be accomplished by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (2.3) as follows:

If we define the degree of $u(\xi)$ as $D[u(\xi)] = M$, then the degree of other expressions is defined by

$$\begin{aligned} D\left[\frac{d^q u}{d\xi^q}\right] &= M + q, \\ D\left[u^r \left(\frac{d^q u}{d\xi^q}\right)^s\right] &= Mr + s(q + M). \end{aligned}$$

Therefore, we can get the value of M in Eq. (2.4).

Step 5: Substituting expansion (2.4) and derivatives of $u(\xi)$ into Eq. (2.3) and collecting all terms with the same order of (G'/G) together, then setting each coefficient of this polynomial to zero, it yields a set of algebraic equations for a_i, c, λ and μ .

Step 6: Substituting a_i, c, λ and μ obtained in step 5 and the general solutions of Eq. (2.5) into (2.4). Next, depending on the sign of discriminant $(\lambda^2 - 4\mu)$, we can obtain an explicit solution of Eq. (2.1) immediately.

2.2 Description of the Modified Extended tanh method

We first consider $u(x, t) = u(\xi)$, $\xi = x - ct$, then Eq. (2.1) becomes an ordinary differential equation. In order to seek the travelling wave solutions of Eq. (2.1), we use the following ansatz

$$u(\xi) = S(\phi) = A_0 + \sum_{i=1}^M [A_i \phi^i + B_i \phi^{-i}], \quad (2.7)$$

$$\phi' = b + \phi^2, \quad (2.8)$$

where $A_0, A_i, B_i, 1 \leq i \leq M$ are constants to be determined later and b is a parameter, $\phi = \phi(\xi)$, $\phi' = \frac{d\phi}{d\xi}$. The parameter M can be found by balancing the highest-order derivative term with the nonlinear terms. Inserting (2.7) and (2.8) into the ordinary differential equation (2.3) will yield a system of algebraic equations with respect to A_0, A_i, B_i, b and c (where $1 \leq i \leq M$) because all the coefficients of ϕ^i have to vanish. Solving the resulting system of coefficients of ϕ^i we can then determine A_0, A_i, B_i, b and c . Using the general solutions of Riccati differential equation (2.8) as follows:

(i) If $b < 0$

$$\begin{aligned} \phi &= -\sqrt{-b} \tanh(\sqrt{-b}(\xi + \xi_0)) \\ \text{or } \phi &= -\sqrt{-b} \coth(\sqrt{-b}(\xi + \xi_0)), \end{aligned} \quad (2.9)$$

it depends on initial conditions.

(ii) If $b > 0$

$$\phi = \sqrt{b} \tan(\sqrt{b}(\xi + \xi_0)) \text{ or } \phi = -\sqrt{b} \cot(\sqrt{b}(\xi + \xi_0)), \quad (2.10)$$

it depends on initial conditions.

(iii) If $b = 0$

$$\phi = -\frac{1}{\xi + \xi_0}, \quad (2.11)$$

and substituting A_0, A_i, B_i, b and c into (2.7), we have obtained the solutions of Eq. (2.1).

3 Connection between the $\left(\frac{G'}{G}\right)$ - expansion method and the modified extended tanh - method

Theorem:

All the solutions obtained by the $\left(\frac{G'}{G}\right)$ - expansion method can be obtained by the modified extended tanh - method using following equations:

$$u(\xi) = \sum_{i=0}^M a_i \left(\frac{G'}{G}\right)^i = A_0 + \sum_{i=1}^M [A_i \phi^i + B_i \phi^{-i}],$$

$$A_n = \sum_{i=n}^M \left((-1)^i \binom{i}{i-n} a_i \left(\frac{\lambda}{2}\right)^{i-n} \right)$$

$$B_n = 0,$$

where u is the travelling wave solution of the PDE. Therefore the $\left(\frac{G'}{G}\right)$ - expansion method is a special case of the modified extended tanh - method.

Proof. If we divide the Eq. (2.5) by $G(\xi)$ and then use the simple equality $\frac{G''(\xi)}{G(\xi)} = \left(\frac{G'(\xi)}{G(\xi)}\right)' + \left(\frac{G'(\xi)}{G(\xi)}\right)^2$, we get

$$\left(\frac{G'(\xi)}{G(\xi)}\right)' + \left(\frac{G'(\xi)}{G(\xi)}\right)^2 + \lambda \frac{G'(\xi)}{G(\xi)} + \mu = 0. \quad (3.1)$$

The last equation can be written as follows:

$$\left(\frac{G'(\xi)}{G(\xi)}\right)' + \left(\frac{G'(\xi)}{G(\xi)} + \frac{\lambda}{2}\right)^2 + \frac{4\mu - \lambda^2}{4} = 0. \quad (3.2)$$

As it can be seen that Eq. (3.2) is completely equal to Eq. (2.8) when considered $\phi(\xi) = -\frac{\lambda}{2} - \frac{G'(\xi)}{G(\xi)}$ and $b = \frac{4\mu - \lambda^2}{4}$. From here, it has been reached the connection between the Eqs. (2.5) and (2.8) that they are essentially the same equation.

Namely, the solutions in (2.6) can be easily converted the solutions (2.9)–(2.11) by taking the following hyperbolic and trigonometric identities into consideration

$$\begin{aligned} \tanh(a(\xi + \xi_0)) &= \frac{\sinh a\xi \cosh a\xi_0 + \sinh a\xi_0 \cosh a\xi}{\cosh a\xi \cosh a\xi_0 + \sinh a\xi \sinh a\xi_0} \\ &= \frac{c_1 \sinh a\xi + c_2 \cosh a\xi}{c_1 \cosh a\xi + c_2 \sinh a\xi}, \end{aligned}$$

$$\begin{aligned} \coth(a(\xi + \xi_0)) &= \frac{\cosh a\xi \cosh a\xi_0 + \sinh a\xi \sinh a\xi_0}{\sinh a\xi \cosh a\xi_0 + \sinh a\xi_0 \cosh a\xi} \\ &= \frac{c_1 \cosh a\xi + c_2 \sinh a\xi}{c_1 \sinh a\xi + c_2 \cosh a\xi}, \end{aligned}$$

where $\tanh a\xi_0 = \frac{c_2}{c_1}$,

$$\begin{aligned} \tan(a(\xi + \xi_0)) &= \frac{\sin a\xi \cos a\xi_0 + \sin a\xi_0 \cos a\xi}{\cos a\xi \cos a\xi_0 - \sin a\xi \sin a\xi_0} \\ &= -\frac{c_1 \sin a\xi + c_2 \cos a\xi}{c_1 \cos a\xi + c_2 \sin a\xi}, \end{aligned}$$

$$\begin{aligned} \cot(a(\xi + \xi_0)) &= \frac{\cos a\xi \cos a\xi_0 - \sin a\xi \sin a\xi_0}{\sin a\xi \cos a\xi_0 + \sin a\xi_0 \cos a\xi} \\ &= -\frac{c_1 \cos a\xi + c_2 \sin a\xi}{-c_1 \sin a\xi + c_2 \cos a\xi}, \end{aligned}$$

where $\tan b\xi_0 = -\frac{c_2}{c_1}$.

Now, using the binomial expansion and auxiliary transformation $\frac{G'}{G} = -\phi(\xi) - \frac{\lambda}{2}$, we can gain a general formula that gives a connection between the coefficients of these two methods. We consider the expansion in (2.4) and (2.7) as follows:

$$\begin{aligned} u(\xi) &= \sum_{i=0}^M a_i \left(\frac{G'}{G}\right)^i = \sum_{i=0}^M a_i \left(-\phi(\xi) - \frac{\lambda}{2}\right)^i \\ &= A_0 + \sum_{i=1}^M [A_i \phi^i + B_i \phi^{-i}], \end{aligned} \tag{3.3}$$

so that gives

$$\begin{aligned} A_n &= \sum_{i=n}^M \left((-1)^i \binom{i}{i-n} a_i \left(\frac{\lambda}{2}\right)^{i-n} \right) \\ B_n &= 0. \end{aligned} \tag{3.4}$$

From here, the solutions obtained by the $\left(\frac{G'}{G}\right)$ - expansion method can be converted easily to the some solutions of the modified extended tanh method via (3.4). However, when $B_n \neq 0$, the modified extended tanh method gives many more solutions which can not be obtained by the $\left(\frac{G'}{G}\right)$ - expansion method. Particularly, the solutions consist of combinations of tanh-coth and tan-cot functions.

□

In order to indicate these common and different solutions, we will give a concrete example as solving the Lonngren wave equation.

4 The Lonngren Wave Equation as an illustrative example

The Lonngren wave equation is given by

$$\frac{\partial^2}{\partial t^2} (u_{xx} - \alpha u + \beta u^2) + u_{xx} = 0. \tag{4.1}$$

Using the wave variable $\xi = x - ct$ in Eq. (4.1), then integrating this equation and considering the integration constant to be zero, we obtain

$$c^2 U'' + (1 - \alpha c^2) U + \beta c^2 U^2 = 0. \tag{4.2}$$

Balancing U^2 and U'' gives $M = 2$.

4.1 Solutions by using $\left(\frac{G'}{G}\right)$ expansion method

The solutions of Eq. (4.2) can be written in the form as in (2.4)

$$U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \tag{4.3}$$

where a_0, a_1 and a_2 are constants which are unknowns to be determined later.

Substituting Eq. (4.3) and its derivatives into Eq. (4.2) and equating each coefficient of $\left(\frac{G'}{G}\right)$ to zero, we obtain a set of nonlinear algebraic equations for a_0, a_1, a_2 and c . Solving this system using Maple, we obtain

Set 1.

$$c = \mp \frac{1}{\sqrt{\alpha + \lambda^2 - 4\mu}}, a_2 = -\frac{6}{\beta}, a_1 = -\frac{6\lambda}{\beta}, a_0 = -\frac{\lambda^2 + 2\mu}{\beta};$$

Set 2.

$$c = \mp \frac{1}{\sqrt{\alpha - \lambda^2 + 4\mu}}, a_2 = -\frac{6}{\beta}, a_1 = -\frac{6\lambda}{\beta}, a_0 = -\frac{6\mu}{\beta}.$$

Using these values, when $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions respectively:

$$\begin{aligned} u_1(x, t) &= \frac{(\lambda^2 - 4\mu)}{2\beta} \\ &\left(1 - 3 \left(\frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + c_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)} \right)^2 \right), \end{aligned} \tag{4.4}$$

where $\xi = x \mp \frac{t}{\sqrt{\alpha + \lambda^2 - 4\mu}}$,

$$u_2(x, t) = \frac{3(\lambda^2 - 4\mu)}{2\beta} \left(1 - \frac{c_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + c_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{c_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + c_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)} \right)^2, \tag{4.5}$$

where $\xi = x \mp \frac{t}{\sqrt{\alpha - \lambda^2 + 4\mu}}$.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solutions respectively:

$$u_3(x, t) = \frac{(\lambda^2 - 4\mu)}{2\beta} \left(1 + 3 \frac{-c_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + c_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{c_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + c_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} \right)^2, \tag{4.6}$$

where $\xi = x \mp \frac{t}{\sqrt{\alpha + \lambda^2 - 4\mu}}$.

$$u_4(x, t) = \frac{3(\lambda^2 - 4\mu)}{2\beta} \left(1 + \frac{-c_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + c_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{c_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + c_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} \right)^2, \tag{4.7}$$

where $\xi = x \mp \frac{t}{\sqrt{\alpha - \lambda^2 + 4\mu}}$.

When $\lambda^2 - 4\mu = 0$, we obtain the rational function travelling wave solution:

$$u_5(x, t) = -\frac{6}{\beta} \left(\frac{c_2}{c_1 + c_2 \xi} \right)^2, \tag{4.8}$$

where $\xi = x \mp \frac{t}{\sqrt{\alpha}}$.

4.2 Solutions by using modified extended tanh method

The solutions of Eq. (4.2) can be written in the form as in (2.7)

$$U(\xi) = B_2 \phi^{-2} + B_1 \phi^{-1} + A_0 + A_1 \phi + A_2 \phi^2, \tag{4.9}$$

where B_2, B_1, A_0, A_1 and A_2 are constants which are unknowns to be determined later.

Substituting Eq. (4.9) and its derivatives into Eq. (4.2) and equating each coefficient of Y to zero, we obtain a set of nonlinear algebraic equations for B_2, B_1, B_0, B_1, B_2 and c . Solving this system using Maple, we obtain

Set 1.

$$c = \mp \frac{1}{\sqrt{\alpha + 4b}}, B_2 = -\frac{6b^2}{\beta}, B_1 = 0, A_2 = A_1 = 0, A_0 = -\frac{6b}{\beta};$$

Set 2.

$$c = \mp \frac{1}{\sqrt{\alpha - 4b}}, B_2 = -\frac{6b^2}{\beta}, B_1 = 0, A_2 = 0 = A_1, A_0 = -\frac{2b}{\beta};$$

Set 3.

$$c = \mp \frac{1}{\sqrt{\alpha - 4b}}, B_2 = B_1 = 0, A_2 = -\frac{6}{\beta}, A_1 = 0, A_0 = -\frac{2b}{\beta};$$

Set 4.

$$c = \mp \frac{1}{\sqrt{\alpha + 4b}}, B_2 = B_1 = 0, A_2 = -\frac{6}{\beta}, A_1 = 0, A_0 = -\frac{6b}{\beta};$$

Set 5.

$$c = \mp \frac{1}{\sqrt{\alpha + 16b}}, B_2 = -\frac{6b^2}{\beta}, B_1 = 0 = A_1, A_2 = -\frac{6}{\beta}, A_0 = -\frac{12b}{\beta};$$

Set 6.

$$c = \mp \frac{1}{\sqrt{\alpha - 16b}}, B_2 = -\frac{6b^2}{\beta}, B_1 = 0 = A_1, A_2 = -\frac{6}{\beta}, A_0 = \frac{4b}{\beta};$$

Using these values, we obtain following general solutions, respectively:

$$u_1(x, t) = -\frac{6b}{\beta} + \frac{6b}{\beta} \coth^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha + 4b}} + \xi_0 \right) \right), \tag{4.10}$$

$$u_2(x, t) = -\frac{6b}{\beta} + \frac{6b}{\beta} \tanh^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha + 4b}} + \xi_0 \right) \right), \tag{4.11}$$

$$u_3(x, t) = -\frac{6b}{\beta} - \frac{6b}{\beta} \cot^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha + 4b}} + \xi_0 \right) \right), \tag{4.12}$$

$$u_4(x, t) = -\frac{6b}{\beta} - \frac{6b}{\beta} \tan^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha + 4b}} + \xi_0 \right) \right), \tag{4.13}$$

$$u_5(x, t) = -\frac{2b}{\beta} + \frac{6b}{\beta} \coth^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha - 4b}} + \xi_0 \right) \right), \quad (4.14)$$

$$u_6(x, t) = -\frac{2b}{\beta} + \frac{6b}{\beta} \tanh^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha - 4b}} + \xi_0 \right) \right), \quad (4.15)$$

$$u_7(x, t) = -\frac{2b}{\beta} - \frac{6b}{\beta} \cot^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha - 4b}} + \xi_0 \right) \right), \quad (4.16)$$

$$u_8(x, t) = -\frac{2b}{\beta} - \frac{6b}{\beta} \tan^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha - 4b}} + \xi_0 \right) \right), \quad (4.17)$$

$$u_9(x, t) = -\frac{2b}{\beta} + \frac{6b}{\beta} \tanh^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha - 4b}} + \xi_0 \right) \right), \quad (4.18)$$

$$u_{10}(x, t) = -\frac{2b}{\beta} + \frac{6b}{\beta} \coth^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha - 4b}} + \xi_0 \right) \right), \quad (4.19)$$

$$u_{11}(x, t) = -\frac{2b}{\beta} - \frac{6b}{\beta} \tan^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha - 4b}} + \xi_0 \right) \right), \quad (4.20)$$

$$u_{12}(x, t) = -\frac{2b}{\beta} - \frac{6b}{\beta} \cot^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha - 4b}} + \xi_0 \right) \right), \quad (4.21)$$

$$u_{13}(x, t) = -\frac{6b}{\beta} + \frac{6b}{\beta} \tanh^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha + 4b}} + \xi_0 \right) \right), \quad (4.22)$$

$$u_{14}(x, t) = -\frac{6b}{\beta} + \frac{6b}{\beta} \coth^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha + 4b}} + \xi_0 \right) \right), \quad (4.23)$$

$$u_{15}(x, t) = -\frac{6b}{\beta} - \frac{6b}{\beta} \tan^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha + 4b}} + \xi_0 \right) \right), \quad (4.24)$$

$$u_{16}(x, t) = -\frac{6b}{\beta} - \frac{6b}{\beta} \cot^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha + 4b}} + \xi_0 \right) \right), \quad (4.25)$$

$$u_{17}(x, t) = \frac{4b}{\beta} + \frac{6b}{\beta} \left(\coth^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha - 16b}} + \xi_0 \right) \right) + \tanh^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha - 16b}} + \xi_0 \right) \right) \right),$$

$$u_{18}(x, t) = \frac{4b}{\beta} - \frac{6b}{\beta} \left(\cot^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha - 16b}} + \xi_0 \right) \right) + \tan^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha - 16b}} + \xi_0 \right) \right) \right),$$

$$u_{19}(x, t) = -\frac{12b}{\beta} + \frac{6b}{\beta} \left(\coth^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha + 16b}} + \xi_0 \right) \right) + \tanh^2 \left(\sqrt{-b} \left(x \mp \frac{t}{\sqrt{\alpha + 16b}} + \xi_0 \right) \right) \right),$$

$$u_{20}(x, t) = -\frac{12b}{\beta} - \frac{6b}{\beta} \left(\cot^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha + 16b}} + \xi_0 \right) \right) + \tan^2 \left(\sqrt{b} \left(x \mp \frac{t}{\sqrt{\alpha + 16b}} + \xi_0 \right) \right) \right),$$

where $b \neq 0$;

$$u_{21}(x, t) = -\frac{6}{\beta \left(x \mp \frac{t}{\sqrt{\alpha}} + \xi_0 \right)^2}, \quad (4.26)$$

where $b = 0$.

Taking the hyperbolic and trigonometric identities in section 3 into consideration, the solutions (4.4)–(4.7) and (4.18)–(4.25) are same. Besides, using the formula (3.10) the coefficients of modified extended method in set 3 and set 4 are equal to the coefficients of $\left(\frac{G'}{G}\right)$ -expansion method in set 1 and set 2, respectively. On the other hand, while the modified extended method gives us five solution set, $\left(\frac{G'}{G}\right)$ -expansion method gives only two solution set for the Lonngren wave equation. Therefore, the modified extended tanh method is more powerful than the other.

5 Conclusion

In this paper, giving the connection between the $\left(\frac{G'}{G}\right)$ -expansion method and the modified extended tanh method via transformation, we have shown that the solutions of the $\left(\frac{G'}{G}\right)$ -expansion method can also be obtained using the modified extended tanh method. Solving the Lonngren wave equation as an example with these two methods, we were not able to obtain the solutions of the combination of

tanh-coth and tan-cot function by using the $\left(\frac{G'}{G}\right)$ -expansion method. Consequently, the modified extended tanh method is both more powerful and brief. The computations in this work were performed by using Maple 12.

References

- [1] W. Malfliet, Solitary wave solutions of nonlinear wave equations, *Am. J. Phys.* 60 (7) (1992) 650-654.
- [2] W. Malfliet, W. Hereman, The tanh method. I: Exact solutions of nonlinear evolution and wave equations, *Phys. Scr.* 54 (1996) 563-568.
- [3] W. Malfliet, W. Hereman, The tanh method. II: Perturbation technique for conservative systems, *Phys. Scr.* 54 (1996) 569-575.
- [4] W. Hereman, W. Malfliet, in: *The tanh method: a tool to solve nonlinear partial differential equations with symbolic software*, Proceedings 9th World Multi-Conference on Systemics, Cybernetics and Informatics, Orlando, FL, (2005), 165-168.
- [5] A. M. Wazwaz, The tanh method for travelling wave solutions of nonlinear equations, *Appl. Math. Comput.* 154 (3) (2004) 713-723.
- [6] A. M. Wazwaz, The tanh method: exact solutions of the Sine-Gordon and the Sinh-Gordon equations, *Appl. Math. Comput.* 49 (2005) 565-574.
- [7] A. M. Wazwaz, The tanh and the sine-cosine methods for a reliable treatment of the modified equal width equation and its variants, *Comm. Nonlinear Sci. Numer. Simul.* 11 (2) (2006) 148-160.
- [8] A. M. Wazwaz, The tanh and the sine-cosine methods for compact and noncompact solutions of the nonlinear Klein-Gordon equation, *Appl. Math. Comput.* 167 (2) (2005) 1179-1195.
- [9] A. M. Wazwaz, The tanh method: solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tzitzeica-Dodd-Bullough equations, *Chaos, Solitons Fractals* 25 (1) (2005) 55-63.
- [10] A. M. Wazwaz, The tanh method for generalized forms of nonlinear heat conduction and Burgers-Fisher equations, *Appl. Math. Comput.* 169 (2005) 321-338.
- [11] A. M. Wazwaz, Travelling wave solutions of generalized forms of Burgers, Burgers-KdV and Burgers-Huxley equations, *Appl. Math. Comput.* 169 (2005) 639-656.
- [12] A. M. Wazwaz, The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations, *Appl. Math. Comput.*, 184, (2007), 1002-1014.
- [13] A. M. Wazwaz, New solitary wave solutions to the modified forms of Degasperis-Procesi and Camass-Holm equations, *Appl. Math. Comput.*, 186, (2007), 130-141.
- [14] S. A. Khuri, A complex tanh-function method applied to nonlinear equations of Schrodinger type, *Chaos, Solitons and Fractals* 20 (2004), 1037-1040.
- [15] M. Wang, X. Li, J. Zhang, The $\left(\frac{G'}{G}\right)$ expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, *Physics Letter A* 372 (2008), 417-423.
- [16] A. Bekir, Application of the $\left(\frac{G'}{G}\right)$ expansion method for nonlinear evolution equations, *Physics Letter A* 372 (2008) 3400-3406.
- [17] J. Zhang, X. Wei, A generalized $\left(\frac{G'}{G}\right)$ expansion method and its applications, *Physics Letter A* 372 (2008), 3653-3658.
- [18] I. Aslan, T. Öziş, Analytical study on two nonlinear evolution equations by using $\left(\frac{G'}{G}\right)$ expansion method, *Appl. Math. Comp.* 209 (2009), 425-429.
- [19] E. M. E. Zayed, The $\left(\frac{G'}{G}\right)$ expansion method and its application to some nonlinear evolution equations, *J. Appl. Math. Comp.* 30 (2009), 89-103.
- [20] E. M. E. Zayed, K. A. Gepreel, Some applications of the $\left(\frac{G'}{G}\right)$ expansion method to nonlinear partial differential equations, *Appl. Math. Comp.* 212 (2009), 1-13.
- [21] A. Borhanifar, A. M. Zamiri, Application of the $\left(\frac{G'}{G}\right)$ expansion method for the Zhiber-Shabat equation and other related equations, *Math. Comp. Model.* 54 (2011), 2109-2116.
- [22] Z. Ayati, Comparing between G'/G expansion method and tanh method, *Open Engineering*, 4, (2014), 334-340.
- [23] M. Shakeel, S. T. Mohyud-Din, Improved (G'/G) -expansion and extended tanh methods for $(2 + 1)$ -dimensional Calogero-Bogoyavlenskii-Schiff equation, *Alexandria Engineering Journal*, 54 (2015), 27-33.
- [24] E. M. E. Zayed, A further improved (G'/G) -expansion method and the extended tanh-method for finding exact solutions of nonlinear PDEs, *Journal WSEAS Transactions on Mathematics*, 10 (2011), 56-64.
- [25] N. A. Kudryashov, N. B. Loguinova, Be careful with the Exp-function method, *Commun. Nonlinear Sci. Numer. Simulat.* 14 (2009) 1881-1890.
- [26] N. A. Kudryashov, On "new travelling wave solutions" of the KdV and KdV-Burgers equations, *Commun. Nonlinear Sci. Numer. Simulat.* 14 (2009) 1891-1900.
- [27] N. A. Kudryashov, Seven common errors in finding exact solutions of nonlinear differential equations, *Commun. Nonlinear Sci. Numer. Simulat.* 14 (2009) 3507-3529.
- [28] E. J. Parkes, Observations on the basic $\left(\frac{G'}{G}\right)$ expansion method for finding solutions to nonlinear evolution equations, *Applied Mathematics and Computation* 217 (2010) 1759-1763.
- [29] M. I. Rabinovich, NO TITLE, *Izv. Vyssh. Uchebn. Zaved. Radiofizika* 17 (1974), 477-510.
- [30] M. I. Rabinovich and D. I. Trubetskoy, *Introduction in theory of waves*, Nauka, Moscow, 1984, in Russian.
- [31] K. E. Lonngren, H. C. S. Hsuan, and W. F. Ames, On the soliton, invariant, and shock solutions of a fourth-order nonlinear equation, *J. Math. Anal. Appl.* 52 (1975), 538-545.
- [32] M. O. Korpusov, Blow up in nonclassical nonlocal equations, *URSS, Moscow*, 2011, in Russian 8.