Received: 16 October 2019
DOI: 10.1002/mma.6430

SPECIAL ISSUE PAPER

WILEY

A natural way to construct an almost complex B-metric structure

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Communicated by: M. Tosun

In this work, we construct an almost complex B-metric structure on a manifold $M \times \mathbb{R}$, where M is equipped with an almost contact B-metric manifold. Moreover, we give some relations and conditions for existence of structures between the classes of almost contact B-metric structures and the obtained almost complex B-metric structures.

KEYWORDS

almost complex B-metric structure, almost contact B-metric structure, Norden metric

MSC CLASSIFICATION

53C15; 53C55; 53D15

1 | INTRODUCTION

The differential geometry of the manifolds equipped with almost contact structures and almost complex metric structures are well studied in Blair.[1] In the literature, many researchers have studied the classifications of these structure with given certain metrics. For instance in other studies,[2-4] almost contact metric structures and their classifications are studied. Also, many authors studied these classes and the relations with other structures. The existence of almost contact metric structures of certain classes that were obtained from G_2-structures are investigated in Özdemir et al.[5] Almost contact metric structures on nilpotent Lie groups of dimension five and the relations between the classification of these structures are studied in Özdemir et al.[6] Almost hermitian structures were classified in Gray and Hervella[7]; almost contact structures with B-metric are classified in Ganchev et al,[8] and in a similar vein, almost complex B-metric structures are classified in Ganchev and Borisov.[9] In this work, we will construct an almost complex B-metric structure from a given almost contact B-metric structure and evaluate some relations between the classes of these structure. In section 2, the definition of almost contact B-metric structure on a manifold and the defining relations of the classes are given. Section 3 is dedicated for almost complex B-metric structures and the classification of them. In the next section, an almost complex B-metric structure will be constructed from a given almost contact B-metric structure. Also, some relations between these two structures are expressed. The last section is devoted for the results of the paper.

2 | PRELIMINARIES

In this section, we will give the definition of almost contact B-metric structure on a manifold and consider the classification.

Definition 1. Let M be a $2n+1$- dimensional smooth manifold and ξ, φ, and η be a nowhere-zero vector field, a $(1,1)$ tensor field and a 1-form respectively on M satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{1}$$

then (φ, ξ, η) is called an almost contact structure on M and M is said to be an almost contact manifold, being denoted by (M, φ, ξ, η).

If g is a semi-Riemannian metric (with the signature $(n+1, n)$) on (M, φ, ξ, η) satisfying

$$g(\varphi(X), \varphi(Y)) = -g(X, Y) + \eta(X)\eta(Y) \tag{2}$$

for any $X, Y \in \mathfrak{X}(M)$, then M is said to be an almost contact B-metric manifold with the almost contact B-metric structure (φ, ξ, η, g). The metric g is said to be a compatible metric with the structure. For a given manifold $(M, \varphi, \xi, \eta, g)$, the following hold:

$$\varphi^2 = -id + \eta \otimes \xi \quad ; \quad \eta(\xi) = 1 \tag{3}$$

It follows immediately,

$$\eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(X) = g(X, \xi), \quad g(\varphi X, Y) = g(X, \varphi Y), \tag{4}$$

where $X \in \mathfrak{X}(M)$. Furthermore, the tensor field φ has rank $2n$. [8]

Definition 2. The tensor field G of type $(0, 3)$ on the manifold M, called the fundamental tensor of the almost contact B-metric structure, is given with

$$G(X, Y, Z) = g((D_X\varphi)Y, Z), \tag{5}$$

where D is the Levi-Civita connection of g and $X, Y, Z \in \mathfrak{X}(M)$.

Proposition 1. *By the definition of covariant derivative, we have*

$$(D_X\eta)(Y) = g(D_x\xi, Y) = G(X, \varphi Y, \xi). \tag{6}$$

Also from equations (1) and (4), it follows immediately

$$\begin{aligned} G(X, Y, Z) &= G(X, Z, Y) \\ G(X, \varphi Y, \varphi Z) &= G(X, Y, Z) - \eta(Y)G(X, \xi, Z) - \eta(Z)G(X, Y, \xi) \\ G(X, \xi, \xi) &= 0, \end{aligned} \tag{7}$$

where $X, Y, Z \in \mathfrak{X}(M)$. The following 1-forms, called "Lee forms," are associated with G:

$$\theta(x) = g^{ij}G(e_i, e_j, x); \quad \theta^*(x) = g^{ij}G(e_i, \varphi e_j, x); \quad \omega(x) = G(\xi, \xi, x), \tag{8}$$

where $x \in T_pM$, $\{e_i, \xi\}(i = 1, 2, \ldots, 2n)$ is a basis of T_pM, and (g^{ij}) is the inverse matrix of (g_{ij}).

In Ganchev et al,[8] almost contact B-metric structures classified with regard to the fundamental tensor G is studied, where the defining relations of the 11 basic classes $\mathcal{F}_i, (i = 1, \ldots, 11)$ are given. These defining relations are as follows:

$\mathcal{F}_1$	$G(X, Y, Z) = \frac{1}{2n}[g(X, \varphi Y)\theta(\varphi Z) + g(X, \varphi Z)\theta(\varphi Y) + g(\varphi X, \varphi Y)\theta(\varphi^2 Z) + g(\varphi X, \varphi Z)\theta(\varphi^2 Y)]$
$\mathcal{F}_2$	$G(\xi, Y, Z) = G(X, \xi, Z) = 0, G(X, Y, \varphi Z) + G(Y, Z, \varphi X) + G(Z, X, \varphi Y) = 0, \theta = 0$
$\mathcal{F}_3$	$G(\xi, Y, Z) = G(X, \xi, Z) = 0, G(X, Y, Z) + G(Y, Z, X) + G(Z, X, Y) = 0$
$\mathcal{F}_4$	$G(X, Y, Z) = -\frac{\theta(\xi)}{2n}[g(\varphi X, \varphi Y)\eta(Z) + g(\varphi X, \varphi Z)\eta(Y)]$
$\mathcal{F}_5$	$G(X, Y, Z) = -\frac{\theta^*(\xi)}{2n}[g(X, \varphi Y)\eta(Z) + g(X, \varphi Z)\eta(Y)]$
$\mathcal{F}_6$	$G(X, Y, Z) = -G(\varphi X, \varphi Y, Z) - G(\varphi X, Y, \varphi Z) = -G(Y, Z, X) + G(Z, X, Y) - 2G((\varphi X, \varphi Y, Z),\ \theta(\xi) = \theta^*(\xi) = 0$
$\mathcal{F}_7$	$G(X, Y, Z) = -G(\varphi X, \varphi Y, Z) - G(\varphi X, Y, \varphi Z) = -G(Y, Z, X) - G(Z, X, Y)$
$\mathcal{F}_8$	$G(X, Y, Z) = G(\varphi X, \varphi Y, Z) + G(\varphi X, Y, \varphi Z) = -G(Y, Z, X) + G(Z, X, Y) + 2G(\varphi X, \varphi Y, Z)$
$\mathcal{F}_9$	$G(X, Y, Z) = G(\varphi X, \varphi Y, Z) + G(\varphi X, Y, \varphi Z) = -G(Y, Z, X) - G(Z, X, Y)$
$\mathcal{F}_{10}$	$G(X, Y, Z) = \eta(X)G(\xi, \varphi Y, \varphi Z)$
$\mathcal{F}_{11}$	$G(X, Y, Z) = \eta(X)[\eta(Y)\omega(Z) + \eta(Z)\omega(Y)]$

The class $\mathcal{F}_0$ is given with $G = 0$.

3 | ALMOST COMPLEX B-METRIC STRUCTURES

In this section, we will give the definition of almost complex B-metric structures and mention the classification of them.

Definition 3. Let M^{2n} be a smooth manifold endowed with an almost complex structure J and a metric h of signature (n, n) satisfying

$$J^2(X) = -X, \quad h(JX, JY) = -h(X, Y), \tag{9}$$

for $X, Y \in \mathfrak{X}(M)$. Then, (M, J, h) is said be an almost complex B-metric manifold.[8]

Here, It can be seen that the metric h satisfies the following:

$$h(JX, Y) = h(X, JY). \tag{10}$$

Definition 4. The fundamental tensor W of (M, J, h) is given with

$$W(X, Y, Z) = h((\tilde{D}_X J)Y, Z), \tag{11}$$

where $X, Y, Z \in \mathfrak{X}(M)$ and $\tilde{D}$ is the Levi-Civita connection of the metric h.

The fundamental tensor W satisfies the following properties:

$$\begin{aligned} W(X, Y, Z) &= W(X, Z, Y) \\ W(X, Y, Z) &= W(X, JY, JZ). \end{aligned} \tag{12}$$

Definition 5. Let $p \in M$ be a point and $\{e_1, \ldots, e_n\}$ be a basis of T_pM. Then, the Lee form ψ associated with the fundamental tensor W is given with

$$\psi(x) = h^{ij} W(e_i, e_j, x), \tag{13}$$

where $x \in T_pM$ and h^{ij} is the components of the inverse matrix of h.

A classification of almost complex B-metric structures with regard to the covariant derivative of J is given in Ganchev and Borisov.[9] According to these classifications, there are three basic classes $\mathcal{W}_i, (i = 1, 2, 3)$ and thus 2^3 invariant subspaces. For any $X, Y, Z \in \mathfrak{X}(M)$, the defining relations of these subspaces are as follows:

$\mathcal{W}_0$ (Kaehlerian B-metric manifolds):

$$W(X, Y, Z) = 0.$$

$\mathcal{W}_1$ (conformally Kaehlerian B-metric manifolds):

$$\begin{aligned} W(X, Y, Z) = &\frac{1}{2n}[h(X, Y)\psi(Z) + h(X, Z)\psi(Y) \\ &+ h(X, JY)\psi(JZ) + h(X, JZ)\psi(JY)]. \end{aligned}$$

$\mathcal{W}_2$ (special complex B-metric manifolds):

$$W(X, Y, JZ) + W(Y, Z, JX) + W(Z, X, JY) = 0, \quad \psi = 0.$$

$\mathcal{W}_3$ (quasi-Kaehlerian B-metric manifolds):

$$W(X, Y, Z) + W(Y, Z, X) + W(Z, X, Y) = 0.$$

$\mathcal{W}_1 \oplus \mathcal{W}_2$ (complex B-metric manifolds):

$$W(X, Y, JZ) + W(Y, Z, JX) + W(Z, X, JY) = 0.$$

$\mathcal{W}_2 \oplus \mathcal{W}_3$ (semi-Kaehlerian B-metric manifolds):

$$\psi = 0.$$

$\mathcal{W}_1 \oplus \mathcal{W}_3$:

$$W(X,Y,Z) + W(Y,Z,X) + W(Z,X,Y) = \frac{1}{n}[h(X,Y)\psi(Z) + h(Z,X)\psi(Y) + h(Y,Z)\psi(X) + h(X,JY)\psi(JZ) + h(Y,JZ)\psi(JX) + h(Z,JX)\psi(JY)].$$

$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$: (the whole class).

4 | AN ALMOST COMPLEX B-METRIC STRUCTURE DERIVED FROM AN ALMOST CONTACT B-METRIC STRUCTURE

In this section, an almost complex B-metric structure on a manifold $M \times \mathbb{R}$ will be constructed by the given almost contact B-metric structure (φ, ξ, η, g) on M. Also, we will express some relations between these two structures. For a given almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$ of dimension $2n+1$, a smooth vector field on $M \times \mathbb{R}$ will be denoted by $(X, a\frac{d}{dt})$, where $X \in \mathfrak{X}(M)$, a is a differentiable real valued function on $M \times \mathbb{R}$, and t is the coordinate of $\mathbb{R}$. Let us define $J : M \times \mathbb{R} \to M \times \mathbb{R}$ as follows:

$$J\left(X, a\frac{d}{dt}\right) := \left(\varphi X - a\xi, \eta(X)\frac{d}{dt}\right). \tag{14}$$

Obviously, J is well-defined. Now, we will show that J is also an almost complex structure on $M \times \mathbb{R}$.

Proposition 2. *The function J given with the equation (14) is an almost complex structure on $M \times \mathbb{R}$.*

Proof. We need to show that J satisfies the equation $J^2 = -Id$ on $M \times \mathbb{R}$. Let $(X, a\frac{d}{dt}) \in \mathfrak{X}(M \times \mathbb{R})$. Then,

$$\begin{aligned}
J^2\left(X, a\frac{d}{dt}\right) &= J\left(J\left(X, a\frac{d}{dt}\right)\right) = J\left(\varphi X - a\xi, \eta(X)\frac{d}{dt}\right) \\
&= \left(\varphi(\varphi X - a\xi) - \eta(X)\xi, \eta(\varphi X - a\xi)\frac{d}{dt}\right) \\
&= \left(\varphi^2 X - a\varphi\xi - \eta(X)\xi, (\eta(\varphi X) - a\eta(\xi))\frac{d}{dt}\right) \\
&= \left(-X + \eta(X)\xi - \eta(X)\xi, -a\frac{d}{dt}\right) \\
&= -\left(X, a\frac{d}{dt}\right).
\end{aligned}$$

□

Now, we shall define a metric, say it h, on $M \times \mathbb{R}$. Let $\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right) \in \mathfrak{X}(M \times \mathbb{R})$. We define the metric h as

$$h\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) := g(X,Y) - ab. \tag{15}$$

Obviously, the metric h given with equation(15) is of signature $(n+1, n+1)$.

Proposition 3. *Let $(M, \varphi, \xi, \eta, g)$ be an almost contact B-metric manifold of dimension $2n+1$. Then, $(M \times \mathbb{R}, J, h)$ is an almost complex B-metric manifold, where J and h are given with the equations (14) and (15), respectively.*

Since we have proposition (2), only need to show that the second part of equality (9) holds.

Proof.

$$
\begin{aligned}
h\left(J\left(X,a\frac{d}{dt}\right),J\left(Y,b\frac{d}{dt}\right)\right) =& h\left(\left(\varphi X-a\xi,\eta(X)\frac{d}{dt}\right),\left(\varphi Y-b\xi,\eta(Y)\frac{d}{dt}\right)\right)\\
=& g(\varphi X-a\xi,\varphi Y-b\xi)-\eta(X)\eta(Y)\\
=& g(\varphi X,\varphi Y)-bg(\varphi X,\xi)-ag(\varphi Y,\xi)\\
&+abg(\xi,\xi)-\eta(X)\eta(Y)\\
=& -g(X,Y)+\eta(X)\eta(Y)+ab-\eta(X)\eta(Y)\\
=& -(g(X,Y)-ab)=-h\left(\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right)\right).
\end{aligned}
$$

Therefore, $(M\times\mathbb{R},J,h)$ is an almost complex B-metric manifold. □

Now, we give the relevances between covariant derivatives of the almost contact B-metric structure and the derived almost complex B-metric structure. If $(p_0,t_0)\in M\times\mathbb{R}$, we consider injections $i:M\to M\times\mathbb{R}$ and $j:\mathbb{R}\to M\times\mathbb{R}$ defined by $i(p)=(p,t_0)$ and $j(t)=(p_0,t)$. So if $X\in\chi(M)$ and $a\in C^\infty(M\times\mathbb{R})$, $X(a\circ i)$ and $\frac{d}{dt}(a\circ j)$ will simply denoted by $X(a)$ and $\frac{da}{dt}$.

Let (M,φ,ξ,η,g) be an almost contact B-metric manifold and $(M\times\mathbb{R},J,h)$ be the derived almost complex B-metric manifold and let D and $\tilde{D}$ denote the Levi-Civita connections of the metrics g and h, respectively. Then,

$$
\tilde{D}_{\left(X,a\frac{d}{dt}\right)}\left(Y,b\frac{d}{dt}\right)=\left(D_XY,\left(X[b]+a\frac{db}{dt}\right)\frac{d}{dt}\right), \tag{16}
$$

whence

$$
\left(\tilde{D}_{(X,a\frac{d}{dt}}J\right)\left(Y,b\frac{d}{dt}\right)=\left((D_X\varphi)(Y)-bD_x\xi,(D_X\eta)(Y)\frac{d}{dt}\right). \tag{17}
$$

Now, we are able to give the relation between the fundamental tensors G and W of the structures on M and $M\times\mathbb{R}$ respectively.

Proposition 4.

$$
W\left(\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right),\left(Z,c\frac{d}{dt}\right)\right)=G(X,Y,Z)-bG(X,\xi,\varphi Z)-cG(X,\varphi Y,\xi). \tag{18}
$$

Proof. From the definitions of the fundamental tensors and equations (16) and (17), we have

$$
\begin{aligned}
W\left(\left(X,a\frac{d}{dt}\right),\left(Y,b\frac{d}{dt}\right),\left(Z,c\frac{d}{dt}\right)\right)&=h\left(\left(\tilde{D}_{\left(X,a\frac{d}{dt}J\right)}\left(Y,b\frac{d}{dt}\right),\left(Z,c\frac{d}{dt}\right)\right)\right)\\
&=h\left(\left((D_X\varphi)(Y)-bD_x\xi,(D_X\eta)(Y)\frac{d}{dt}\right),\left(Z,c\frac{d}{dt}\right)\right)\\
&=g((D_X\varphi)(Y)-bD_x\xi,Z)-c(D_X\eta)(Y))\\
&=g((D_X\varphi)(Y),Z)-bg(D_x\xi,Z))-c(D_X\eta)(Y))\\
&=G(X,Y,Z)-bG(X,\varphi Z,\xi)-cG(X,\varphi Y,\xi).
\end{aligned}
$$

□

5 | RESULTS

Let (M,φ,ξ,η,g) be an almost contact B-metric manifold and $(M\times\mathbb{R},J,h)$ be the derived almost complex B-metric manifold and let G and W be their fundamental tensors, respectively. Then we have the following:

Theorem 1. *If (M,φ,ξ,η,g) is of class $\mathcal{F}_0$, then the induced manifold $(M\times\mathbb{R},J,h)$ is class of $\mathcal{W}_0$.*

Proof. It is a natural result of the proposition (4). □

Theorem 2. *If (M,φ,ξ,η,g) is of class $\mathcal{F}_2$, then the induced manifold $(M\times\mathbb{R},J,h)$ is a complex B-metric manifold $(\mathcal{W}_1\oplus\mathcal{W}_2)$.*

Proof. Let $(M, \varphi, \xi, \eta, g)$ is of class $\mathcal{F}_2$. Then for any smooth vector fields $\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)$ on $M \times \mathbb{R}$, we have

$$\begin{aligned}
&W\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), J\left(\left(Z, c\frac{d}{dt}\right)\right)\right) + W\left(\left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right), J\left(\left(X, a\frac{d}{dt}\right)\right)\right)\\
&+ W\left(\left(Z, c\frac{d}{dt}\right), \left(X, a\frac{d}{dt}\right), J\left(\left(Y, b\frac{d}{dt}\right)\right)\right)\\
&= h\left(\left(\tilde{D}_{\left(X, a\frac{d}{dt}\right)}J\right)\left(Y, b\frac{d}{dt}\right), J\left(Z, c\frac{d}{dt}\right)\right) + h\left(\left(\tilde{D}_{\left(Y, b\frac{d}{dt}\right)}J\right)\left(Z, c\frac{d}{dt}\right), J\left(X, a\frac{d}{dt}\right)\right)\\
&+ h\left(\left(\tilde{D}_{\left(Z, c\frac{d}{dt}\right)}J\right)\left(X, a\frac{d}{dt}\right), J\left(Y, b\frac{d}{dt}\right)\right)\\
&= g((D_X\varphi)(Y) - bD_X\xi, \varphi Z - c\xi) - \eta(Z)(D_X\eta)(Y)\\
&+ g((D_Y\varphi)(Z) - cD_Y\xi, \varphi X - a\xi) - \eta(X)(D_Y\eta)(Z)\\
&+ g((D_Z\varphi)(X) - aD_Z\xi, \varphi Y - b\xi) - \eta(Y)(D_Z\eta)(X)\\
&= G(X, Y, \varphi Z) + G(Y, Z, \varphi X) + G(Z, X, \varphi Y) - cG(X, Y, \xi) + cG(Y, X, \xi)\\
&- aG(Y, Z, \xi) + aG(Z, Y, \xi) - bG(Z, X, \xi) + bG(X, Z, \xi) - \eta(Z)G(X, \varphi Y, \xi)\\
&- \eta(X)G(Y, \varphi Z, \xi) - \eta(Y)G(Z, \varphi X, \xi)\\
&= 0.
\end{aligned}$$

Thus, the derived almost complex B-metric manifold is of the class $\mathcal{W}_1 \oplus \mathcal{W}_2$. □

Theorem 3. *If $(M, \varphi, \xi, \eta, g)$ is of class $\mathcal{F}_3$, then the induced manifold $(M \times \mathbb{R}, J, h)$ is a Quasi-Kaehlerian B-metric manifold ($\mathcal{W}_3$).*

Proof. From the proposition (4), the following holds:

$$\begin{aligned}
&W\left(\left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right)\right) + W\left(\left(Y, b\frac{d}{dt}\right), \left(Z, c\frac{d}{dt}\right), \left(X, a\frac{d}{dt}\right)\right)\\
&+ W\left(\left(Z, c\frac{d}{dt}\right), \left(X, a\frac{d}{dt}\right), \left(Y, b\frac{d}{dt}\right)\right) = G(X, Y, Z) + G(Y, Z, X)\\
&+ G(Z, X, Y) - bG(X, \varphi Z, \xi) - bG(Z, \varphi X, \xi) - cG(X, \varphi Y, \xi)\\
&- cG(Y, \varphi X, \xi) - aG(Y, \varphi Z, \xi) - aG(Z, \varphi Y, \xi)\\
&= 0,
\end{aligned}$$

since $(M, \varphi, \xi, \eta, g)$ satisfies the defining relation of the class $\mathcal{F}_3$. □

Theorem 4. *Let $(M, \varphi, \xi, \eta, g)$ be an almost contact B-metric manifold and $(M \times \mathbb{R}, J, h)$ be the derived almost complex B-metric manifold. Then, $(M \times \mathbb{R}, J, h)$ is Semi-Kaehlerian B-metric manifold if and only if $\theta = 0$ and $\delta\eta = 0$.*

Before giving the proof, we want to mention that in a similar vein with.[10] It is known that there exists an orthonormal basis (called φ-basis)

$$\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi\}$$

with

$$g(e_i, e_j) = -g(\varphi e_i, \varphi e_j) = \delta_{ij}, \quad (i, j = 1, \dots, n),$$

for an almost contact B-metric manifold M. Then the basis (J-basis) on the manifold $M \times \mathbb{R}$ occurs as $\{E_1 = (e_1, 0), \dots, E_n = (e_n, 0), JE_1 = (\varphi e_1, 0), \dots, JE_n = (\varphi e_n, 0), (0, \frac{d}{dt})\}$ with

$$h(E_i, E_j) = -h(JEi, JE_j) = -h\left(\left(0, \frac{d}{dt}\right), \left(0, \frac{d}{dt}\right)\right) = \delta_{ij}.$$

We will consider these bases to clear the way to choose the signatures of (g^{ij}) and (h^{ij}).

Proof. For any vector field $\left(X, a\frac{d}{dt}\right)$ on $M \times \mathbb{R}$, we have

$$\begin{aligned}
\psi\left(\left(X, a\frac{d}{dt}\right)\right) &= \sum_{i=1}^{n}\left(W\left((e_i,0),(e_i,0),\left(X,a\frac{d}{dt}\right)\right)\right. \\
&\quad - \sum_{i=1}^{n}\left(W\left((\varphi e_i,0),(\varphi e_i,0),\left(X,a\frac{d}{dt}\right)\right) + W\left(\left(0,\frac{d}{dt}\right),\left(0,\frac{d}{dt}\right),\left(X,a\frac{d}{dt}\right)\right)\right) \\
&= \sum_{i=1}^{n}\left(G(e_i,e_i,X) - a(D_{e_i}\eta)(e_i)\right) - \sum_{i=1}^{n}(G(\varphi e_i,\varphi e_i,X) - a(D_{\varphi e_i}\eta)(\varphi e_i)) \\
&= \theta(X) - a\delta\eta,
\end{aligned}$$

which shows that $M \times \mathbb{R}, J, h)$ is Semi-Kaehlerian B-metric manifold (i.e. $\psi = 0$) if and only if θ and $\delta\eta$ are both identically zero. □

6 | CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

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How to cite this article: Solgun M, Karababa Y. A natural way to construct an almost complex B-metric structure. *Math Meth Appl Sci*. 2021;44:7607–7613. https://doi.org/10.1002/mma.6430