BİLECİK ŞEYH EDEBALİ ÜNİVERSİTESİ	**T.C.** **BİLECİK ŞEYH EDEBALİ ÜNİVERSİTESİ** **BİLİMSEL ARAŞTIRMA PROJESİ** **SONUÇ RAPORU FORMU**

PROJE ADI: HECKE EİGENFORMLAR ÜZERİNE

PROJE YÜRÜTÜCÜSÜ: Doç. Dr. İlker İNAM

PROJE NOSU: 2018-01.BŞEÜ.04-01

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BİLECİK, 2020

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ÖZET

Bu araştırma projesinde matematiğin popüler konularından birisi olan modüler formlar konusu ele alınmıştır. Daha kesin olarak özel bir modüler form sınıfı olan yarım tamsayı ağırlıklı Hecke eigenformlar üzerinde çeşitli açık problemler çalışılmıştır. Yarım tamsayı ağırlıklı Hecke eigenformların sistematik seçimi ve bu eigenformların Fourier katsayılarının hızlı şekilde hesaplanması problemlerinin çözümleri bu araştırma projesinin öncelikli sonuçlarıdır. Elde edilen örnekler yardımıyla Ramanujan-Petersson sanısı yardımıyla normalleştirilen Fourier katsayılarının dağılımı konusunda bir çalışma yapılmış ve yayına sunulmuştur.

Anahtar kelimeler: Modüler formlar; Hecke eigenformlar, Ramanujan-Petersson sanısı, Sato-Tate sanısı

ABSTRACT

In this research project, modular forms, which is one of the popular subjects of mathematics, is discussed. More precisely, various open problems have been studied on Hecke eigenforms of half integral weight, which is a specially modular form class. The systematic selection of half integral weight Hecke eigenforms and the solutions of the fast computation of the Fourier coefficients of these eigenforms are the primary results of this research project. With the help of the obtained samples, an article was made on the distribution of the Fourier coefficients normalized with the Ramanujan-Petersson Conjecture and submitted.

Keywords: Modular forms; Hecke eigenforms, Ramanujan-Petersson conjecture, Sato-Tate conjecture

1.SONUÇ RAPORU

Proje yürütücülüğünü yaptığım Bilecik Şeyh Edebali Üniversitesi 2018-01.BŞEÜ.04-01 nolu "Hecke Eigenformlar Üzerine" isimli bilimsel araştırma projesi 2 Ocak 2020 tarihi itibariyle sona ermiştir. Projenin bilimsel çıktılarına bakılacak olursa bunlar iki adet SCI-E kapsamında taranan dergilerde yayına sunulan makaleler, bir adet TR-Dizin'de taranan dergilerde basılmış makale ve iki adet davetli konuşma ve bir adet uluslararası bildiridir. Daha kesin olarak:

1. İnam, İ. , Wiese, G. : On the distribution of coefficients of half-integral weight modular forms and the Ramanujan-Petersson Conjecture, Yayına sunuldu, (2020),

2. İnam, İ. , Wiese, G. : Fast computation of half-integral weight modular forms, Yayına sunuldu, (2020),

3. İnam, İ. , Cıvgın, E. : 3/2 ağırlıklı Hecke eigenformlar üzerine, Bitlis Eren Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 8(4), (2019), 1203-1208,

4. İnam, İ. , Cıvgın, E. : From quadratic forms to modular forms, Journées Arithmétiques XXXI, İstanbul, 1-5 Temmuz 2019, Sözlü sunum,

5. İnam, İ. : Modular forms: 9 Eylül Üniversitesi Matematik Bölümü Aritmetik Seminerleri, 5 Nisan 2019, Davetli konuşmacı,

6. İnam, İ. : The Sato-Tate Conjecture: History and Consequences, International Conference on Mathematical Studies and Applications, Karaman, 5-8 Ekim 2018, Davetli konuşmacı.

İlgili proje devam ederken 15 Kasım 2018 tarihi itibariyle proje yürütücüsü olduğum 118F148 nolu ve "Yarım Tamsayı Ağırlıklı Modüler Formlar Üzerinde Sato-Tate Benzeri Problemler Üzerine" isimli TÜBİTAK 1001 Araştırma Projesi devreye girmiştir. Lüksemburg Üniversitesi öğretim üyesi Prof. Dr. Gabor Wiese 118F148 nolu projede Yurtdışı Araştırmacı sıfatıyla görev yapmaktadır. Çalışma ekibinin bir araya gelmesi amacıyla sadece Lüksemburg'a yurtdışı görevlendirme yolluk kaleminden oluşan 2018-01.BŞEÜ.04-01 nolu Bilecik Şeyh Edebali Üniversitesi Bilimsel Araştırma Projesi, TÜBİTAK projesinin tamamlayıcı projesi haline gelmiştir. Çünkü TÜBİTAK projesinin ilk aşamalarında çalışma ekibinin bir araya gelmesini sağlayacak herhangi bir bütçe kalemi yer almamaktadır, ilgili bütçe kalemi 2020 itibariyle kullanıma açılmıştır. İki proje arasındaki organik bağ bu şekildedir ve iki projenin çakışan bütçe kalemi bulunmamaktadır. İlgili projenin tamamlayıcı proje olarak kullanılmasına fırsat veren ve bu sayede iki projenin de başarısını yukarıya çeken kurumuma teşekkürü bir borç bilirim. Bu projenin Proje Öneri Formu'na göz atıldığında amaç kısmında TÜBİTAK projesinden söz edilmiş ve ikisinin bağlantısı ortaya konmuştu.

Proje önerinde 3 amaç ortaya konmuştu. İlk amaç "Bruinier-Kohnen İşaret Eş dağılım Konjektürü boyunca yeni sonuçlar elde etmek" olup üç makale ile istenen sonuçlar elde edilmiştir. Buna göre (İnam ve Wiese 2020a)'da yarım tamsayı ağırlıklı Hecke eigenformlar üzerinde Sato-Tate sanısına benzer bir sonuç için bir model ortaya konmuştur. Daha açık olarak yeterince sayıda ve farklı ağırlıktaki yarım tamsayı ağırlıklı Hecke eigenformların her bir örnek için en az 10^8 Fourier katsayısı hesaplanıp Ramanujan-Petersson sınırı tarafından normalleştirilen katsayılar üzerinde istatistiksel çalışmalar yapılmıştır. Buna göre bu normalleştirilmiş katsayıların genelleştirilmiş Gauss dağılımına sahip olabileceği öngörülmüştür. Laplace ve Cauchy dağılımlarının da kısmen bu veriyi açıklayabildiği gözlemlenmiştir. Gözlemler grafikler ve dağılımların parametreleri ile desteklenmiş ve hata analizi yapılmıştır. Makalenin temel hedefi sonraki çalışmalara ilham vermek ve bunun için gerekli soruları sormak olduğu için konunun yaygın çalışılması da göz önüne alındığında olumlu reaksiyonlar alınması ve çeşitli atıflar almasını bekliyoruz.

(İnam ve Wiese 2020b) çalışmasında ise yarım tamsayı ağırlıklı modüler form uzayının (sonlu boyutlu vektör uzayı) Rankin-Cohen tabanı olarak tanımladığımız bir sonuç elde edilmiştir. Daha kesin olarak (İnam ve Wiese 2020a) çalışmasının hazırlığı sırasında literatürde boşluk bulunan yarım tamsayı ağırlıklı Hecke eigenformların seçimi problemi bu formlar üzerinde tanımlı bir özel türev operatörü olan Rankin-Cohen parantezinin etkin kullanımıyla bu problem çözülmüştür. Bu makalede ise mevcut diğer bazlarla kıyaslamalar yapılıp yarım tamsayı ağırlıklı Hecke eigenformlar üzerinde "hızlı" hesaplama nasıl yapılacağı açıklanmıştır.

(İnam ve Cıvgın 2019) ise konuyla ilgili Türkçe çalışma yer almadığı için oldukça basit bir sonuçla beraber 3/2 ağırlıklı Hecke eigenformlar tanıtılmış ve kuadratik formlarla çok iyi bilinen bağlantısı açıklanmıştır.

İki davetli konuşma ve bir sözlü bildiride ise konuyla ilgili sunumlar yapılmıştır. Proje önerisinde yaygın etki kısmında "Uluslararası saygınlığı olan (örneğin Journées Arithmétiques, 2019'da İstanbul'da düzenlenecektir) bir bilimsel toplantıda proje sonuçlarının yer aldığı bir bildiri sunulması da bir diğer hedeftir." olarak belirlenmiş olup bu hedefe net olarak ulaşılmıştır.

Projenin ikinci amacında Hecke eigenformların Fourier katsayılarının belirli modlarda çalışılması amaçlanmıştır. Ancak TÜBİTAK projesinin devreye girmesi ve bu projenin ilk amacının zaman alması nedeniyle bu problemin alt yapısı olan Hecke eigenformların Fourier katsayıları hazır olmasına rağmen bu problem proje sonrasına bırakılmıştır.

Projenin üçüncü amacı olan Dedekind-eta çarpımları üzerine olan problem danışmanlığını yürüttüğüm Bilecik Şeyh Edebali Üniversitesi Lisansüstü Eğitim Enstitüsü Matematik Anabilim Dalı Doktora Öğrencisi için bir tez problemi olarak belirlenmiş olup çalışmalar sürmektedir.

Yukarıda künyeleri verilen çalışmalar bu sonuç raporunun doğal eki olup detaylı tüm bilgiler ilgili çalışmalar da yer almaktadır. Yayına sunulan makaleler basıldıktan sonra son halleri otomasyona yüklenecektir.

Projenin başarıyla sonuçlandığı kanaatindeyim, proje çıktılarının yeni proje fikirlerine ve akademik çalışmalara yol açabileceği açıktır. Tüm proje çıktılarında destek belirtilmiş olup bu destek için üniversiteme bir kez daha teşekkür ederim.

2. SONUÇLAR, ÖNERİLER VE TARTIŞMALAR

Bilecik Şeyh Edebali Üniversitesi 2018-01.BŞEÜ.04-01 nolu "Hecke Eigenformlar Üzerine" isimli bilimsel araştırma projesi 2 Ocak 2020 tarihi itibariyle sona ermiştir. Projenin bilimsel çıktılarına bakılacak olursa, bunlar iki adet SCI-E kapsamında taranan dergilerde yayına sunulan makaleler, bir adet TR-Dizin'de taranan dergilerde basılmış makale ve iki adet davetli konuşma ve bir adet uluslararası bildiridir.

İlgili sonuçlar, öneri ve tartışmalar bu sonuç raporunun doğal eki olan proje çıktılarından takip edilebilir. Projenin başarıyla sonuçlandığı kanaatindeyim.

3. KAYNAKLAR

1. İnam, İ. , Wiese, G. : On the distribution of coefficients of half-integral weight modular forms and the Ramanujan-Petersson Conjecture, Yayına sunuldu, (2020),

2. İnam, İ. , Wiese, G. : Fast computation of half-integral weight modular forms, Yayına sunuldu, (2020),

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4. EKLER

Proje bütçesi yolluklardan oluşmakta olup 15.000 TL bütçeden toplam 12.152,18 TL harcanmıştır. Bu bütçeyle projenin paydaşı Lüksemburg Üniversitesi öğretim üyesi Prof.Dr.Gabor Wiese ile Lüksemburg'da çalışmalar yapılmıştır.

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Fast computation of half-integral weight modular forms

Ilker Inam* and Gabor Wiese†

Abstract

To study statistical properties of modular forms, including for instance Sato-Tate like problems, it is essential to have a large number of Fourier coefficients. In this article, we exhibit three bases for the space of modular forms of any half-integral weight and level 4, which have the property that many coefficients can be computed (relatively) quickly on a computer.
MSC (2020): 11F30 (primary); 11F37.
Keywords: Modular forms of half-integral weight, Fourier coefficients, computation, Rankin-Cohen operators

Coefficients of modular forms, especially of Hecke eigenforms, carry a lot of arithmetic information. This is true for both integral weight and half-integral weight. Many aspects that could be completely understood in the case of integral weight, such as the resolution of the Sato-Tate Conjecture [BLGHT11], are open in the half-integral weight case. Even the equidistribution of signs is still open despite many recent results (for example, [IW13], [AdRIW15], [IW16], [KKT18], [Kum13]). In the absence of many tools that work in integral weight, such as the description of modular forms as differential forms on modular curves that admit models over the rational field, a natural approach in view of clarifying the expectations in half-integral weight is to perform numerical studies of modular forms. Consequently, it is very important to know modular forms to a very high precision. For most cases, however, it is very time (and memory) consuming to compute q-expansions up to a high power of q. This statement already applies to integral weight modular forms. It is even more true in the half-integral weight case, which in many treatments is reduced to the integral one.

The purpose of this article is to introduce and study bases of spaces of half-integral weight modular forms such that 'many' coefficients of the standard q-expansions for each form in the basis can be calculated relatively quickly. We focus on achieving high weights, but work in the lowest possible level $\Gamma_0(4)$. This is comparable to the classical case of integral weight modular forms of level 1, where a 'fast' basis is given by the standard Eisenstein series (see, for instance, the Miller basis in [Ste07, Lemma 2.20]). These bases do not easily generalise to the case of higher levels, which can be considered an open question.

We build on existing results from the literature, particularly, the papers of Cohen [Coh75] and Kohnen [Koh80], and exploit them in view of our aims. The general approach chosen here, contrary to the way half-integral weight modular forms are implemented in Magma [BCP97] and Pari/GP [The19] (the latter has more features, such as Hecke operators; see [BC18]), is to use only modular forms that are easy and quick to write down, such as Eisenstein series and theta series, and to multiply power series. Note that the theta series used are lacunary and that computing Eisenstein series costs little.

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Hence, this is very simple to implement and does not rely on more complex algorithms such as the modular symbols algorithms (see e.g. [Ste07] or [Wie19])). However, the bottleneck will be the multiplication of power series for which good algorithms (e.g. those depending on the Fast Fourier Transformation) should be used. The algorithms have been implemented in Magma, which provides such fast algorithms. See [Wie20] for the corresponding Magma package.

We chose to present three kinds of bases, which we name the *standard basis* (see Section 2), the *Kohnen basis* (see Section 3) and the *Rankin-Cohen basis* (see Section 4), respectively. The systematic study of the distribution of Fourier coefficients in [IDOTW21] is a concrete example of the use of the Rankin-Cohen basis allowing the relatively quick computation of $2 \cdot 10^8$ Fourier coefficients of some modular form of half-integral weight.

In Section 5, we make a simple theoretical analysis of the three algorithms and also compare their performance experimentally. One can summarise the findings by stating that in small weights $k + \frac{1}{2}$ with even k, the Rankin-Cohen basis performs best for computing the Kohnen plus-space. In all other cases, the plus-space is best computed by the Kohnen basis. For the computation of the full space, the standard basis always behaves very well.

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1 Background on modular forms

Let k be a non-negative integer. Denote by $M_k(N)$ (resp. $S_k(N)$) the $\mathbb{C}$-vector space of modular forms (resp. cusp forms) of weight k and level $\Gamma_0(N)$ for a positive integer N. Furthermore, write $M_{k+1/2}(4)$ (resp. $S_{k+1/2}(4)$) for the $\mathbb{C}$-vector space of modular forms (resp. cusp forms) of half-integral weight $k + 1/2$ and level $\Gamma_0(4)$. These contain the *Kohnen plus-space* $M^+_{k+1/2}(4)$ (resp. $S^+_{k+1/2}(4)$) consisting of those modular forms (resp. cusp forms) f such that $a_n(f) = 0$ whenever $(-1)^k n \equiv 2, 3 \pmod 4$. Here $a_n(f)$ denotes the n-th Fourier coefficient (for $n \in \mathbb{Z}$) of the Fourier expansion of f at the standard cusp ∞. The Kohnen plus-space can be described as the eigenspace for a certain eigenvalue of an explicit linear operator. See [Koh80, Proposition 2]. We shall, however, not need this operator in our computations.

Let $q := \exp(2\pi i z)$ for the complex variable $z \in \mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$, the upper half-plane. For an even integer $k \geq 4$, we let

$$E_k(q) = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

denote the (q-expansion of the) standard normalised Eisenstein series of weight k and level 1, where $\sigma_{k-1}(n) = \sum_{0<d|n} d^{k-1}$ and B_k denotes the standard k-th Bernoulli number. More generally, for

any positive integer k and any pair of primitive Dirichlet characters χ_1, χ_2, let

$$E_k^{\chi_1,\chi_2}(q) = \frac{-B_k^{\chi_1}}{2k} + \sum_{n=1}^{\infty} \Big(\sum_{0<d|n} \chi_1(d)\chi_2(n/d)d^{k-1} \Big) q^n$$

denote the (q-expansion of the) normalised Eisenstein series of weight k attached to the characters χ_1, χ_2, where $B_k^{\chi_1}$ denotes the k-th generalised Bernoulli number for the character χ_1. The level of $E_k^{\chi_1,\chi_2}$ is the product of the conductors of χ_1 and χ_2. We also let

$$F_2 := \sum_{n \geq 1 \text{ odd}} \sigma_1(n) q^n \in M_2(4)$$

be the standard Eisenstein series of weight 2 and level 4.

We shall also need the result that the algebra of all integral weight modular forms of level 1 is generated by E_4 and E_6 and that a $\mathbb{C}$-basis of the vector space of modular forms of level 1 and weight k is given by $E_4^a E_6^b$ where $a, b \in \mathbb{Z}_{\geq 0}$ run through all possibilities for $k = 4a + 6b$. For more details, see Chapter 8 and Chapter 10.6 of [CS17].

The for our purposes most important modular form of half-integral weight is the *standard ϑ-series* defined as

$$\vartheta := \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}.$$

It is a modular form of weight $1/2$ for the group $\Gamma_0(4)$. Finally, for every integer $k \geq 2$, let $H_{k+1/2}$ be the modular form which is explicitly described in the proof of [Coh75, Theorem 3.1] as a linear combination of two linearly independent Eisenstein series in $M_{k+1/2}(4)$.

2 The standard basis

We recall [Coh75, Proposition 1.1].

Proposition 2.1. *The natural embedding*

$$\mathbb{C}[\vartheta, F_2] \to \bigoplus_{\ell \in \frac{1}{2}\mathbb{Z}} M_\ell(4)$$

is an isomorphism of graded algebras, where ϑ and F_2 are the modular forms of weight $1/2$ and 2, respectively, that are described above.

Corollary 2.2. *Let $\ell \in \frac{1}{2}\mathbb{Z}$. Then the modular forms*

$$\vartheta^a F_2^b \text{ for all } a, b \in \mathbb{Z}_{\geq 0} \text{ such that } \ell = \frac{a}{2} + 2b$$

form a basis of $M_\ell(4)$, which we call the standard basis.

The standard basis is computed by writing down ϑ and F_2 explicitly as power series (using their definition) and then multiplying power series.

The standard basis is a basis for the full space of modular forms of weight $\ell \in \frac{1}{2}\mathbb{Z}$ and level 4. We now describe how to compute the Kohnen plus-space as a subspace for $\ell = k + \frac{1}{2}$ with $k \in \mathbb{Z}$. Instead of using the operator introduced in [Koh80], we solve this as a linear algebra problem. Take a basis $f_1, \dots, f_m$ of the full space (with precision D), e.g. the standard basis, and, for each $1 \le i \le m$, write the coefficients $a_n(f_i)$ for all $0 \le n < D$ such that $(-1)^k n \equiv 2, 3 \pmod 4$ into a vector v_i. Then take the matrix M of these vectors and compute a basis $b_1, \dots, b_r$ of its kernel. Then a basis of the Kohnen plus-space is given by $g_i = \sum_{j=1}^m b_{i,j} f_j$ for $1 \le i \le r$ where $b_{i,j}$ is the j-th entry of the vector b_i.

It is quite fast to compute the standard basis, and the linear algebra step required for calculating a basis for the Kohnen plus-space is also very fast; note that for computing the b_i, one can usually work with a smaller precision than the one that one might like to obtain in the end. However, we will see in the next two sections that there are direct ways to compute the Kohnen plus-space, which do not require the computation of the full space. They are, of course, still faster.

3 The Kohnen basis for the plus-space

The first 'fast' basis for the Kohnen plus-space we present has been studied by Kohnen in the fundamental paper [Koh80], in which he defines the plus-space.

Proposition 3.1 (Kohnen basis – even case). *Let $k \in \mathbb{Z}_{\ge 2}$ be even. Let $a_0 \in \{0, 1, 2\}$ satisfy $k \equiv a_0 \pmod 3$ and put $m = \frac{k-4a_0}{6} - 1$. Then the set consisting of the modular forms*

$$E_4^{a_0+3a+1}(4z) \cdot E_6^{m-2a}(4z) \cdot H_{5/2}(z), \quad E_4^{a_0+3a}(4z) \cdot E_6^{m-2a+1}(4z) \cdot \vartheta(z) \text{ for } 0 \le a \le \lfloor \frac{m}{2} \rfloor,$$
$$E_4^{\frac{k}{4}}(4z) \cdot \vartheta(z) \text{ if } 4 \mid k,$$
$$E_6^{\frac{k-2}{6}}(4z) \cdot H_{5/2}(z) \text{ if } 6 \mid (k-2)$$

forms a basis of $M^+_{k+1/2}(4)$.

Proof. The modular forms $E_4^{a_0+3a} E_6^{m-2a}$ for $0 \le a \le \lfloor \frac{m}{2} \rfloor$ form a basis of $M_{k-6}(1)$. Multiplying by E_4, this space is mapped injectively into $M_{k-2}(1)$ hitting all standard basis elements of the target space except $E_6^{\frac{k-2}{6}}$ if $6 \mid (k-2)$. Similarly, multiplying by E_6, we obtain a subspace of $M_k(1)$ containing all standard basis elements except $E_4^{\frac{k}{4}}$ if $4 \mid k$. Now it suffices to apply [Koh80, Proposition 1]. □

Proposition 3.2 (Kohnen basis – odd case). *Let $k \in \mathbb{Z}_{\ge 2}$ be odd. Let $a_0 \in \{0, 1, 2\}$ satisfy $k \equiv a_0 \pmod 3$ and put $m = \frac{k-4a_0-9}{6}$. Then the set consisting of the modular forms*

$$E_4^{a_0+3a+1}(4z) \cdot E_6^{m-2a}(4z) \cdot H_{11/2}(z), \quad E_4^{a_0+3a}(4z) \cdot E_6^{m-2a+1}(4z) \cdot H_{7/2}(z) \text{ for } 0 \le a \le \lfloor \frac{m}{2} \rfloor$$
$$E_4^{\frac{k-3}{4}}(4z) \cdot H_{7/2}(z) \text{ if } 4 \mid (k-3)$$
$$E_6^{\frac{k-5}{6}}(4z) \cdot H_{11/2}(z) \text{ if } 6 \mid (k-5)$$

forms a basis of $M^+_{k+1/2}(4)$.

Proof. The modular forms $E_4^{a_0+3a} E_6^{m-2a}$ for $0 \le a \le \lfloor \frac{m}{2} \rfloor$ form a basis of $M_{k-9}(1)$. Multiplying by E_4, this space is mapped injectively into $M_{k-5}(1)$ hitting all standard basis elements of the target space except $E_6^{\frac{k-5}{6}}$ if $6 \mid (k-5)$. Similarly, multiplying by E_6, we obtain a subspace of $M_{k-3}(1)$ containing all standard basis elements except $E_4^{\frac{k-3}{4}}$ if $4 \mid (k-3)$. Now it suffices to apply [Koh80, Proposition 1]. □

In both cases, the Kohnen bases can be obtained by multiplying power series that can be easily computed. In particular, we use that Cohen's modular forms $H_{5/2}$, $H_{7/2}$ and $H_{11/2}$ can be explicitly given in terms of the standard basis (see [Coh75, Corollary 3.2]).

4 The Rankin-Cohen modular forms

It is well known that the dimension of the cusp space of the Kohnen plus-space of weight $k + 1/2$ equals one for $k = 6, 8, 9, 10, 11, 13$. Therefore, any form in that space is a Hecke eigenform of half-integral weight. We got the inspiration to work with Rankin-Cohen brackets from the nice example $\delta(z)$ [KZ81, p. 177]. That example was good enough to obtain large number of Fourier coefficients. The natural idea is to seek for such nice examples in higher weights.

We recall the definition of the Rankin-Cohen bracket [CS17, Def. 5.3.23], [Coh75] and [Zag94]: Let f, g be two modular forms of weights k and ℓ, respectively, and let $n \in \mathbb{Z}_{\geq 1}$. Put

$$[f, g]_n := \sum_{j=0}^{n} (-1)^j \binom{n+k-1}{j} \binom{n+\ell-1}{n-j} f^{(n-j)} g^{(j)},$$

where $f^{(i)}$ denotes the i-th derivative of f.

For a non-negative integer k, we now define *Rankin-Cohen modular forms* of weight $k + \frac{1}{2}$ and level 4, as follows.

Case 1: k is even. Let $n \in \mathbb{Z}$ satisfy $0 \leq n \leq \frac{k-4}{2}$. Put

$$\Phi_{k,n} := [E_{k-2n}(4z), \vartheta(z)]_n \in M^+_{k+1/2}(4).$$

Case 2: k is odd. Let $n \in \mathbb{Z}$ satisfy $0 \leq n \leq \frac{k-2}{2}$. Put

$$\Phi_{k,n,1} := [E^{1,\chi}_{k-2n}(z), \vartheta(z)]_n, \Phi_{k,n,2} := [E^{\chi,1}_{k-2n}(z), \vartheta(z)]_n \in M^+_{k+1/2}(4)$$

where χ is the Kronecker character of conductor 4 corresponding to $\mathbb{Q}(\sqrt{-1})$.

Definition 4.1. *Let $k \in \mathbb{N}$ be even. Let $d = \dim M^+_{k+1/2}(4)$. If the modular forms*

$$\Phi_{k,n} \text{ for } 0 \leq n \leq d - 1$$

are linearly independent, then we call them the Rankin-Cohen basis *of $M^+_{k+1/2}(4)$.*

Let now $k \in \mathbb{N}$ be odd. Let $d = \dim M_{k+1/2}(4)$. If the first d of the following modular forms

$$\Phi_{k,n,1}, \Phi_{k,n,2} \text{ for } 0 \leq n \leq \lceil \frac{d}{2} \rceil$$

are linearly independent, then we call them the Rankin-Cohen basis *of $M_{k+1/2}(4)$.*

We have been unable to prove that the modular forms in the definition are always linearly independent. However, this was true in all cases we computed. It does not seem entirely evident how to write down a basis via Rankin-Cohen brackets for the plus-space if k is odd which is as simple as the one for even k. The Rankin-Cohen bases are straight forward to compute by multiplying and differentiating power series. The sparseness of ϑ positively effects the speed of the computation.

5 Comparison of the complexity and the running time of the algorithms and final comments

The main cost in all three algorithms is the multiplication of power series. It takes significantly more time than adding power series, differentiating them or creating modular forms such as theta series and the simple Eisenstein series we need as power series. We thus use the number of power series multiplications as our measure for the complexity of the algorithms. We disregard finer effects such as how lacunary are the power series (note that the theta series and its derivatives are very lacunary) and the size of the coefficients in the power series. The latter depend on the weight and so similar effects are noticable in all three algorithms.

Straight forward counting for each of the three algorithms (as currently implemented in the package FastBases[Wie20]) yields the following table, showing the number of multiplications of power series (with fixed precision) as a function of the weight $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ (we assume $k - \frac{1}{2}$ even for the Rankin-Cohen basis) with only minor approximations:

Standard Basis of $M_k(4)$	Kohnen Basis of $M_k(4)^+$	Rankin-Cohen Basis of $M_k(4)^+$
$\frac{3}{2}k + 4$	$\frac{5}{12}k + 12$	$\frac{1}{72}k^2 + \frac{3}{4}k + 10$

We can conclude that the number of multiplications performed for the standard basis and for the Kohnen basis behaves linearly with respect to the weight, whereas the dependence is quadratic for the RC algorithm. Moreover, the Kohnen basis needs roughly a third of the multiplications of the standard basis.

However, the actual running time is not simply proportional to the number of power series multiplications. Other effects play a role. For instance, the size of the coefficients of the power series is important since the computations are exact computations over the rational numbers, and the average size is known to grow with the weight. Moreover, the time needed for a single multiplication of two power series can be significantly lower when at least one of the power series is sparse, which is the case for the powers of θ and their derivatives. In order to see how the algorithms behave in practice, we ran our Magma implementation on a standard laptop computer[1] and obtained the following comparison of the computation times (in seconds) of all coefficients up to the indicated bound for some selected weights.

	Standard Basis			Kohnen Basis			Rankin-Cohen Basis		
weight	10^4	10^5	10^6	10^4	10^5	10^6	10^4	10^5	10^6
25/2	0.16	3.02	55.34	0.12	2.05	34.92	0.13	1.76	31.71
41/2	0.40	6.86	134.19	0.22	3.39	61.96	0.25	4.08	73.91
61/2	0.81	19.60	287.14	0.31	7.26	85.90	0.70	15.11	181.21
81/2	1.39	31.92	563.40	0.54	12.00	200.58	1.19	24.01	381.79
101/2	2.31	50.82	934.79	0.76	17.47	283.91	2.58	51.81	796.42
121/2	3.09	76.02	1226.15	1.04	25.62	380.82	4.15	94.60	1213.17
141/2	4.18	102.95	1917.40	1.37	36.21	565.46	5.20	132.32	2266.65
161/2	4.97	128.07	2339.31	1.75	45.48	796.49	8.62	190.40	3129.62
181/2	7.46	171.61	2882.44	1.85	55.21	869.71	14.44	283.70	4065.21
201/2	8.65	193.83	3486.27	3.58	60.30	1015.79	20.63	358.83	5104.97

[1] Intel Core i5 Dual Core CPU 1.80 GHz, 8 GB 1600 MHz DDR3 RAM

A clear conclusion is that the Kohnen basis is the one to choose for the computation of the Kohnen plus-space unless the weights are small, in which case the Rankin-Cohen basis has a slightly better performance. If the weight is sufficiently high, then even the standard basis with subsequent linear algebra reduction to the plus-space outperforms the Rankin-Cohen basis. The ratio of the number of multiplications between the standard and the Kohnen basis almost becomes visible in the highest weight in the table.

In order to obtain a clearer idea of the complexity of the three algorithms with respect to the weight and also with respect to the number of coefficients to be computed, we used gnuplot [WKm17] for computing functions approximating the computation times. For the behaviour with respect to the weight, the above table was used. In order to understand the behviour with respect to the number of coefficients, also computation times for other numbers of coefficients were measured.

We approximated the running time as a function of the weight $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ by $f(k) = b \cdot k^a$ with a particular interest in the exponent a. The following table shows the calculated exponents a for the cases of 10^5 and 10^6 coefficients.

nb. coeff.	Standard Basis	Kohnen Basis	Rankin-Cohen Basis
10^5	1.93	1.74	2.77
10^6	1.93	1.81	2.57

These exponents clearly make the advantage of the Kohnen basis over the Rankin-Cohen basis visible. Whereas the exponent a seems to be quite stable with respect to the number of coefficients in the first two cases, one notices a decrease for the Rankin-Cohen basis. For fixed modular forms spaces, we also approximated the running time as a function of the number of coefficients x by the function $g(x) = b \cdot x^a$, again with particular interest in the exponent a. The following table shows the calculated value of a for three different weights.

weight	Standard Basis	Kohnen Basis	Rankin-Cohen Basis
$41/2$	1.16	1.17	1.17
$101/2$	1.31	1.21	1.22
$201/2$	1.22	1.18	1.13

We see that the three algorithms present a similar behaviour of the computation time with respect to the number of coefficients, which is surely only due to the fact that all three rely essentially on multiplications of power series. The data suggests a slight advantage for the Rankin-Cohen basis, which might be caused by the lacunarity of the powers of θ and their derivatives.

We close the paper with some final remarks on higher level cases. The standard and the Kohnen basis rely on an explicit description of a basis in terms of modular forms the q-expansion of which can be computed efficiently to a high precision. We do not know of any such description in any higher level. However, even if only generators (which might have some linear dependence) can be described in such a way, similar algorithms as those presented here will be direct consequences. For the moment, this remains an open problem.

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On the distribution of coefficients of half-integral weight modular forms and the Bruinier-Kohnen Conjecture

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Abstract

This work represents a systematic computational study of the distribution of the Fourier coefficients of cuspidal Hecke eigenforms of level $\Gamma_0(4)$ and *half-integral* weights. Based on substantial calculations, the question is raised whether the distribution of normalised Fourier coefficients with bounded indices can be approximated by a generalised Gaussian distribution. Moreover, it is argued that the apparent symmetry around zero of the data lends strong evidence to the Bruinier-Kohnen Conjecture on the equidistribution of signs and even suggests the strengthening that signs and absolute values are distributed independently.

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Keywords: Modular forms of half-integer weight, Fourier coefficients of automorphic forms, Ramanujan-Petersson conjecture, Sato-Tate conjecture, distribution of coefficients, sign changes.

1 Introduction

This article represents a systematic computational study of the Fourier coefficients of *half-integral weight* cuspidal Hecke eigenforms with the aim of experimentally shedding new light on their distribution, particularly focusing on signs.

Size and normalisation of coefficients and the Ramanujan-Petersson Conjecture. Let $f = \sum_{n=1}^{\infty} a(n)q^n$ be a cuspidal Hecke eigenform of weight k. The Ramanujan-Petersson Conjecture (see e.g. [Koh94]) claims

$$a(n) = O(n^{(k-1)/2+\epsilon})$$

for any $\epsilon > 0$. By Deligne's famous proof of the Weil Conjectures [Del74], the Ramanujan–Petersson Conjecture is true with $\epsilon = 0$ in the *integer* weight case. Motivated by the Ramanujan-Petersson

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Conjecture we define the *normalised coefficients* to be

$$b(n) := \frac{a(n)}{n^{(k-1)/2}}.$$

It seems that the Ramanujan-Petersson Conjecture has not been proved for a single cuspform of *non-integral* weight. However, it is known by work of Gun and Kohnen in [GK19] that the Ramanujan–Petersson Conjecture would fail for $\epsilon = 0$ in half-integral weight. Their argument uses a sequence of non-squarefree indices coming from the Shimura lift to construct a counter example. In recent work of Gun, Kohnen and Soundararajan [GKS20], the authors suggest that in half-integral weight 'perhaps' the bound

$$|b(|n|)| \leq \exp(C\sqrt{\log|n| \log\log|n|}),$$

derived from conjectures in [FGH07] and stronger than the stated form of the Ramanujan-Petersson Conjecture might hold.

Known results and conjectures on the distribution in half-integral weight. In half-integral weight $k = \ell + \frac{1}{2}$, there is the crucial relation, due to Waldspurger (Theorem 1 on p. 378 of [Wal81]) and Kohnen-Zagier (Theorem 1 of [KZ81], p. 177), between the *squares* of the Fourier coefficients and central values of L-functions. More precisely, the Shimura lift (see [Shi73] and [Niw75]) relates those Fourier coefficients of f that are indexed by tn^2 with $t \in \mathbb{N}$ squarefree and $n \in \mathbb{N}$ to the n-th Fourier coefficient of a modular form g in integral weight 2ℓ. Then $b(|n|)^2$ is proportional to $L(g, \chi_n, \ell)$ for fundamental discriminants n such that $|n| = (-1)^\ell n$, where $L(g, \chi_n, s)$ is the Hecke L-function of g twisted by the primitive quadratic character χ_n corresponding to n.

This relation is at the basis of most of the results on the absolute value of $b(|n|)$ and has led to a conjectural description of the distribution of the $b(|n|)^2$ for fundamental discriminants n. In that context, we recall that the famous Sato-Tate Conjecture describing the distribution of the normalised Fourier coefficients in the *integral weight* case has been proved by Barnet-Lamb, Geraghty, Harris and Taylor [BLGHT11]. In [CKRS06], Conrey, Keating, Rubinstein and Snaith propose a conjecture on the distribution of coefficients of modular forms of weight $3/2$ attached to elliptic curves. Their Conjecture 4.2 states that for a modular form of weight $3/2$ which is attached to an elliptic curve E, the natural density of fundamental discriminants n such that

$$\alpha \leq (\kappa^{\pm}\sqrt{\log(|n|)}(b(|n|))^2)^{\frac{1}{\sqrt{\log\log|n|}}} \leq \beta$$

equals

$$\frac{1}{\sqrt{2\pi}} \int_\alpha^\beta \frac{1}{t} \exp(-\frac{1}{2}(\log t)^2) dt,$$

where $\kappa^{\pm}$ is a positive constant and $0 \leq \alpha \leq \beta$. K. Soundararajan kindly informed us that similar conjectures are made for higher weights as well. Note also that Conjecture 4.2 of [CKRS06] implies that the normalised coefficients $b(|n|)$ tend to zero almost surely, in the sense that for all $\epsilon > 0$, the set of $d \in S^{\pm}$ such that $|b(|d|)| < \epsilon$ has natural density equal to 1. This prediction is confirmed by a theorem of Radziwiłł and Soundararajan [RS15], as cited in [GKS20] in level $\Gamma_0(4)$:

For every $\epsilon > 0$, there is a constant $C = C(\epsilon, f)$ such that for all but $o(x)$ fundamental discriminants n with $x \leq (-1)^\ell n \leq 2x$, one has

$$|b(|n|)| \leq C \cdot \log(|n|)^{-\frac{1}{4}+\epsilon}. \tag{1.1}$$

This theorem hence implies that the normalised coefficients $b(n)$ tend to 0 with probability 1 and hence follow a Dirac distribution at 0. In the other direction, in recent work of Gun, Kohnen and Soundararajan [GKS20], the existence of large values for normalised Fourier is proved (for level $\Gamma_0(4)$):

For any $\epsilon > 0$ and x large, there are at least $x^{1-\epsilon}$ fundamental discriminants n with $x < (-1)^\ell n < 2x$ such that

$$|b(|n|)| \geq \exp\left(\frac{1}{82}\frac{\sqrt{\log|n|}}{\sqrt{\log\log|n|}}\right). \tag{1.2}$$

Note that the relation with central values of L-functions only gives information about the squares of the coefficients and hence no information about their signs. This is where the conjecture of Bruinier and Kohnen ([BK08], [KLW13]) enters, claiming that exactly half of the non-zero coefficients are positive, the other half negative.

Contributions of this work. Here are the main points that we want to make with this article.

(1) Even though it is known by (1.1) that the absolute values of the normalised coefficients $b(|n|)$ tend to zero with probability 1, we do observe a very neat non-trivial distribution of the coefficients up to (computationally reachable) bounds. The distribution seems to follow a generalised Gaussian distribution.

(2) The histograms of the distribution of the normalised Fourier coefficients up to varying bounds and for varying Hecke eigenforms of half-integral weights all seem to present a similar 'global shape' in the sense that they can be well approximated by a single type of density function, and that only the parameters depend on the modular form and on the bound.

(3) The symmetry around 0 of the observed distributions of the coefficients up to bounds can be interpreted as very strong evidence towards the Bruinier-Kohnen Conjecture. In fact, it suggests a strengthening of the conjecture to the point that the absolute value and the signs are independently distributed (see Conjecture 6.1). To the best of our knowledge, the calculations in this article can be seen as the most systematic and largest computational support for the Bruinier-Kohnen Conjecture to date. Furthermore, if Question 5.1 has a positive answer then the Bruinier-Kohnen Conjecture is true and this links the two topics in the title.

Short overview over the article. In §2, we report on the examples of half-integral weight modular cuspforms used for our experimental study and how they were computed. The main point is that we chose to stay in the lowest possible level $\Gamma_0(4)$ and considered weights up to $61/2$. In view of studying the distribution of the normalised Fourier coefficients of the examples, we created histograms and report on them in §3. We take into account the specific nature of half-integral weight forms that distinguishes them significantly from the well understood integral weight ones: we disregard all coefficients that via the Shimura lift come from the integral case and, consequently, study only squarefree indexed coefficients. The similar shape that the histograms exhibit suggests that they can be described by distribution functions. We consider four types of such functions in §4, namely, the Laplace and the Cauchy distribution as well as two generalisations of the Gaussian distribution. We also report on data obtained when fitting the aforementioned distribution functions with the histograms. It turns out that one of the generalised Gaussian distributions is clearly the best one. In view of the fact that the normalised coefficients tend to zero almost surely, in §5 we explicitly seek for dependencies of the best fit parameters with data such as the number of coefficients used. We also formulate the explicit question if the normalised Fourier coefficients up to any bound indeed follow such a generalised Gaussian distribution, see Question 5.1. Finally, in §6, we recall the Bruinier-Kohnen Conjecture and some previous results on it. We make the point that the observed symmetry of the histograms and of the studied distribution functions around zero is strong computational evidence in favour of the conjecture and even suggests a strengthening of it.

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2 Examples of Hecke eigenforms in half-integral weights for $\Gamma_0(4)$

For studying their distribution computationally, we need as many Fourier coefficients of modular forms as possible. Since we are interested in higher weights, we choose to work in the smallest possible level $\Gamma_0(4)$. As described in the article [IW20] by two of the authors, the Kohnen-plus space in half-integral weight admits bases that can be computed relatively quickly up to some high precision. For this article, we worked with the Rankin-Cohen basis as described in *loc. cit.* We performed exact computations over the rationals in order not to lose any precision and only converted the normalised coefficients to real numbers in the end. A disadvantage of this choice is a huge consumption of memory, when the q-expansions are computed up to a high power of q.

To give some more details, we use Pari/GP (see [The19]) to express Hecke eigenforms with respect to the Rankin-Cohen basis. Here the *mf package* of Pari/GP [BC18] provides us with the necessary tools. Then, we export the basis coefficients to Magma [BCP97], where we construct the Hecke eigenform as a power series (in general, over a number field). This is done in Magma because it provides very fast algorithms for the multiplication of power series. In a final step, we compute the normalised coefficients over the reals. Since all previous computations are exact computations, 10 digits of real precision are enough.

We only recorded coefficients at squarefree indices which are not known to be zero by the fact that we look only in the Kohnen-plus space (e.g. when $k - 1/2$ is even, $a(n)$ is zero when n is 2 or 3 modulo 4). We also normalised all modular forms in such a way that the first recorded normalised coefficient equals 1. This is the natural way of normalisation if we consider the definition of the Kohnen-plus space, but it is in no way canonical.

We compute all Hecke eigenforms of weights $13/2, 17/2, 19/2, \ldots, 61/2$ (level $\Gamma_0(4)$) with 10^7 Fourier coefficients. By this, we mean all normalised coefficients $b(n)$ with squarefree index $n < 10^7$. We reached 10^8 Fourier coefficients for some examples and for the weight $13/2$, we could go up to $2 \cdot 10^8$. Text files containing the normalised coefficients can be downloaded[1]. The total amount of data used for the study exceeds 4 GB. In all tables below, a label such as $25/2(2)$ stands for the second cuspidal Hecke eigenform (with respect to an internal ordering) in weight $25/2$ and level $\Gamma_0(4)$. The reader is referred to [IW20] for some more details on the computations.

Note that, under the Shimura lift, any half-integral weight Hecke eigenform (in weight k) corresponds to an integral weight Hecke eigenform in level 1 and weight $2k - 1 \in 2\mathbb{Z}$. By [Koh85, p. 241], the Shimura lift is a Hecke equivariant isomorphism between the Kohnen plus space and the corresponding space in integral weight. This means that the eigenforms in half-integral weight are in bijection with those in integral weight. By Maeda's Conjecture (see [HM97]), in weight $2k - 1$ there is only a single Hecke orbit of eigenforms. Assuming Maeda's conjecture (which is known up to high weight by [GM12], far exceeding our examples), it follows that the number of half-integral weight Hecke eigenforms in the Kohnen-plus space equals the degree of the number field generated by the coefficients of the integral weight form.

3 Histograms for the distribution of normalised coefficients

The principal aim of this article is to understand the distribution of normalised coefficients of half-integral weight Hecke eigenforms. More precisely, the point we want to make is the following. Even though the normalised coefficients $b(n)$ tend to zero almost surely by the cited result of Radziwiłł and Soundararajan, the coefficients up to bounds that we can computationally reach do follow a very interesting non trivial distribution, which can be well approximated by density functions. They turn out to be symmetric around zero.

[1] http://math.uni.lu/wiese/FourierData.html

In order to study the distribution, we created histograms for the distribution of the normalised coefficients for all the modular forms mentioned in the previous section. We restricted our attention to coefficients with squarefree indices that are not known to be zero by the fact that the modular form lies in the Kohnen plus-space. The reason for only considering squarefree indices is the following: coefficients at indices of the form tn^2 with t squarefree and $n \in \mathbb{Z}_{\geq 2}$ are governed by the n-th coefficient of the Shimura lift, which is an integral weight eigenform and thus behaves with respect to the proved Sato-Tate law (if it is not CM). So, if we did not restrict to squarefree indices, we would 'mix' two distributions, making the pictures harder to analyse.

We created histograms for the distribution of the normalised coefficients using gnuplot [WKm17]. One choice that one has to make is that of the box size for the histograms. In order to understand dependencies, we created the histograms with different box sizes. Some box sizes are more pleasing to the eye than others (sometimes depending on the modular form). The graphs in Figure 1 are the histograms of the normalised Fourier coefficients $b(n)$ for the Hecke eigenform of weight $13/2$ with box sizes 0.001, 0.0001 and 0.00001, respectively, with 10^8 Fourier coefficients.

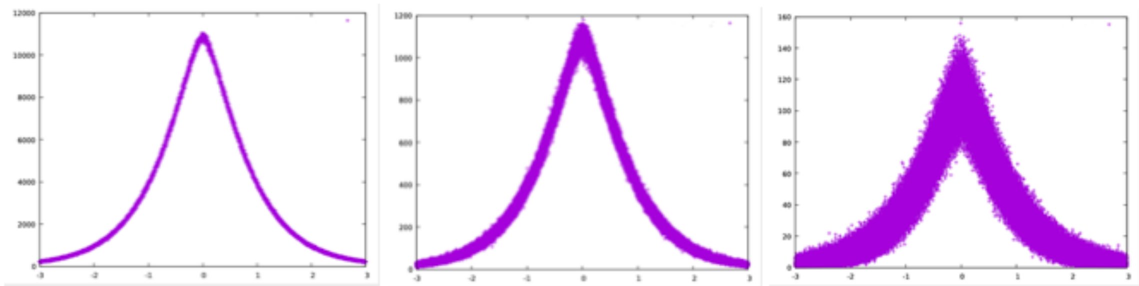

Figure 1: Histogram of 10^8 normalised Fourier coefficients of the Hecke eigenform of weight $13/2$ with box sizes 0.001, 0.0001 and 0.00001, respectively

We observed that the choice of box size does not influence the parameters for best fits in any significant way. To confirm this, we give Table 1 for the parameter a for the GG-distribution for five different forms (indicated by their weights) and three different box widths. We hence disregard box sizes in our discussions and, throughout the paper, we use graphs with box size 0.001 since in this case, the graphs seem the most pleasing to the eyes.

4 Candidate distribution functions and fits

The point that we want to make in this section is the following: The 'global shape' of the histograms is independent of the modular form and of the bound for the coefficients. More precisely, our computations suggest that the histograms for the normalised Fourier coefficients of any half-integral weight Hecke eigenforms up to a given bound can be described by a single type of density function, and that only the parameters depend on the modular form as well as on the bound.

	0.001	0.0001	0.00001
$13/2$	0.634	0.634	0.633
$17/2$	0.553	0.553	0.553
$21/2$	0.558	0.558	0.557
$25/2(1)$	0.504	0.504	0.504
$29/2(1)$	0.506	0.506	0.506

Table 1: Best fit parameters with different box widths for the parameter a in the GG-distribution

Looking at the histograms, one immediately notices that the histograms look symmetric around 0. Even though they present some kind of bell shape, one also sees very quickly that they do not follow a standard Gaussian. Instead, we tried the following two generalisations of the standard Gaussian and also the Laplace and the Cauchy distributions:

$$\mathrm{GGG}(x) := b \cdot \exp(-\frac{(d+x^2)^a}{c}) \tag{4.3}$$

$$\mathrm{GG}(x) := b \cdot \exp(-\frac{(x^2)^a}{c}) \tag{4.4}$$

$$\mathrm{Laplace}(x) := b \cdot \exp(-|x|/c) \tag{4.5}$$

$$\mathrm{Cauchy}(x) := \frac{a}{b+(cx)^2}. \tag{4.6}$$

Of course, GG is a special case of GGG (with $d = 0$) and $\mathrm{Laplace}$ is a special case of GG (with $a = 0.5$). Since the a-parameter in GG is quite close to 0.5 (the data is given in the appendix) and because the Laplace distribution is much simpler than the Generalised Gaussians, we took it up into our considerations. Since we did not normalise our histograms so that the area under it is 1, we also did not normalise the above distribution functions to be probability distributions (even though we think of them this way).

Graphically, all four distribution functions describe the histograms pretty well, the GGG-distribution being clearly the best. The Cauchy distribution seems to be systematically too high in the tails, whence we consider it the worst of the four. The reader is referred to the appendix for the graphs with inscribed best fit distribution functions. Sample graphs of some fits are given in Figures 2, 3 and 4.

Since our histograms are not uniformly normalised (recall that we normalised the coefficients of the half-integral weight form in such a way that the first non-zero coefficient is 1) in the sense that generally they present the same shape, but some are wider, some are steeper, etc., it is very hard to compare the quality of the fits between different histograms. We will measure the quality of the fits by the Root Mean Square (RMS) value as output by gnuplot. Of course, the GGG-fit will always be better than the GG-fit and that one will always be better than the $\mathrm{Laplace}$-fit as they are special cases of each other.

We illustrate the results of the fits performed using gnuplot by giving the tables of all best fit values for all examples for which we reached 10^8 Fourier coefficients. The results for computations up to 10^7 Fourier coefficients are included in the appendix.

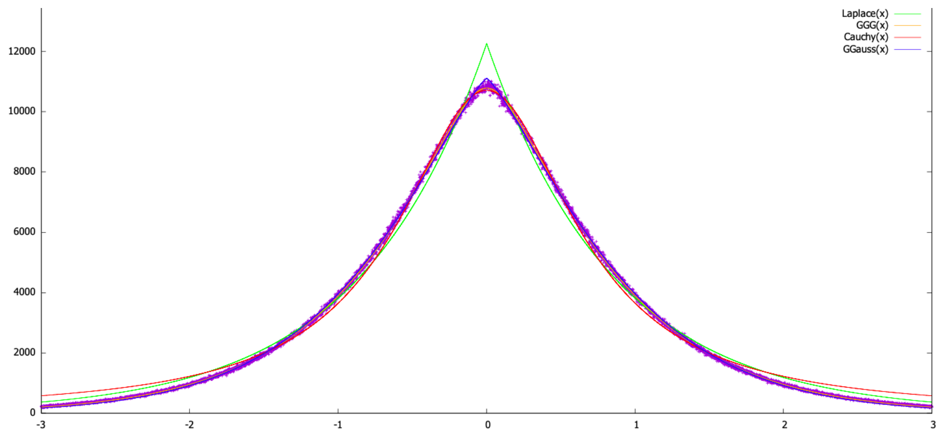

Figure 2: Histogram and distributions of Hecke eigenform of weight $13/2$ with 10^8 coefficients

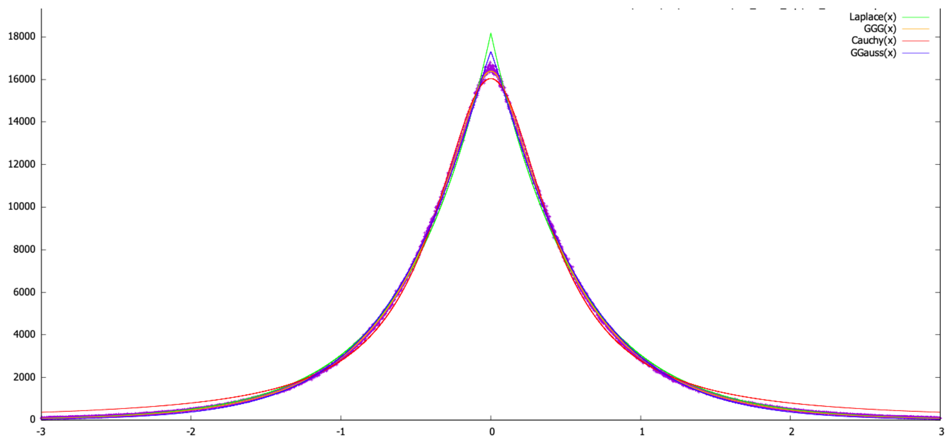

Figure 3: Histogram and distributions of Hecke eigenform of weight $25/2$ with 10^8 coefficients

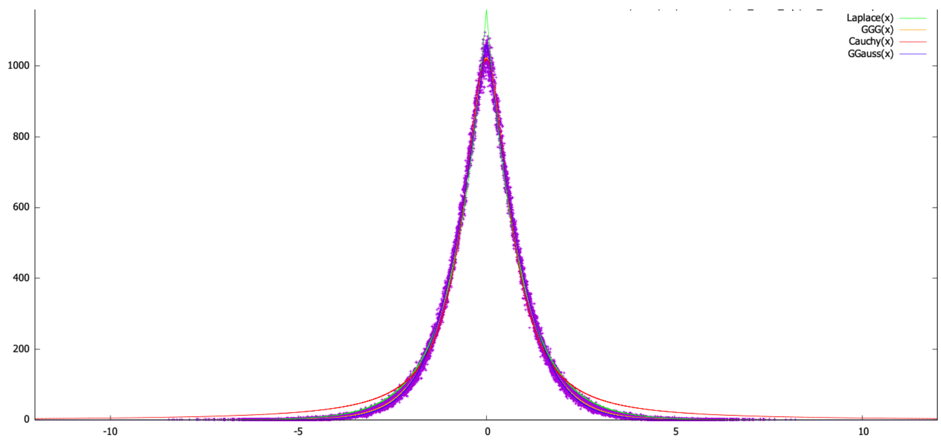

Figure 4: Histogram and distributions of the second Hecke eigenform of weight $43/2$ with 10^7 coefficients

Best fit parameters (rounded)
for the GGG-distribution:

	a	b	c	d
13/2	0.581	12538	0.872	0.030
17/2	0.453	20421	0.550	0.030
19/2	0.385	44105	0.317	0.012
21/2	0.460	23411	0.485	0.022
23/2	0.494	14866	0.725	0.034
25/2(1)	0.391	19462	0.577	0.035
25/2(2)	0.237	88927	0.284	0.033
27/2	0.513	22681	0.471	0.014
29/2(1)	0.364	11886	0.812	0.092
29/2(2)	0.423	30278	0.402	0.016

Best fit parameters (rounded)
for the GG-distribution:

	a	b	c
13/2	0.634	11105	0.969
17/2	0.553	14999	0.663
19/2	0.515	26822	0.363
21/2	0.558	17300	0.566
23/2	0.573	11997	0.857
25/2(1)	0.504	13107	0.752
25/2(2)	0.453	20676	0.485
27/2	0.584	18721	0.514
29/2(1)	0.506	7555	1.256
29/2(2)	0.532	20884	0.466

Best fit parameters (rounded)
for the Laplace distribution:

	b	c
13/2	12264	0.860
17/2	15711	0.650
19/2	27201	0.367
21/2	18184	0.560
23/2	12746	0.805
25/2(1)	13161	0.750
25/2(2)	19699	0.484
27/2	20071	0.514
29/2(1)	7597	1.244
29/2(2)	21515	0.470

Best fit parameters (rounded)
for the Cauchy distribution:

	a	b	c
13/2	183	0.017	0.181
17/2	194	0.014	0.222
19/2	2455	0.010	0.340
21/2	194	0.012	0.240
23/2	208	0.019	0.204
25/2(1)	194	0.017	0.212
25/2(2)	1376	0.080	0.725
27/2	233	0.013	0.272
29/2(1)	813	0.122	0.340
29/2(2)	342	0.018	-0.350

RMS values:

	GG	GGG	Laplace	Cauchy
13/2	76	59	286	252
17/2	126	64	187	219
19/2	208	75	215	295
21/2	133	66	209	265
23/2	97	60	190	189

	GG	GGG	Laplace	Cauchy
25/2(1)	134	62	135	178
25/2(2)	283	79	320	238
27/2	111	63	266	347
29/2(1)	95	56	95	157
29/2(2)	161	65	191	273

5 Dependence or independence of parameters

Even though the distribution for normalised coefficients up to the bounds we reached computationally can be pretty well described by the GGG function, as seen in the previous section, the result of Radziwiłł and Soundararajan shows that, for a fixed Hecke eigenform, the parameters must depend on the bound. We investigate this dependence in this section. More precisely, we look at how the parameters behave with respect to x, when we compute with all coefficients up to x.

In order to study this, we consider the case where we have the biggest number of normalised Fourier coefficients, namely $2 \cdot 10^8$, in weight $13/2$. We broke the list of coefficients into 20 subsequent lists of equal size and did the fitting for each of these sublists separately, leading to the results in the following table.

	GGG				GG			Laplace		Cauchy		
	a	b	c	d	a	b	c	b	c	a	b	c
1	0.622	1177	0.967	0.045	0.677	1038	1.08	1172	0.908	142.8	0.140	0.488
2	0.601	1206	0.917	0.031	0.650	1082	1.009	1205	0.876	142.0	0.135	0.498
3	0.591	1221	0.896	0.027	0.638	1101	0.981	1219	0.864	143.4	0.135	0.506
4	0.587	1230	0.886	0.025	0.633	1110	0.969	1226	0.858	144.1	0.135	0.510
5	0.569	1299	0.846	0.037	0.631	1117	0.960	1232	0.852	144.3	0.134	0.513
6	0.580	1252	0.867	0.025	0.627	1123	0.952	1236	0.848	144.8	0.134	0.515
7	0.561	1326	0.828	0.039	0.628	1126	0.948	1239	0.846	144.8	0.134	0.516
8	0.566	1292	0.841	0.030	0.622	1133	0.940	1243	0.843	145.6	0.134	0.520
9	0.572	1273	0.850	0.026	0.621	1136	0.936	1246	0.840	145.6	0.134	0.520
10	0.559	1317	0.825	0.032	0.619	1141	0.930	1249	0.837	146.0	0.134	0.522
11	0.567	1289	0.839	0.027	0.619	1141	0.930	1250	0.837	146.0	0.134	0.522
12	0.559	1317	0.824	0.031	0.618	1144	0.926	1252	0.835	146.2	0.133	0.523
13	0.557	1326	0.819	0.032	0.617	1146	0.924	1254	0.833	146.4	0.133	0.524
14	0.566	1275	0.840	0.020	0.610	1154	0.915	1257	0.830	147.3	0.134	0.524
15	0.551	1349	0.807	0.034	0.616	1150	0.919	1257	0.830	146.6	0.133	0.526
16	0.555	1332	0.814	0.031	0.616	1151	0.918	1258	0.828	146.5	0.133	0.526
17	0.555	1327	0.814	0.029	0.613	1156	0.913	1261	0.828	146.9	0.133	0.528
18	0.560	1309	0.822	0.025	0.612	1158	0.911	1262	0.826	147.0	0.133	0.529
19	0.553	1333	0.811	0.029	0.612	1157	0.911	1262	0.827	147.1	0.133	0.529
20	0.555	1323	0.814	0.026	0.610	1161	0.907	1264	0.825	147.3	0.133	0.530

Here is the corresponding table of the RMS-values.

	GGG	GG	Laplace	Cauchy
1	18.4	18.9	39.0	33.6
2	18.1	18.6	35.4	31.6
3	18.7	19.2	34.1	31.2
4	18.2	18.8	33.3	30.7
5	18.0	18.8	33.2	30.0
6	18.5	19.0	32.7	30.4
7	18.4	19.3	33.0	29.9
8	18.2	19.0	31.9	29.7
9	18.3	18.8	31.7	29.9
10	18.3	19.1	31.6	29.5

	GGG	GG	Laplace	Cauchy
11	18.4	19.0	31.6	29.8
12	18.3	19.2	31.5	29.5
13	18.3	19.2	31.5	29.5
14	18.2	18.7	30.1	29.4
15	17.7	18.7	31.0	28.9
16	18.3	19.2	31.3	29.4
17	18.0	18.8	30.6	29.1
18	18.3	19.0	30.7	29.4
19	18.4	19.2	30.8	29.2
20	18.4	19.2	30.5	29.2

One clearly observes some dependence. For instance, for the Laplace distribution (4.5), the b value seems to be slowly, but strictly increasing, whereas the c-value slowly, but strictly decreases (with one exception). The values for the Cauchy distribution (4.6) and the GG-distribution (4.4) also suggest a dependence. For the GGG-distribution (4.3), there is a clear dependence of the values for the first couple of sets. However, all four values seem to stabilise for the last sets. The range of data we investigated hence does not allow us to illustrate that the limit distribution is known to be a Dirac delta function.

Recall that we restricted our efforts to squarefree indices n such that the coefficient is not automatically known to be zero. We investigated further if the distribution seems to change significantly when considering prime indexed coefficients only. In the integral weight case, there are huge differences and the semi-circular distribution of Sato-Tate for non-CM eigenforms is only valid for prime indices. Moreover, the coefficients of integral weight Hecke eigenforms are multiplicative functions, hence the distribution of the coefficients at prime indices determines the rest. We do not know of any reason to believe that similar things happen in the half-integral weight case. Indeed, the shapes of the distribution graphs do not change significantly if we restrict to prime indices (see Figure 5 for an example). Of course, some of the best fit parameters move, as we can see in this table:

	GGG				GG			Laplace		Cauchy		
	a	b	c	d	a	b	c	b	c	a	b	c
Sqfree	0.570	25666	0.850	0.030	0.623	22621	0.942	24837	0.843	140	0.006	0.113
Prime	0.550	3892	0.759	0.030	0.614	3329	0.856	3635	0.784	196	0.062	0.380

The b values differ just because there are far more squarefree numbers than prime numbers. For the same reason also the RMS values are different. However, the most important parameters, i.e. the a-parameters in GGG and GG are very similar. It is not clear if the slight change of parameters can be explained by the fact that the set of primes among squarefree numbers is biassed towards small values, or if the change is not significant. This gives us confidence in our belief that prime indexed normalised coefficients are not distributed differently from those with squarefree indices, when considering coefficients with indices up to the bounds we could reach.

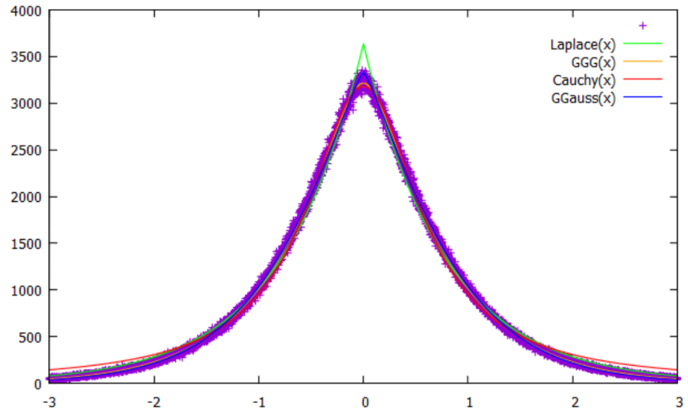

Figure 5: Histogram and distributions of Hecke eigenform of weight $13/2$ for prime indexed coefficients only

The observed very good approximations by the GGG-distribution of the normalised Fourier coefficients up to the bounds we could computationally reach lead us to ask the question whether this holds for all bounds.

Question 5.1. *Let f be a half-integral weight cuspidal Hecke eigenform in the Kohnen plus-space in half-integral weight $k = \ell + \frac{1}{2}$ and let $b(n)$ be its normalised coefficients. Let $x \in \mathbb{R}_{>0}$.*

Can the distribution of the $b(n)$ for $n \leq x$ squarefree and $n \equiv (-1)^\ell \mod 4$ be approximated by the GGG-distribution with parameters depending on f and x?

More precisely, are there constants a, c, d, depending on f and x, such that for all intervals $I = [\alpha, \beta] \subseteq \mathbb{R}$ the quotient

$$\frac{\#\{n \in \mathbb{N} \text{ squarefree} \mid n \leq x, n \equiv (-1)^\ell \mod 4, b(n) \in I\}}{\#\{n \in \mathbb{N} \text{ squarefree} \mid n \leq x, n \equiv (-1)^\ell \mod 4\}}$$

is 'close to'

$$\frac{1}{b}\int_\alpha^\beta \exp(-\frac{(d+t^2)^a}{c})dt,$$

where $b = \int_{-\infty}^{\infty} \exp(-\frac{(d+t^2)^a}{c})dt$?

6 A strengthening of the Bruinier-Kohnen conjecture

Bruinier and Kohnen conjectured that the signs of coefficients of half-integral weight Hecke eigenforms are equidistributed. More precisely, let $f = \sum_{n=1}^{\infty} a(n)q^n \in S_k(N, \chi)$ be a cusp form of weight $k = \ell + 1/2$ with real Fourier coefficients and assume that f is orthogonal to the unary theta series when $\ell = 1$. Then the Bruinier-Kohnen Conjecture ([BK08] and [KLW13]) asserts that the sets

$\{n \in \mathbb{N} : a(n) > 0\}$ and $\{n \in \mathbb{N} : a(n) < 0\}$ have the same natural density, equal to half of the natural density of $\{n \in \mathbb{N} : a(n) \neq 0\}$.

Combining the Shimura lift with the (proved) celebrated Sato-Tate Conjecture for integral weight Hecke eigenforms, it is not very difficult to prove equidistribution of signs for the coefficients indexed by squares, see [IW13], [AdIW15], [IW16]. The sign equidistribution problem has still received much attention and is widely studied (for instance [Amr19]), and the technique from [IW13] has been extended to more general automorphic forms like Hilbert modular forms in [KKT18]. Note that this is only a partial result and the full proof of the conjecture is still an open problem and, for the moment, it is likely that there is no theoretical tool to attack this problem.

We see the calculations in this article as the most systematic and largest computational support for the Bruinier-Kohnen Conjecture to date. In fact, if the distribution of coefficients up to any bound x follows any distribution function (depending on x) that is symmetric with respect to 0, e.g. any of the four types discussed above, then the Bruinier-Kohnen Conjecture is true.

In fact, the symmetry around 0 suggests that the signs are uniformly distributed and that the distribution of the signs and the distribution of the absolute value of the normalised coefficients are independent. In order to make this precise, we recall that according to (1.2), there are infinitely many 'big' normalised coefficients $|b(n)|$. This suggests that any non-empty interval $I \subseteq \mathbb{R}_{>0}$ will contain infinitely many normalised coefficients $|b(n)|$ for squarefree n. We insist on squarefree because we do not want to deal with the contributions that are understood by the Shimura lift.

We feel that the symmetry around 0 of the distribution of the normalised coefficients up to the bounds that we computationally reached warrant the formulation of the following conjecture, strengthening the Bruinier-Kohnen Conjecture.

Conjecture 6.1 (Independence of sign and absolute value)**.** *Let f be as above. Let $I \subseteq \mathbb{R}_{>0}$ be any interval. Then the following limit exists and we have*

$$\lim_{x\to\infty} \frac{\#\{n \leq x \text{ squarefree } \mid |b(n)| \in I, b(n) > 0\}}{\#\{n \leq x \text{ squarefree } \mid |b(n)| \in I\}} = \frac{1}{2}.$$

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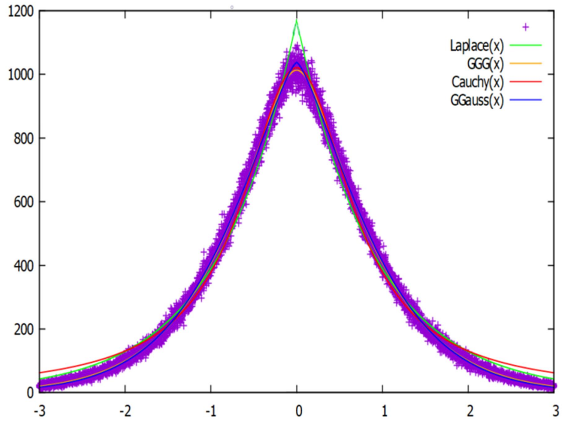
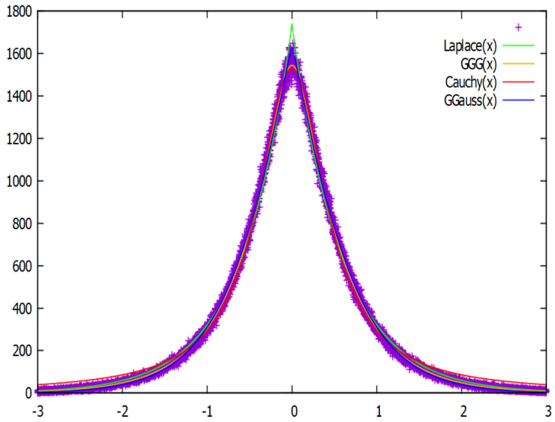
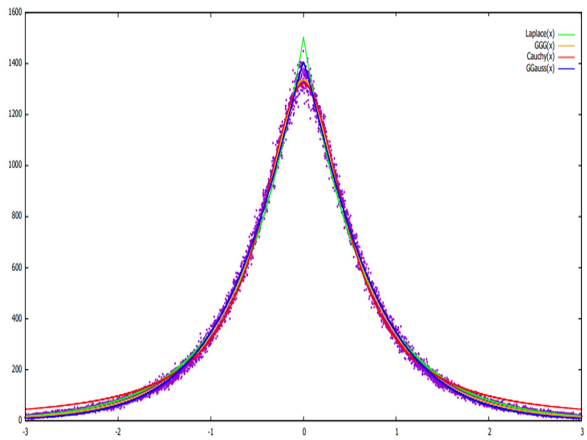
Appendix: Graphs of histograms of all computed examples



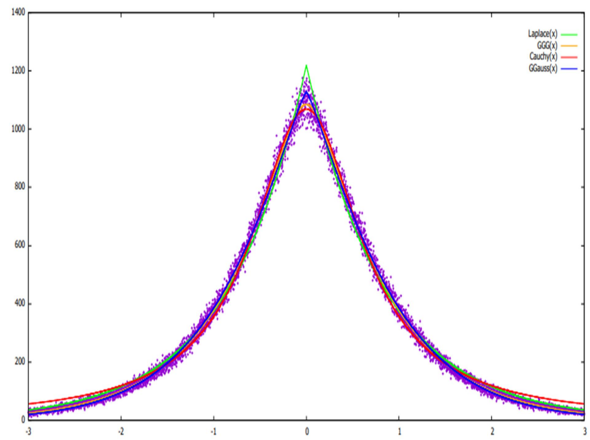
Histogram of 10^7 *normalised Fourier coefficients of Hecke eigenform of weight* $13/2$ *and distributions*



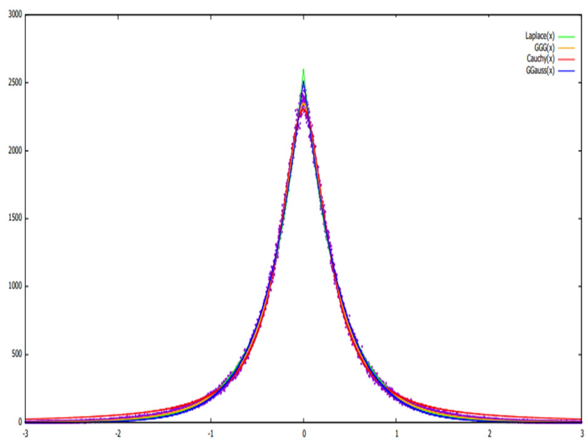
Histogram of 10^7 *normalised Fourier coefficients of Hecke eigenform of weight* $21/2$ *and distributions*



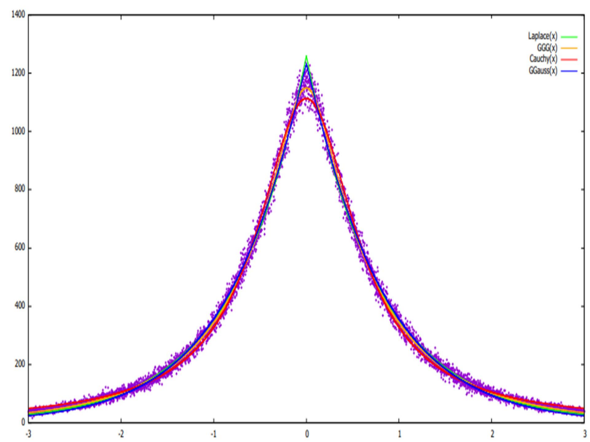
Histogram of 10^7 *normalised Fourier coefficients of Hecke eigenform of weight* $17/2$ *and distributions*



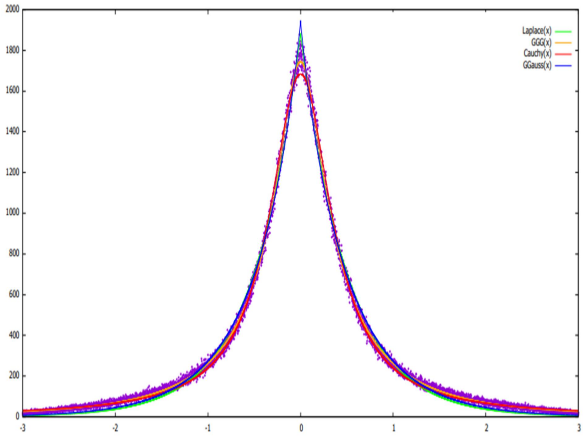
Histogram of 10^7 *normalised Fourier coefficients of Hecke eigenform of weight* $23/2$ *and distributions*



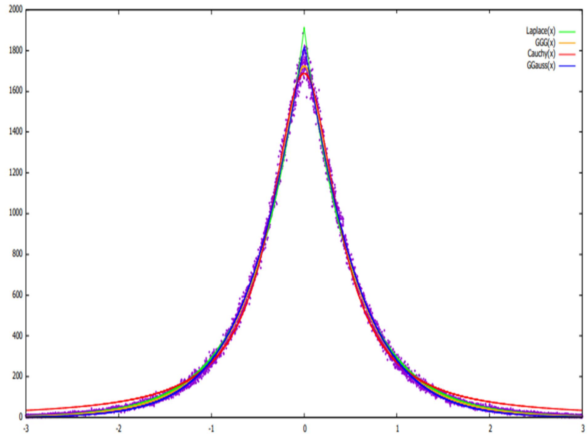
Histogram of 10^7 *normalised Fourier coefficients of Hecke eigenform of weight* $19/2$ *and distributions*



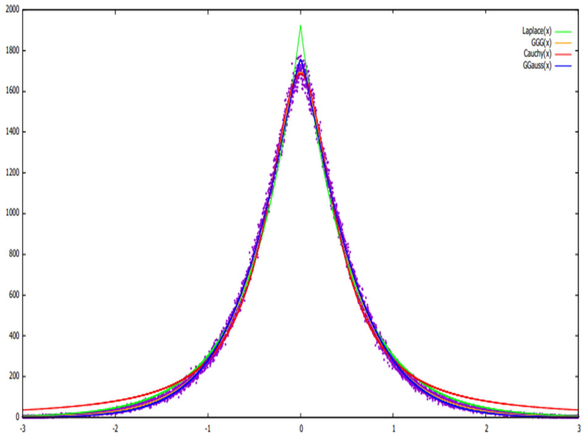
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $25/2$ *and distributions*



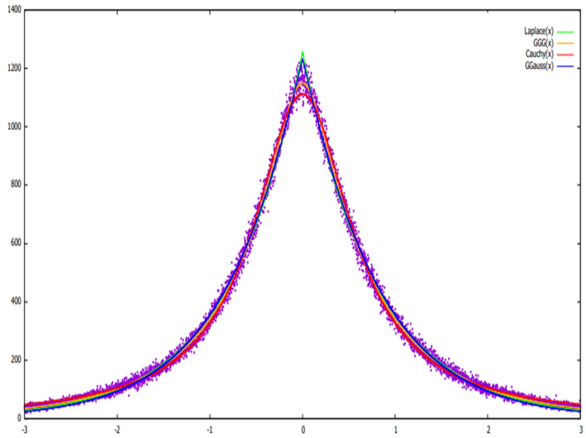
Histogram of 10^7 normalised Fourier coefficients
of second Hecke eigenform of weight $25/2$
and distributions



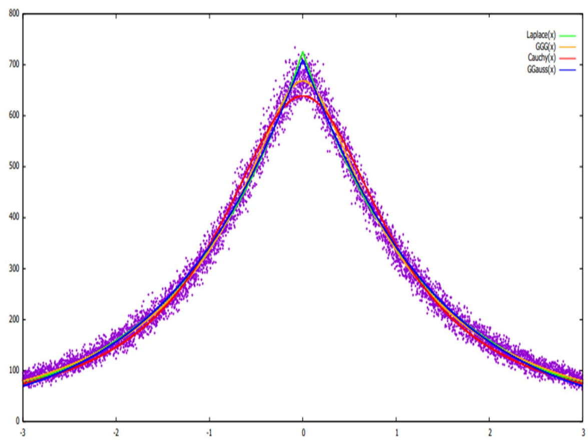
Histogram of 10^7 normalised Fourier coefficients
of first Hecke eigenform of weight $31/2$
and distributions



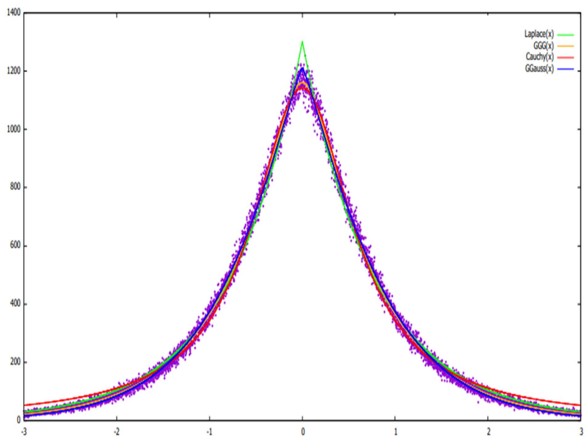
Histogram of 10^7 normalised Fourier coefficients
of Hecke eigenform of weight $27/2$
and distributions



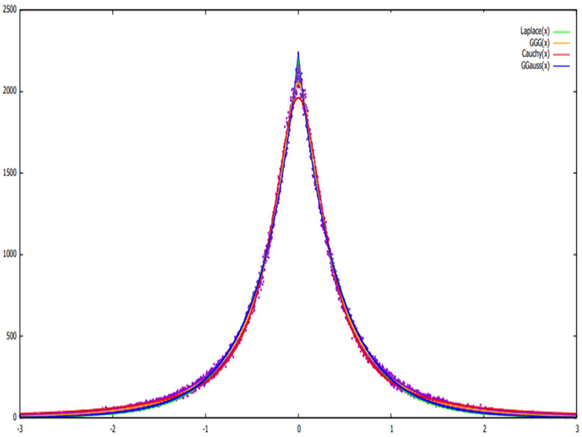
Histogram of 10^7 normalised Fourier coefficients
of second Hecke eigenform of weight $31/2$
and distributions



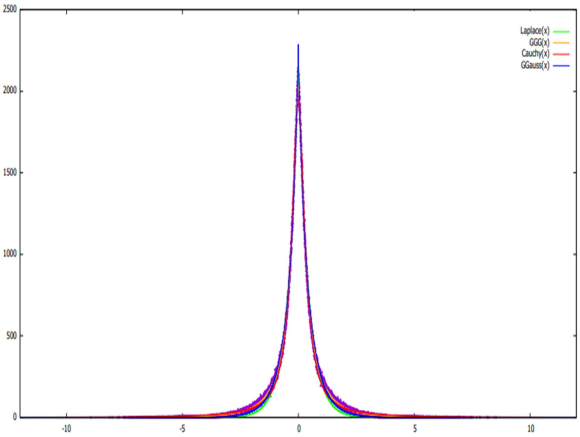
Histogram of 10^7 normalised Fourier coefficients
of Hecke eigenform of weight $29/2$
and distributions



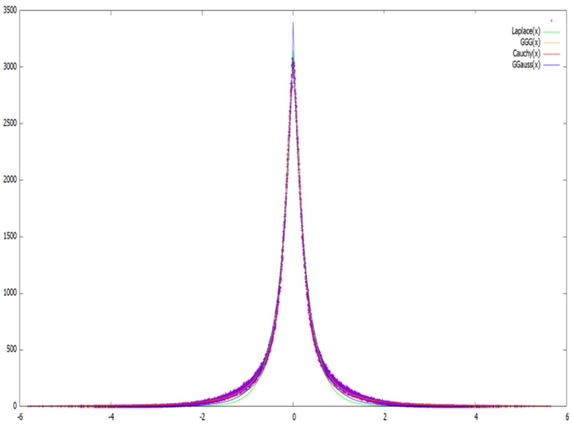
Histogram of 10^7 normalised Fourier coefficients
of first Hecke eigenform of weight $33/2$
and distributions



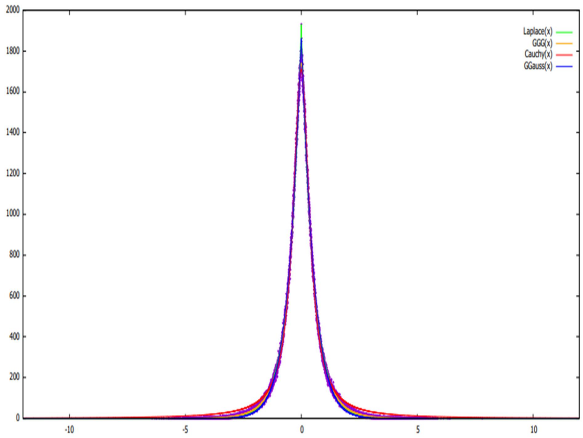
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $33/2$ *and distributions*



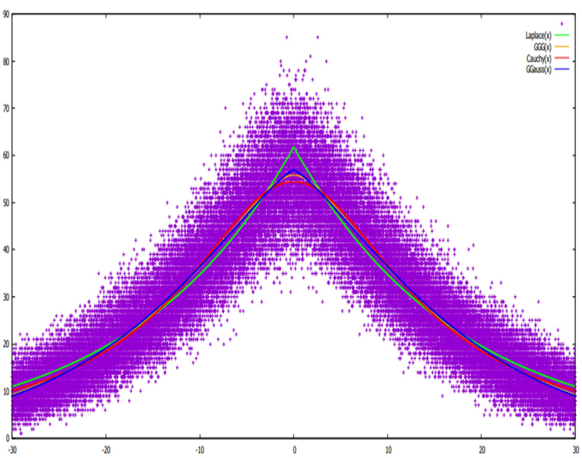
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $37/2$ *and distributions*



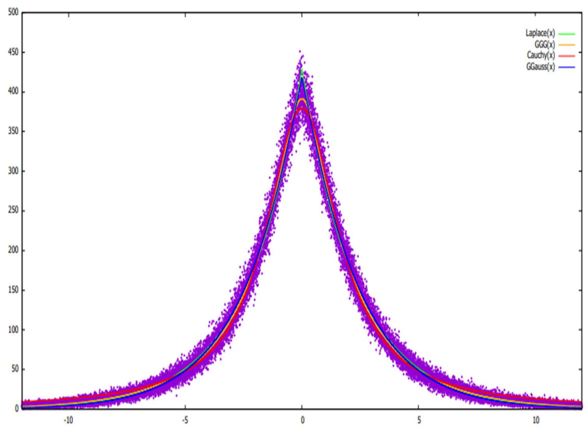
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $35/2$ *and distributions*



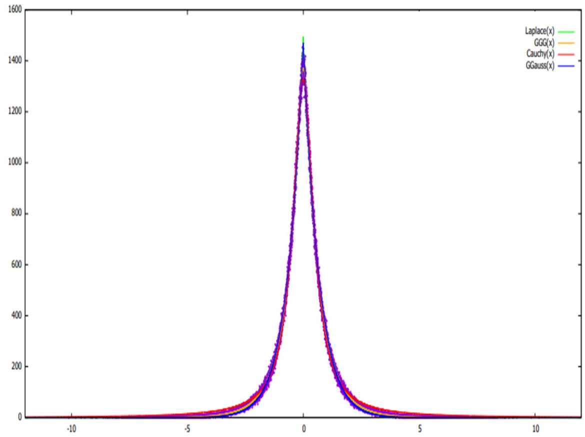
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $37/2$ *and distributions*



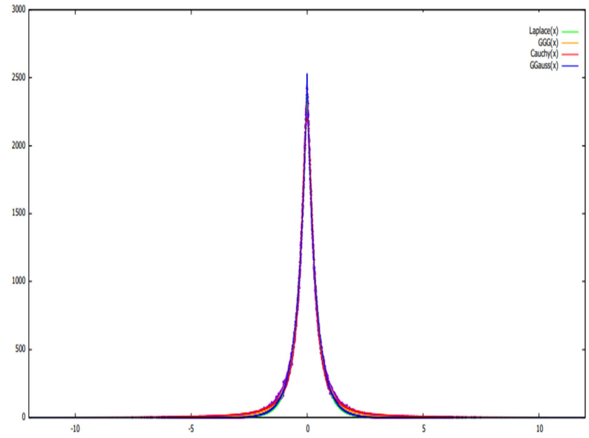
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $35/2$ *and distributions*



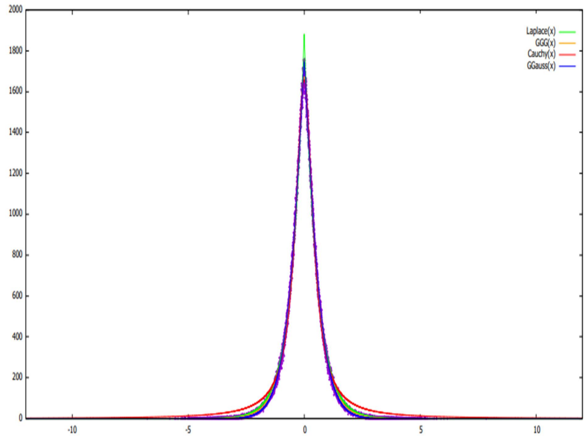
Histogram of 10^7 *normalised Fourier coefficients of third Hecke eigenform of weight* $37/2$ *and distributions*



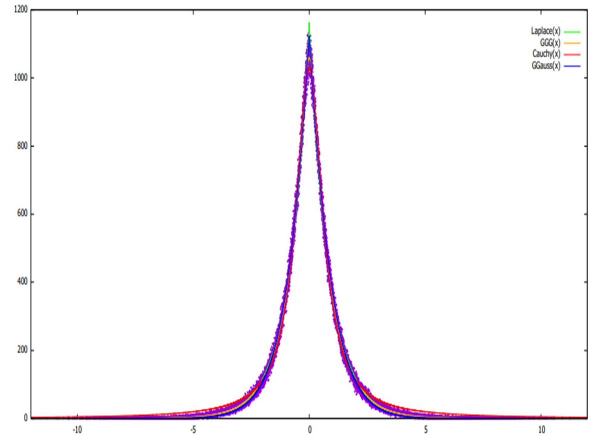
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $39/2$ *and distributions*



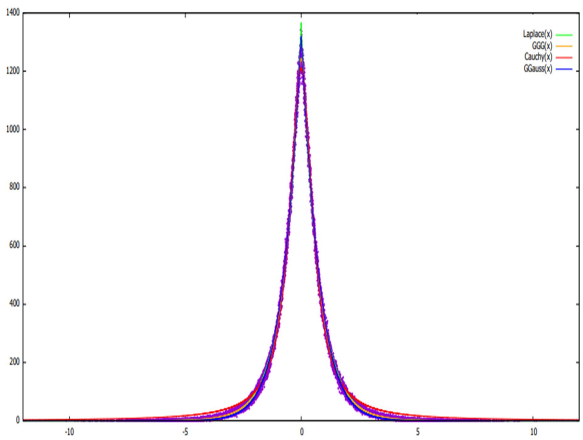
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $41/2$ *and distributions*



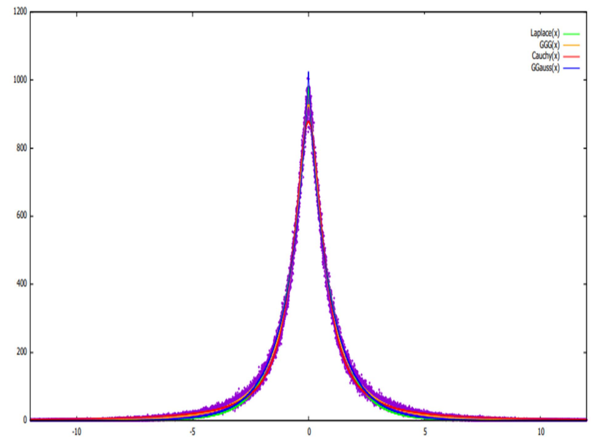
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $39/2$ *and distributions*



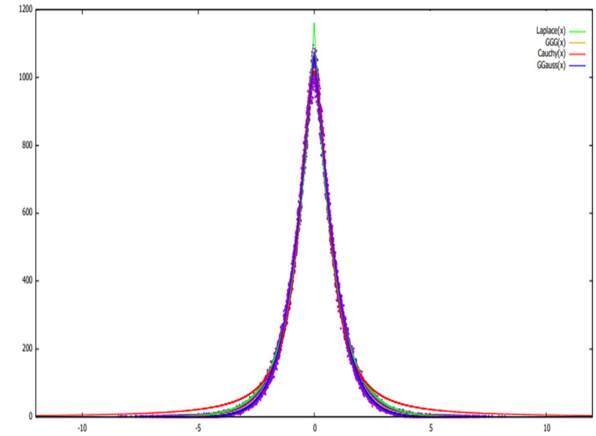
Histogram of 10^7 *normalised Fourier coefficients of third Hecke eigenform of weight* $41/2$ *and distributions*



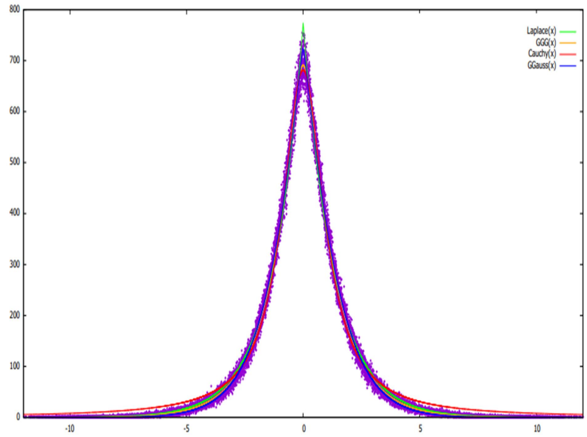
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $41/2$ *and distributions*



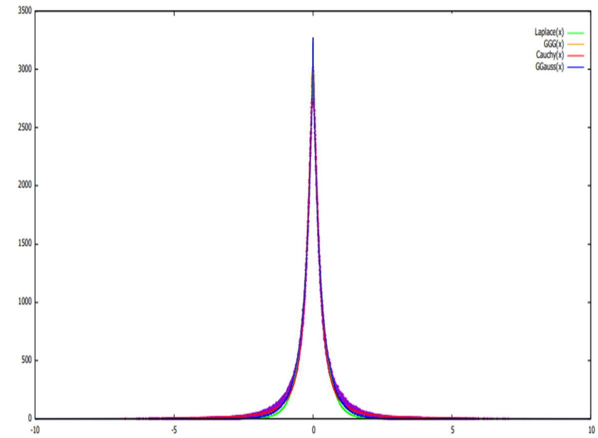
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $43/2$ *and distributions*



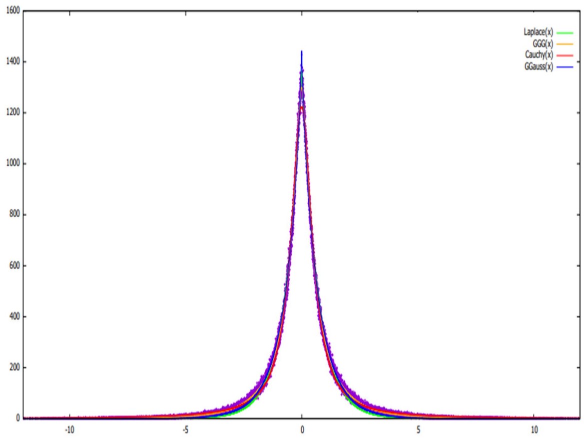
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $43/2$ *and distributions*



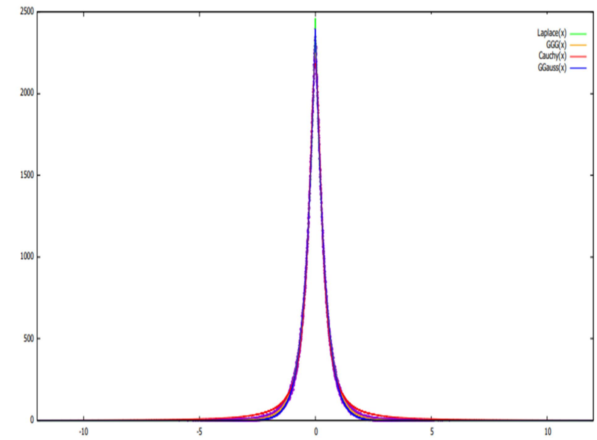
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $45/2$ *and distributions*



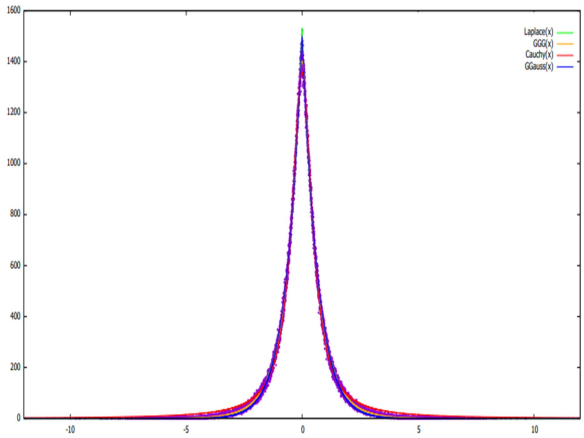
Histogram of 10^7 *normalised Fourier coefficients of third Hecke eigenform of weight* $43/2$ *and distributions*



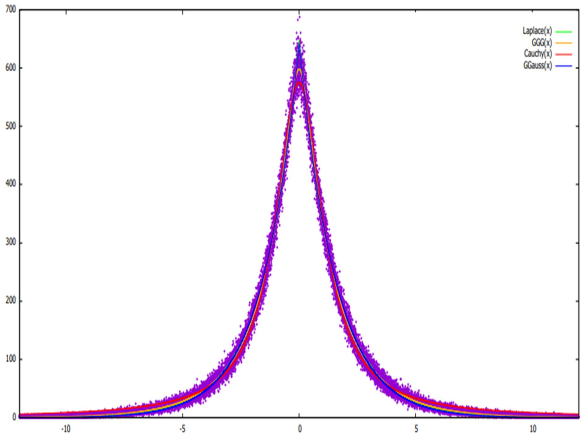
Histogram of 10^7 *normalised Fourier coefficients of third Hecke eigenform of weight* $45/2$ *and distributions*



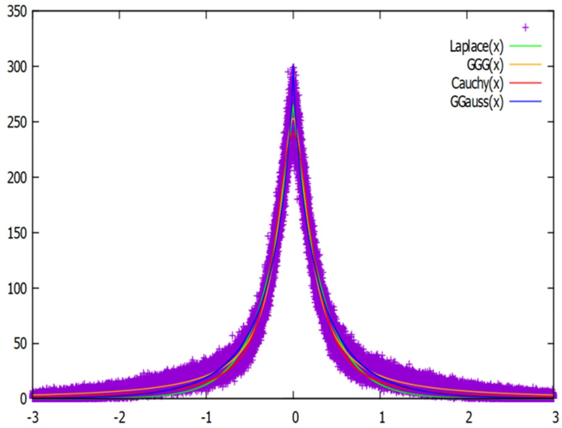
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $45/2$ *and distributions*



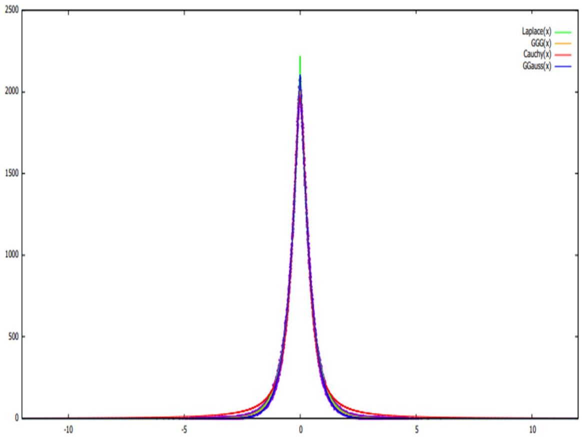
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $47/2$ *and distributions*



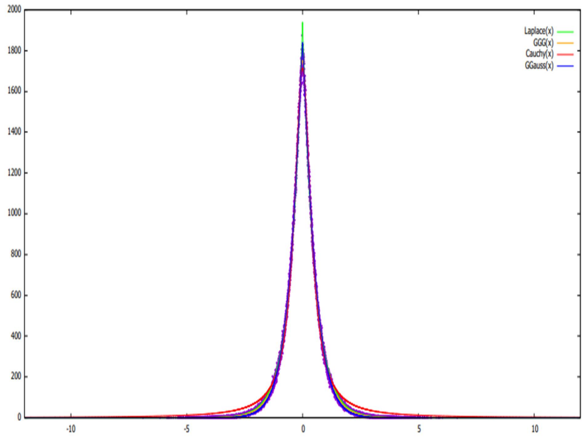
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $47/2$ *and distributions*



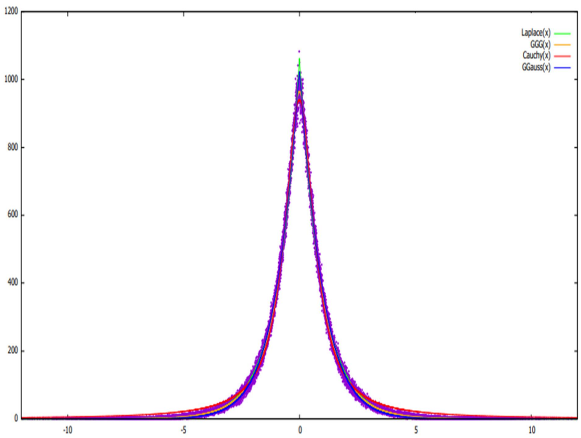
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $49/2$ *and distributions*



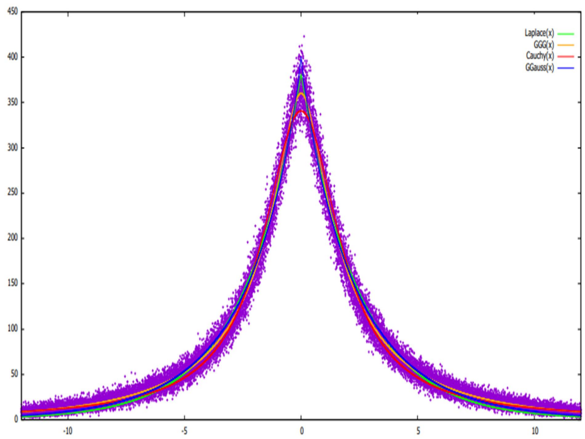
Histogram of 10^7 *normalised Fourier coefficients of third Hecke eigenform of weight* $47/2$ *and distributions*



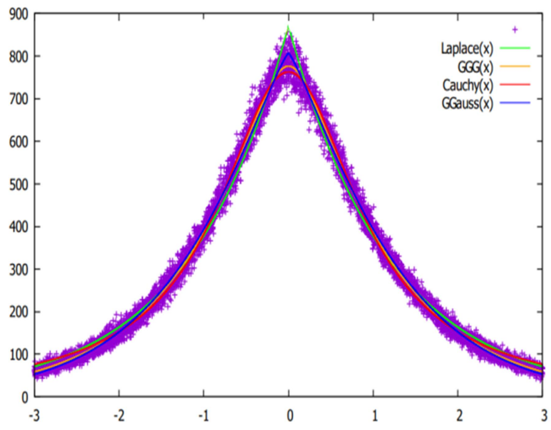
Histogram of 10^7 *normalised Fourier coefficients of third Hecke eigenform of weight* $49/2$ *and distributions*



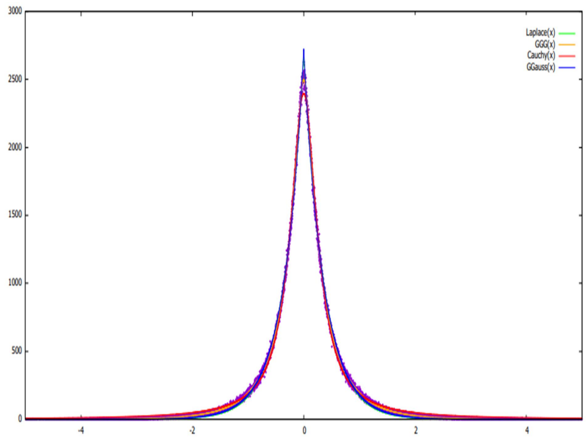
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $49/2$ *and distributions*



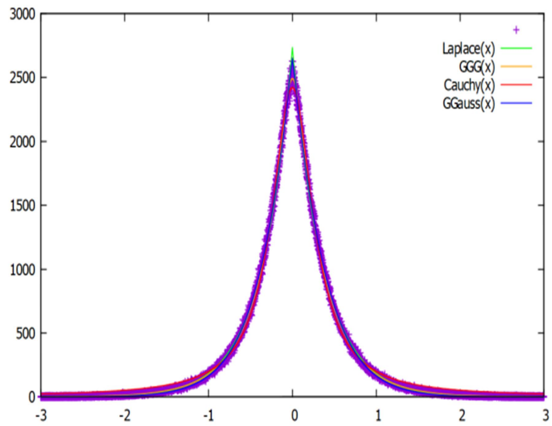
Histogram of 10^7 *normalised Fourier coefficients of fourth Hecke eigenform of weight* $49/2$ *and distributions*



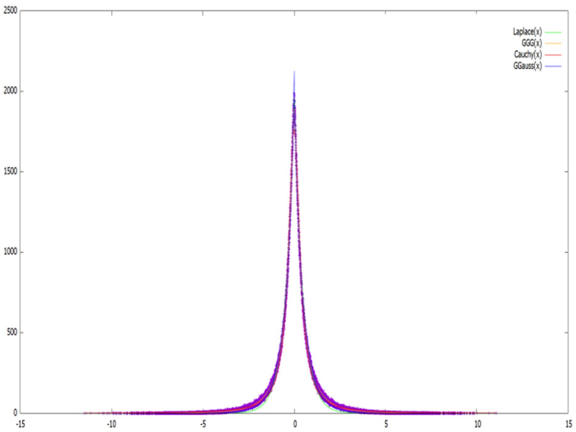
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $51/2$ *and distributions*



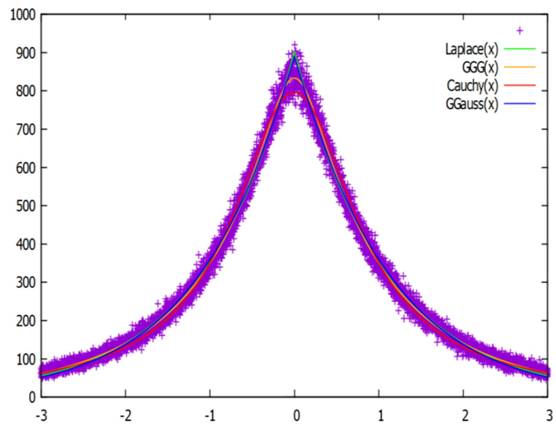
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $53/2$ *and distributions*



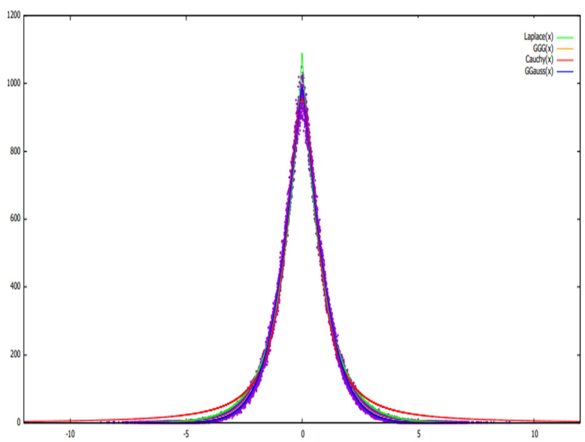
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $51/2$ *and distributions*



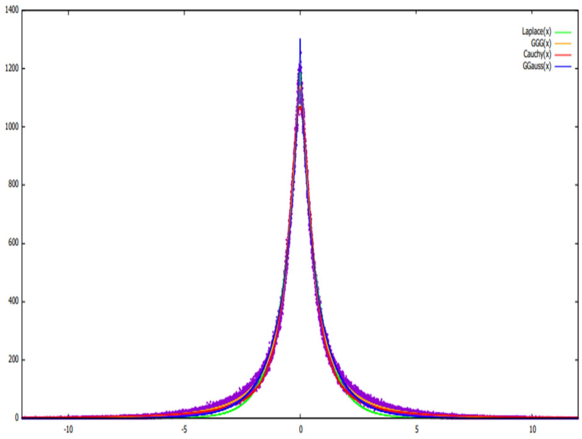
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $53/2$ *and distributions*



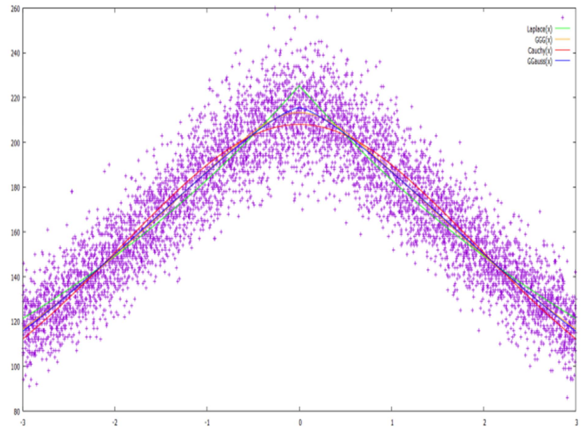
Histogram of 10^7 *normalised Fourier coefficients of third Hecke eigenform of weight* $51/2$ *and distributions*



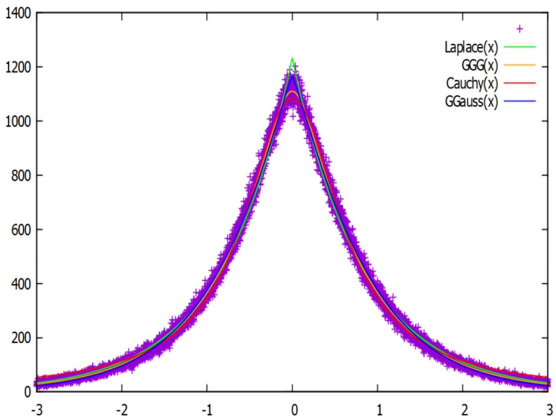
Histogram of 10^7 *normalised Fourier coefficients of third Hecke eigenform of weight* $53/2$ *and distributions*



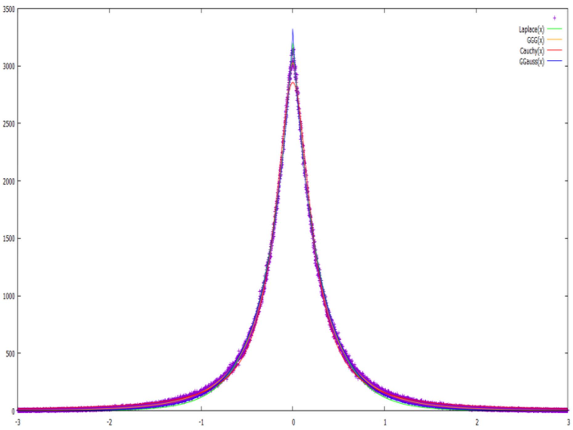
Histogram of 10^7 *normalised Fourier coefficients*
of fourth Hecke eigenform of weight $53/2$
and distributions



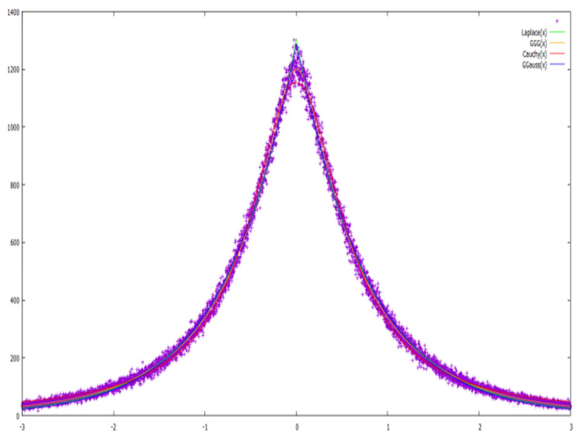
Histogram of 10^7 *normalised Fourier coefficients*
of third Hecke eigenform of weight $55/2$
and distributions



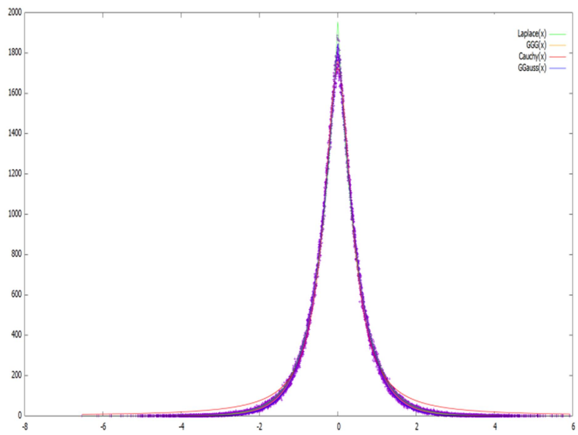
Histogram of 10^7 *normalised Fourier coefficients*
of first Hecke eigenform of weight $55/2$
and distributions



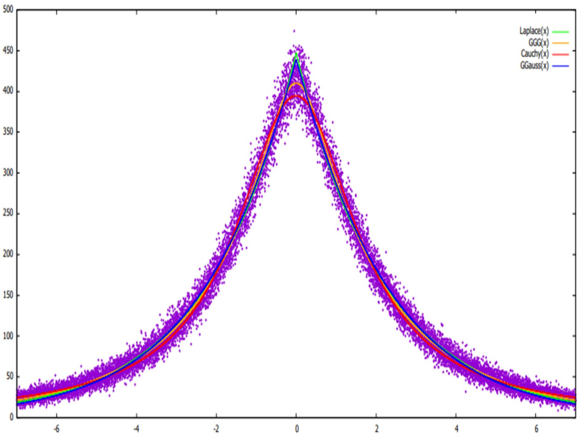
Histogram of 10^7 *normalised Fourier coefficients*
of fourth Hecke eigenform of weight $55/2$
and distributions



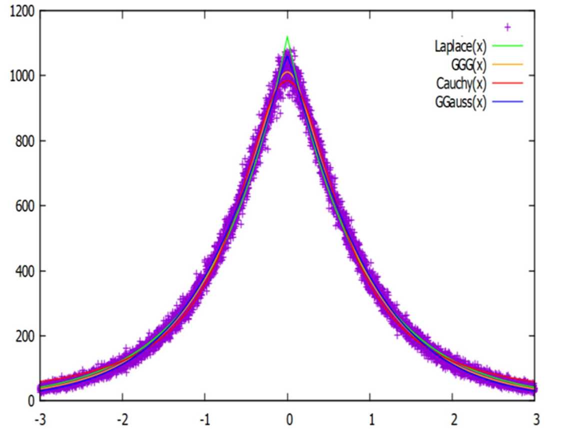
Histogram of 10^7 *normalised Fourier coefficients*
of second Hecke eigenform of weight $55/2$
and distributions



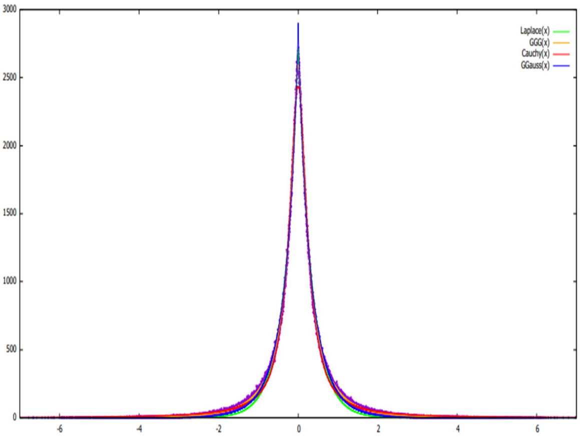
Histogram of 10^7 *normalised Fourier coefficients*
of first Hecke eigenform of weight $57/2$
and distribution



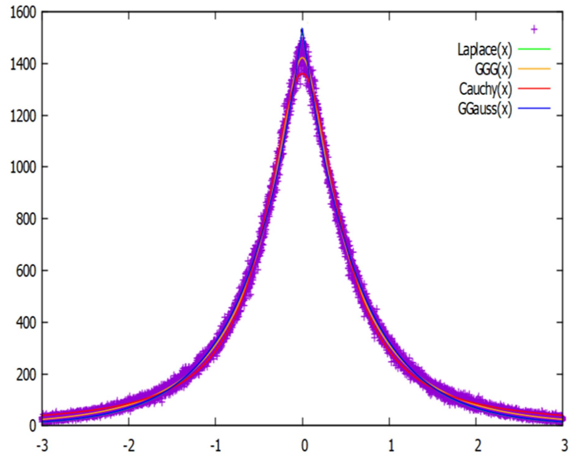
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $57/2$ *and distributions*



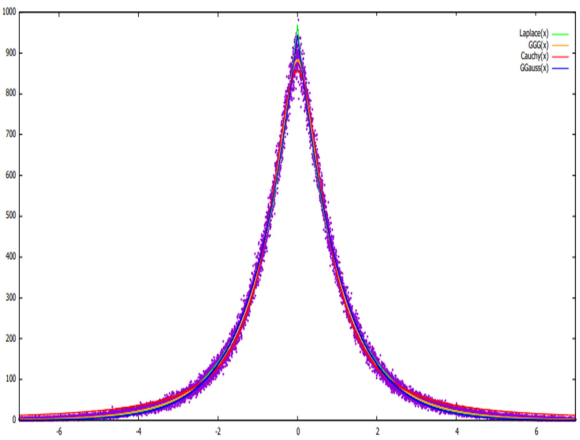
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $59/2$ *and distributions*



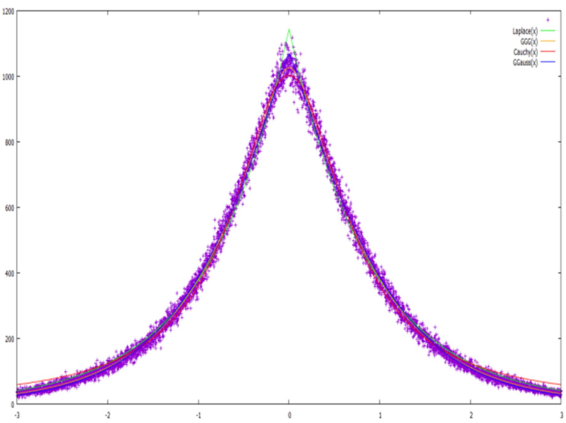
Histogram of 10^7 *normalised Fourier coefficients of third Hecke eigenform of weight* $57/2$ *and distributions*



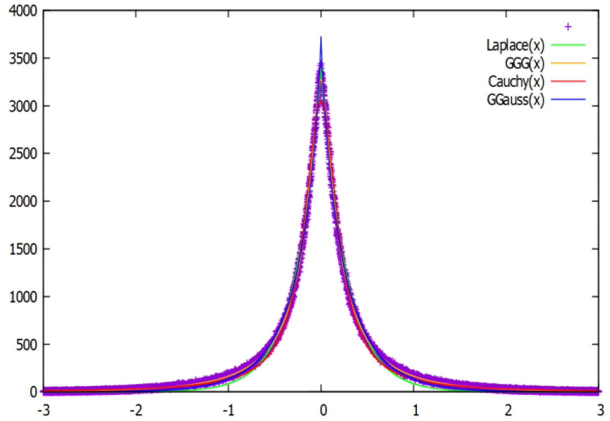
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $59/2$ *and distributions*



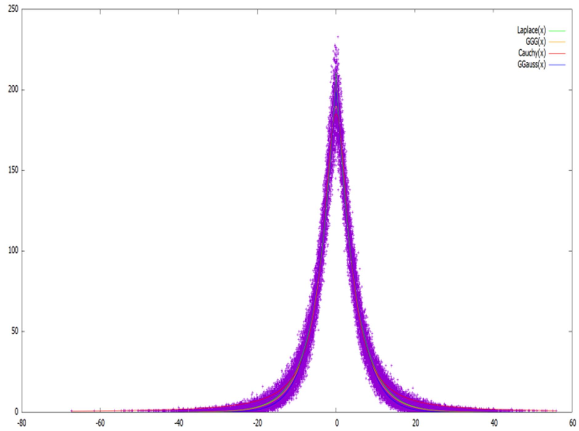
Histogram of 10^7 *normalised Fourier coefficients of fourth Hecke eigenform of weight* $57/2$ *and distributions*



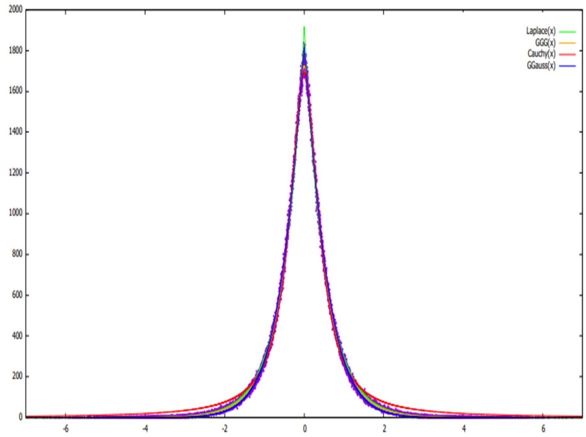
Histogram of 10^7 *normalised Fourier coefficients of third Hecke eigenform of weight* $59/2$ *and distributions*



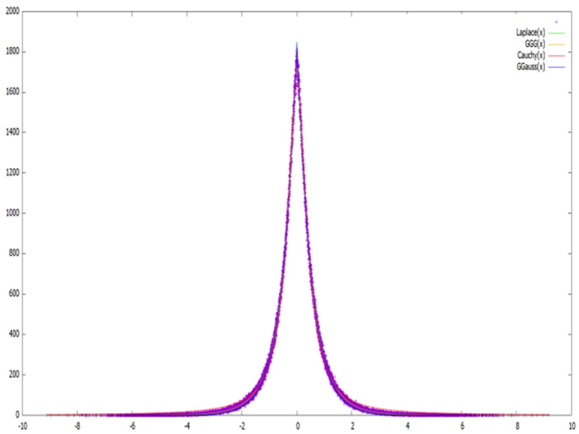
Histogram of 10^7 *normalised Fourier coefficients of fourth Hecke eigenform of weight* $59/2$ *and distributions*



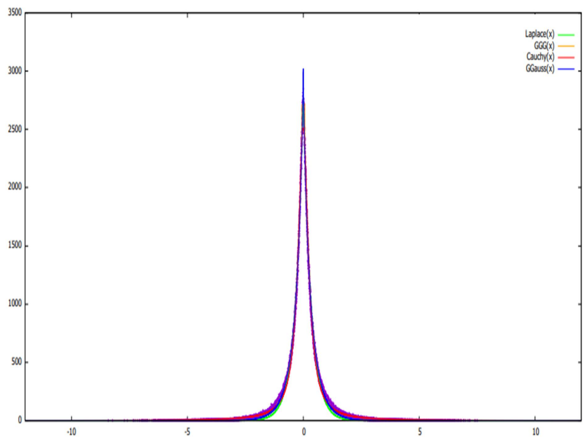
Histogram of 10^7 *normalised Fourier coefficients of third Hecke eigenform of weight* $61/2$ *and distributions*



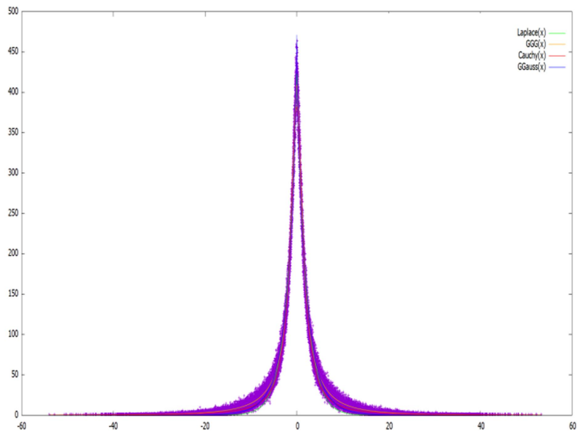
Histogram of 10^7 *normalised Fourier coefficients of first Hecke eigenform of weight* $61/2$ *and distributions*



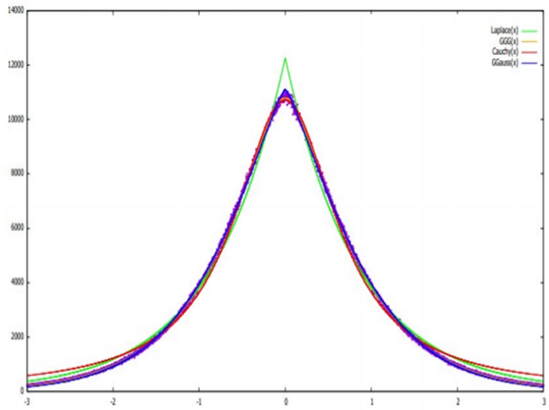
Histogram of 10^7 *normalised Fourier coefficients of fourth Hecke eigenform of weight* $61/2$ *and distributions*



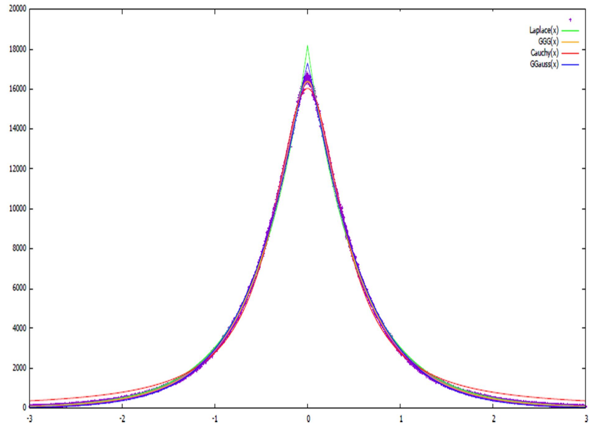
Histogram of 10^7 *normalised Fourier coefficients of second Hecke eigenform of weight* $61/2$ *and distributions*



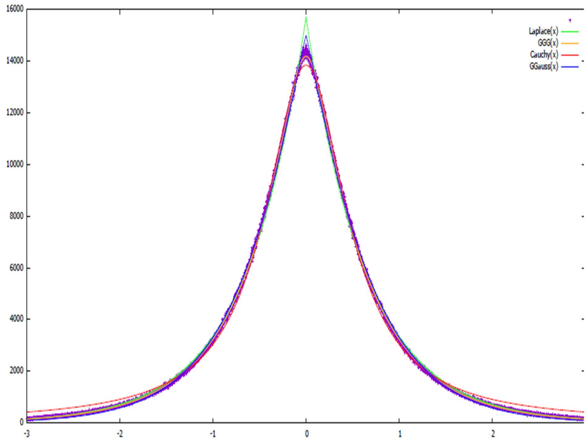
Histogram of 10^7 *normalised Fourier coefficients of fifth Hecke eigenform of weight* $61/2$ *and distributions*



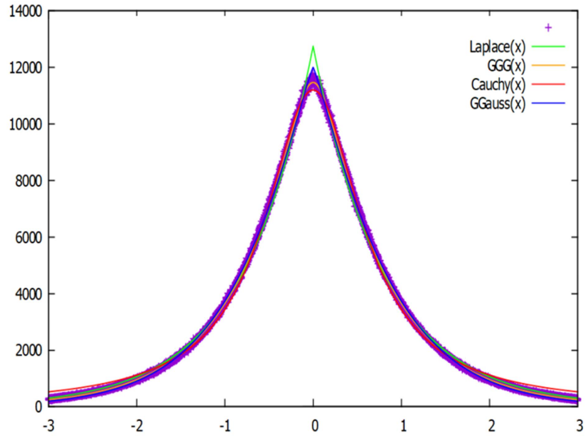
Histogram of 10^8 *normalised Fourier coefficients of Hecke eigenform of weight* $13/2$ *and distributions*



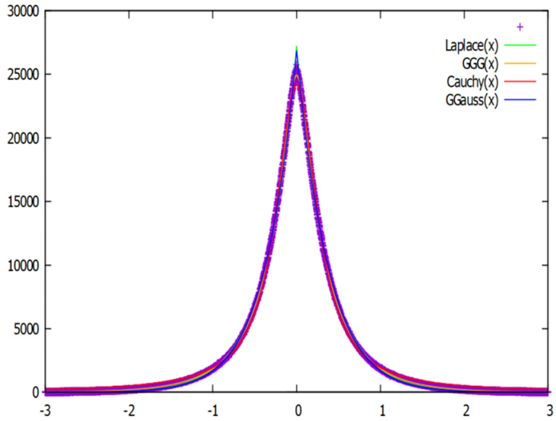
Histogram of 10^8 *normalised Fourier coefficients of Hecke eigenform of weight* $21/2$ *and distributions*



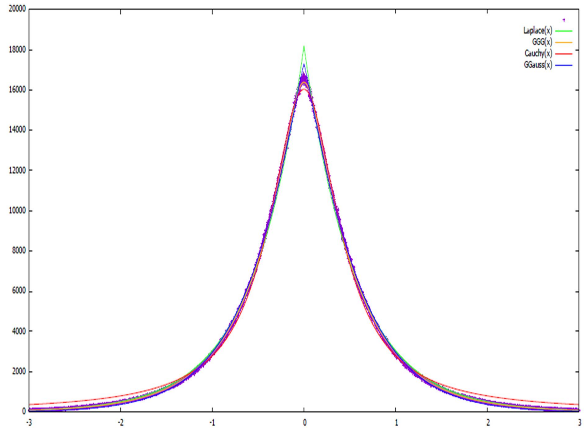
Histogram of 10^8 *normalised Fourier coefficients of Hecke eigenform of weight* $17/2$ *and distributions*



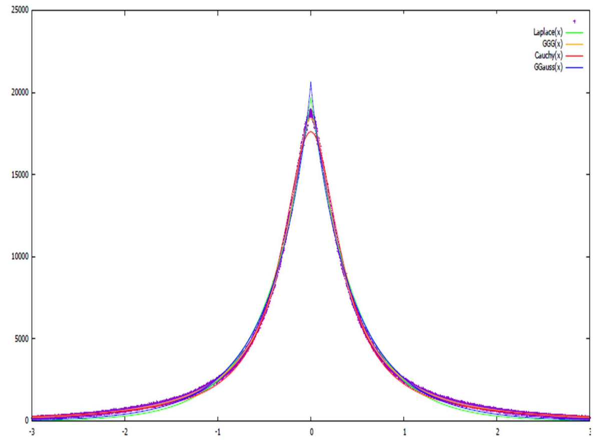
Histogram of 10^8 *normalised Fourier coefficients of Hecke eigenform of weight* $23/2$ *and distributions*



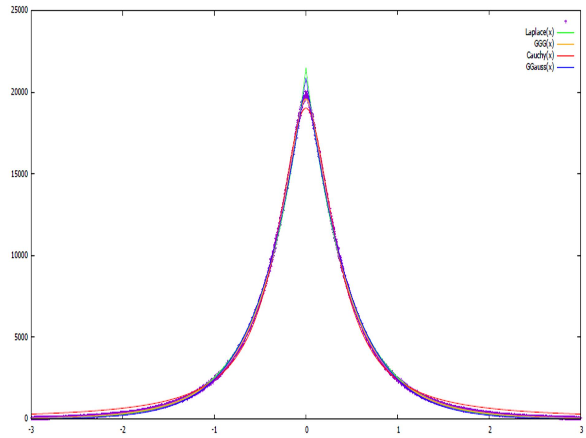
Histogram of 10^8 *normalised Fourier coefficients of Hecke eigenform of weight* $19/2$ *and distributions*



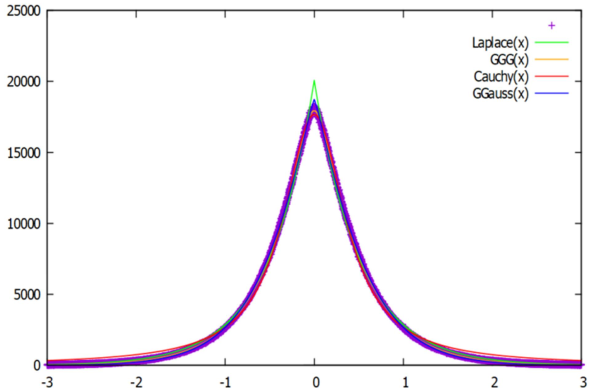
Histogram of 10^8 *normalised Fourier coefficients of first Hecke eigenform of weight* $25/2$ *and distributions*



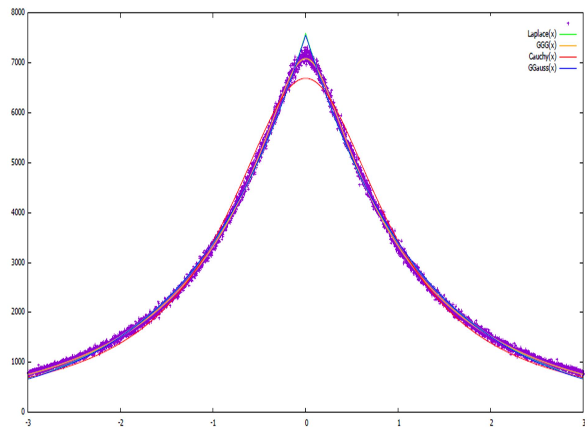
Histogram of 10^8 *normalised Fourier coefficients of second Hecke eigenform of weight* $25/2$ *and distributions*



Histogram of 10^8 *normalised Fourier coefficients of second Hecke eigenform of weight* $29/2$ *and distributions*

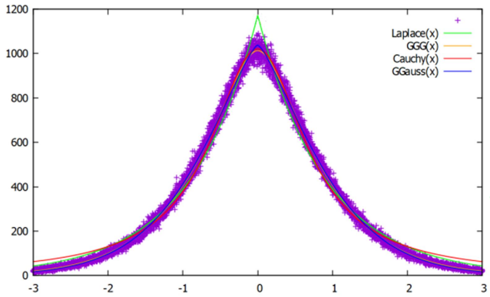


Histogram of 10^8 *normalised Fourier coefficients of Hecke eigenform of weight* $27/2$ *and distributions*

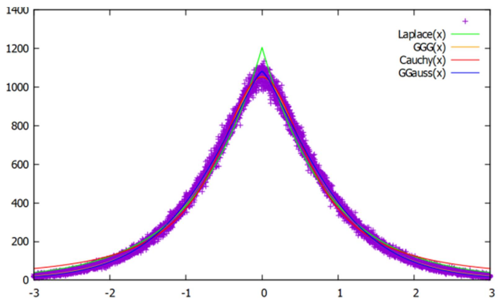


Histogram of 10^8 *normalised Fourier coefficients of first Hecke eigenform of weight* $29/2$ *and distributions*

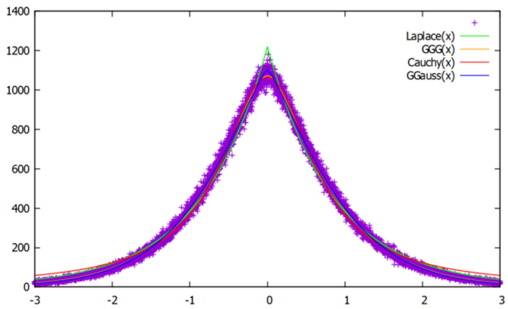
Appendix: Graphs of histograms in weight $13/2$ with $2 \cdot 10^8$ coefficients in 20 subsets



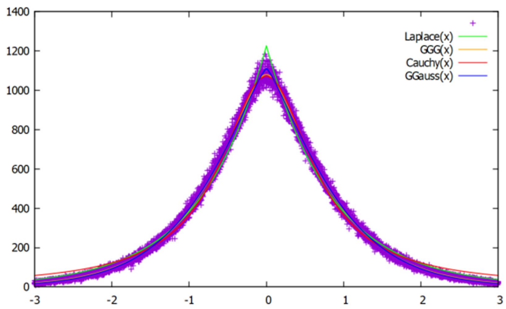
1st part



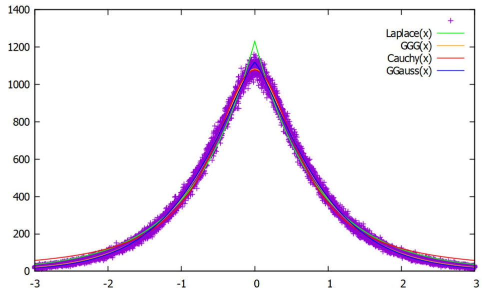
2nd part



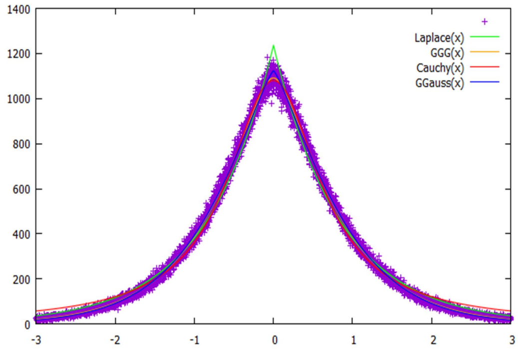
3rd part



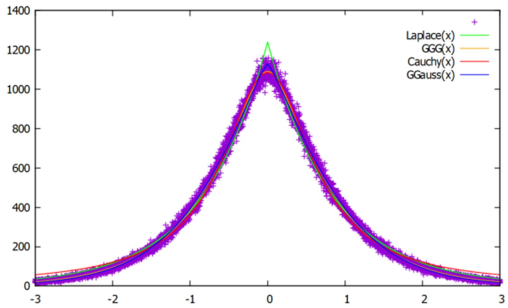
4th part



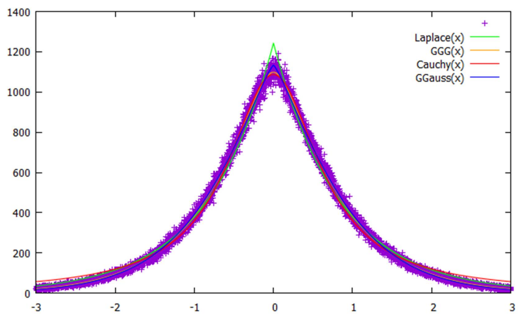
5th part



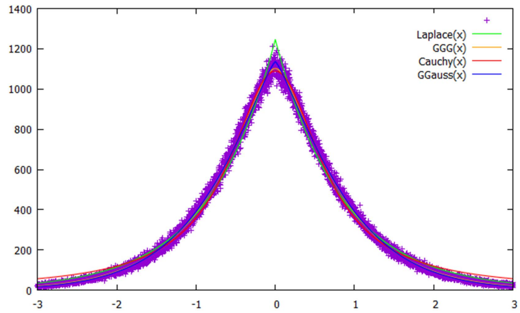
6th part



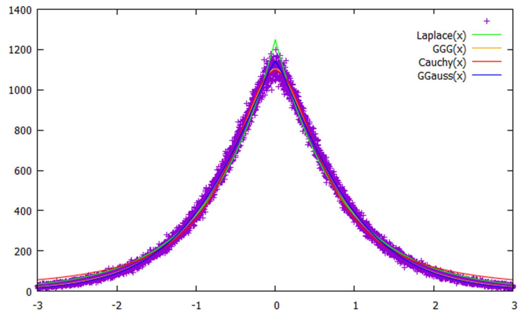
7th part



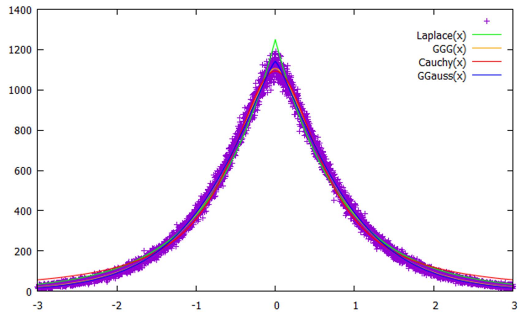
8th part



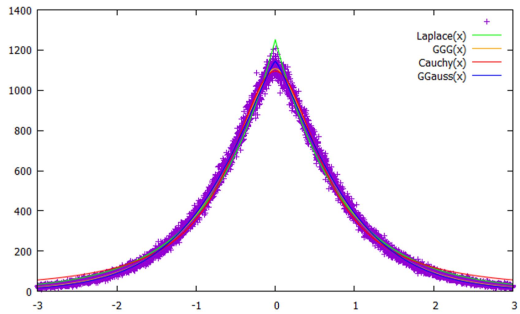
9th part



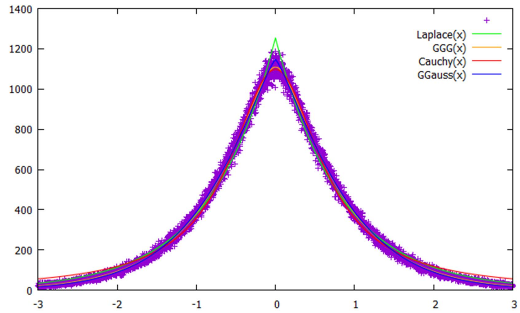
10th part



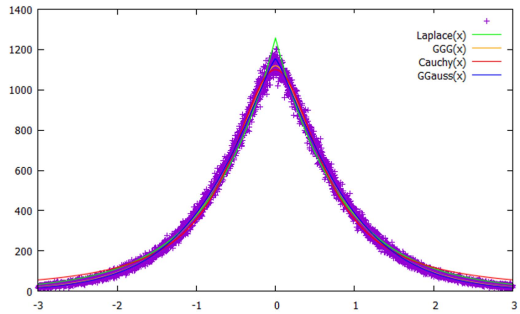
11th part



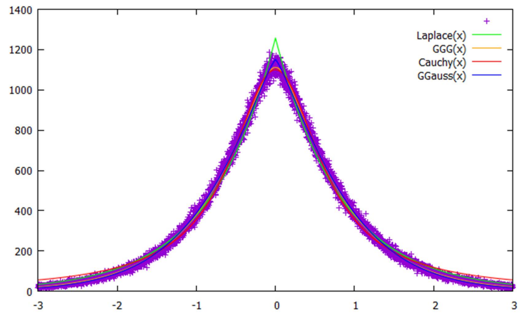
12th part



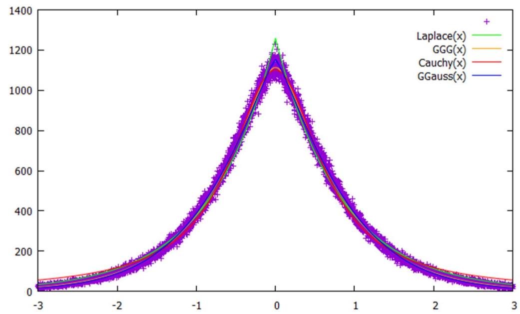
13th part



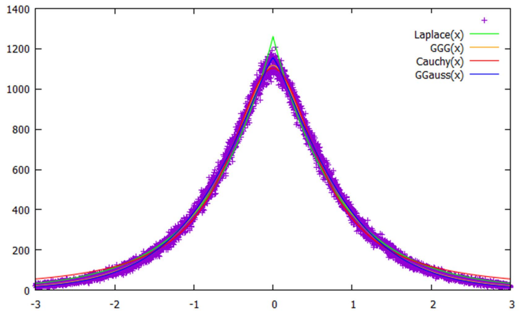
14th part



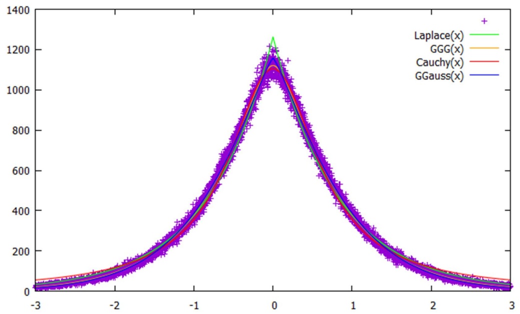
15th part



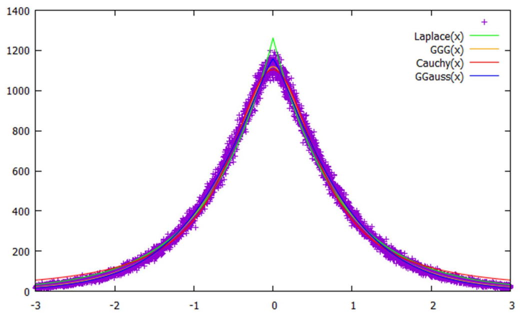
16th part



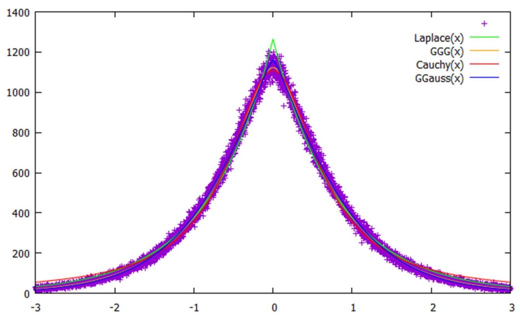
17th part



18th part



19th part



20th part

Appendix: Tables of best fit parameters

Best fit parameters (rounded) for the GGG-distribution for all examples with 10^7 coefficients:

	a	b	c	d
$13/2$	0.622	1177.4	0.967	0.045
$17/2$	0.470	1986.6	0.575	0.043
$19/2$	0.386	4595.4	0.318	0.018
$21/2$	0.477	2268.8	0.506	0.032
$23/2$	0.527	1358.6	0.800	0.039
$25/2(1)$	0.384	2065.3	0.570	0.057
$25/2(2)$	0.219	12428.5	0.262	0.048
$27/2$	0.542	2129.2	0.498	0.018
$29/2$	0.354	1272.2	0.789	0.147
$31/2(1)$	0.468	2404.4	0.469	0.019
$31/2(2)$	0.375	2162.2	0.555	0.061
$33/2(1)$	0.508	1510.6	0.721	0.038
$33/2(2)$	0.338	4206.6	0.3526	0.017
$35/2(1)$	0.185	30546.8	0.195	0.014
$35/2(2)$	0.595	61.6	30.3975	6.408
$37/2(1)$	0.248	7668.0	0.292	0.022
$37/2(2)$	0.384	3432.2	0.397	0.035
$37/2(3)$	0.414	620.5	1.507	0.415
$39/2(1)$	0.397	2286.9	0.519	0.035
$39/2(2)$	0.508	2217.3	0.493	0.021
$41/2(1)$	0.439	1830.7	0.609	0.037
$41/2(2)$	0.334	4608.3	0.329	0.012
$41/2(3)$	0.441	1534.6	0.708	0.048
$43/2(1)$	0.307	2131.8	0.552	0.080
$43/2(2)$	0.548	1252.9	0.879	0.043
$43/2(3)$	0.232	12488.3	0.238	0.011
$45/2(1)$	0.419	3505.8	0.357	0.012
$45/2(2)$	0.492	928.6	1.172	0.116
$45/2(3)$	0.299	3046.2	0.443	0.038
$47/2(1)$	0.408	2253.6	0.521	0.033

	a	b	c	d
$47/2(2)$	0.403	929.1	1.049	0.147
$47/2(3)$	0.475	2714.2	0.412	0.013
$49/2(1)$	0.439	1428.2	0.760	0.064
$49/2(2)$	0.094	798896	0.121	0.022
$49/2(3)$	0.480	2317.8	0.474	0.015
$49/2(4)$	0.269	1124.0	0.782	0.648
$51/2(1)$	0.509	1026.3	1.075	0.094
$51/2(2)$	0.442	3572.5	0.335	0.008
$51/2(3)$	0.369	1488.4	0.713	0.091
$53/2(1)$	0.339	5188.6	0.306	0.012
$53/2(2)$	0.274	4932.5	0.341	0.017
$53/2(3)$	0.570	1143.5	0.979	0.0478
$53/2(4)$	0.220	5509.5	0.350	0.067
$55/2(1)$	0.475	1518.7	0.718	0.043
$55/2(2)$	0.392	1998.2	0.574	0.043
$55/2(3)$	0.567	235.1	5.046	0.286
$55/2(4)$	0.338	5684.9	0.280	0.006
$57/2(1)$	0.488	2320.4	0.473	0.016
$57/2(2)$	0.378	735.0	1.237	0.422
$57/2(3)$	0.276	7174.6	0.281	0.011
$57/2(4)$	0.415	1401.3	0.771	0.083
$59/2(1)$	0.479	1351.5	0.796	0.047
$59/2(2)$	0.365	2562.3	0.481	0.031
$59/2(3)$	0.515	1281.0	0.841	0.038
$59/2(4)$	0.249	10585.4	0.241	0.006
$61/2(1)$	0.465	2445.7	0.465	0.020
$61/2(2)$	0.283	6542.5	0.286	0.008
$61/2(3)$	0.450	285.4	3.280	1.910
$61/2(4)$	0.395	2718.4	0.445	0.018
$61/2(5)$	0.167	4656.1	0.375	0.580

Best fit parameters (rounded) for the GG-distribution for all examples with 10^7 coefficients:

	a	b	c
13/2	0.677	1038	1.08
17/2	0.581	1406	0.71
19/2	0.538	2513	0.37
21/2	0.585	1622	0.60
23/2	0.599	1128	0.93
25/2(1)	0.525	1232	0.80
25/2(2)	0.467	1946	0.51
27/2	0.615	1756	0.54
29/2	0.529	708	1.37
31/2(1)	0.560	1817	0.53
31/2(2)	0.523	1232	0.80
33/2(1)	0.590	1211	0.85
33/2(2)	0.481	2243	0.44
35/2(1)	0.428	3402	0.32
35/2(2)	0.646	57	43.86
37/2(1)	0.437	2291	0.45
37/2(2)	0.540	1862	0.52
37/2(3)	0.529	418	2.55
39/2(1)	0.518	1470	0.68
39/2(2)	0.593	1744	0.56
41/2(1)	0.540	1319	0.76
41/2(2)	0.471	2532	0.40
41/2(3)	0.539	1123	0.91
43/2(1)	0.458	1025	0.93
43/2(2)	0.614	1061	1.02
43/2(3)	0.426	3277	0.33
45/2(1)	0.528	2396	0.40
45/2(2)	0.579	724	1.55
45/2(3)	0.448	1444	0.67
47/2(1)	0.524	1499	0.66

	a	b	c
47/2(2)	0.509	642	1.57
47/2(3)	0.562	2105	0.45
49/2(1)	0.542	1023	1.01
49/2(2)	0.390	3004	0.36
49/2(3)	0.561	1841	0.53
49/2(4)	0.453	400	2.14
51/2(1)	0.606	807	1.39
51/2(2)	0.537	2644	0.36
51/2(3)	0.519	889	1.10
53/2(1)	0.486	2725	0.37
53/2(2)	0.426	2126	0.48
53/2(3)	0.630	986	1.13
53/2(4)	0.417	1304	0.72
55/2(1)	0.568	1165	0.88
55/2(2)	0.516	1287	0.77
55/2(3)	0.670	216	7.00
55/2(4)	0.463	3322	0.31
57/2(1)	0.568	1844	0.53
57/2(2)	0.529	438	2.36
57/2(3)	0.438	2903	0.36
57/2(4)	0.530	943	1.08
59/2(1)	0.568	1060	0.98
59/2(2)	0.496	1539	0.64
59/2(3)	0.586	1068	0.99
59/2(4)	0.416	3728	0.30
61/2(1)	0.562	1818	0.54
61/2(2)	0.430	3023	0.35
61/2(3)	0.579	199	6.74
61/2(4)	0.502	1848	0.54
61/2(5)	0.406	472	1.61

Best fit parameters (rounded) for the Laplace distribution for all examples with 10^7 coefficients:

	b	c
$13/2$	1172	0.908
$17/2$	1503	0.683
$19/2$	2600	0.388
$21/2$	1740	0.592
$23/2$	1220	0.850
$25/2(1)$	1260	0.791
$25/2(2)$	1884	0.511
$27/2$	1923	0.542
$29/2$	725	1.317
$31/2(1)$	1914	0.533
$31/2(2)$	1258	0.789
$33/2(1)$	1302	0.793
$33/2(2)$	2202	0.444
$35/2(1)$	3147	0.295
$35/2(2)$	62	17.400
$37/2(1)$	2140	0.440
$37/2(2)$	1930	0.522
$37/2(3)$	429	2.341
$39/2(1)$	1495	0.669
$39/2(2)$	1881	0.550

	b	c
$41/2(1)$	1367	0.740
$41/2(2)$	2458	0.395
$41/2(3)$	1163	0.897
$43/2(1)$	981	0.980
$43/2(2)$	1160	0.899
$43/2(3)$	3021	0.309
$45/2(1)$	2461	0.409
$45/2(2)$	774	1.331
$45/2(3)$	1366	0.699
$47/2(1)$	1532	0.655
$47/2(2)$	647	1.539
$47/2(3)$	2219	0.460
$49/2(1)$	1063	0.953
$49/2(2)$	2650	0.333
$49/2(3)$	1940	0.527
$49/2(4)$	381	2.493
$51/2(1)$	871	1.196
$51/2(2)$	2734	0.370
$51/2(3)$	903	1.077
$53/2(1)$	2687	0.364

	b	c
$53/2(2)$	1958	0.479
$53/2(3)$	1090	0.962
$53/2(4)$	1188	0.778
$55/2(1)$	1233	0.830
$55/2(2)$	1306	0.760
$55/2(3)$	225	4.846
$55/2(4)$	3197	0.303
$57/2(1)$	1954	0.525
$57/2(2)$	448	2.206
$57/2(3)$	2716	0.349
$57/2(4)$	969	1.039
$59/2(1)$	1121	0.912
$59/2(2)$	1533	0.642
$59/2(3)$	1144	0.902
$59/2(4)$	3389	0.274
$61/2(1)$	1918	0.532
$61/2(2)$	2796	0.336
$61/2(3)$	211	4.830
$61/2(4)$	1852	0.536
$61/2(5)$	423	2.131

Best fit parameters (rounded) for the Cauchy distribution for all examples with 10^7 coefficients:

	a	b	c
$13/2$	143	0.14	0.49
$17/2$	154	0.12	0.61
$19/2$	152	0.07	0.82
$21/2$	150	0.10	0.64
$23/2$	156	0.15	0.54
$25/2(1)$	163	0.15	0.59
$25/2(2)$	169	0.10	0.78
$27/2$	148	0.09	0.66
$29/2$	162	0.25	0.46
$31/2(1)$	156	0.09	0.68
$31/2(2)$	163	0.15	0.59
$33/2(1)$	156	0.14	0.56
$33/2(2)$	163	0.08	0.81
$35/2(1)$	212	0.07	1.17
$35/2(2)$	422	7.75	0.19
$37/2(1)$	200	0.10	0.93
$37/2(2)$	156	0.09	0.71
$37/2(3)$	5310	13.99	1.96
$39/2(1)$	175	0.13	0.67
$39/2(2)$	145	0.09	0.66
$41/2(1)$	161	0.13	0.61
$41/2(2)$	196	0.09	0.95
$41/2(3)$	147	0.14	0.53
$43/2(1)$	192	0.22	0.60
$43/2(2)$	145	0.14	0.51
$43/2(3)$	243	0.09	1.23
$45/2(1)$	287	0.13	1.10
$45/2(2)$	153	0.24	0.44
$45/2(3)$	206	0.17	0.74
$47/2(1)$	164	0.12	0.66

	a	b	c
$47/2(2)$	165	0.29	0.43
$47/2(3)$	151	0.08	0.74
$49/2(1)$	163	0.17	0.54
$49/2(2)$	211	0.09	1.14
$49/2(3)$	152	0.09	0.70
$49/2(4)$	1070	3.14	-0.90
$51/2(1)$	13	0.02	0.13
$51/2(2)$	12	0.00	0.23
$51/2(3)$	952	1.20	1.23
$53/2(1)$	169	0.07	0.91
$53/2(2)$	219	0.12	0.94
$53/2(3)$	144	0.15	0.49
$53/2(4)$	216	0.20	0.73
$55/2(1)$	580	0.54	1.06
$55/2(2)$	1857	1.61	2.04
$55/2(3)$	1459	7.01	0.82
$55/2(4)$	3681	1.29	4.70
$57/2(1)$	149	0.59	0.69
$57/2(2)$	65049	164.83	7.08
$57/2(3)$	208	0.09	1.05
$57/2(4)$	163	0.19	0.52
$59/2(1)$	1383	1.41	1.57
$59/2(2)$	2525	1.86	2.62
$59/2(3)$	385	28.90	24.97
$59/2(4)$	88133	0.30	2.55
$61/2(1)$	151	0.09	0.69
$61/2(2)$	242	0.10	1.17
$61/2(3)$	3403	18.40	-1.07
$61/2(4)$	169	0.10	0.74
$61/2(5)$	1346	3.52	-1.12

RMS values (rounded) for all examples with 10^7 coefficients:

	GG	GGG	Laplace	Cauchy
13/2	19	18	39	33
17/2	21	18	29	28
19/2	27	19	30	34
21/2	21	18	31	33
23/2	19	18	28	26
25/2(1)	21	18	22	22
25/2(2)	33	19	35	25
27/2	19	17	37	41
29/2	18	16	18	20
31/2(1)	22	19	28	33
31/2(2)	21	17	22	21
33/2(1)	19	18	28	27
33/2(2)	27	18	28	30
35/2(1)	48	20	57	40
35/2(2)	5	5	5	5
37/2(1)	26	14	30	25
37/2(2)	20	15	22	25
37/2(3)	9	9	9	10
39/2(1)	16	12	16	20
39/2(2)	17	15	28	34
41/2(1)	14	12	16	20
41/2(2)	22	14	24	27
41/2(3)	13	11	14	18
43/2(1)	13	9	13	14
43/2(2)	14	13	22	25
43/2(3)	41	19	50	41
45/2(1)	21	16	23	31
45/2(2)	11	10	13	16
45/2(3)	15	11	17	17
47/2(1)	16	13	16	21

	GG	GGG	Laplace	Cauchy
47/2(2)	10	9	10	12
47/2(3)	19	16	25	34
49/2(1)	13	11	14	17
49/2(2)	59	24	72	49
49/2(3)	17	15	23	31
49/2(4)	10	10	10	10
51/2(1)	18	17	24	20
51/2(2)	23	18	27	38
51/2(3)	20	17	20	21
53/2(1)	25	16	25	28
53/2(2)	20	12	26	25
53/2(3)	13	13	23	26
53/2(4)	17	10	20	17
55/2(1)	19	17	24	23
55/2(2)	21	18	22	23
55/2(3)	12	12	13	13
55/2(4)	30	18	34	39
57/2(1)	18	16	24	32
57/2(2)	12	12	12	12
57/2(3)	27	15	33	31
57/2(4)	13	11	14	16
59/2(1)	19	18	24	22
59/2(2)	23	17	23	25
59/2(3)	19	18	26	24
59/2(4)	40	19	54	48
61/2(1)	17	15	23	29
61/2(2)	34	18	43	42
61/2(3)	9	9	9	9
61/2(4)	17	13	17	23
61/2(5)	11	9	13	12