

# A new approach for soft topology and soft function via soft element

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In this article, we give some new properties of elementary operations on soft sets and then we introduce a new soft topology by using elementary operations over a universal set with a set of parameters called elementary soft topology. Also, we define a topology, members of which are collections of the soft elements and give the relation between this topology and elementary soft topology. We show that this new soft topology is different from those previously defined soft topologies. We prove some of the properties of the topological concepts we investigate in this topology. Finally, we describe soft function and soft continuity and give an application of the soft function as soft set approach to the rotation in  $\mathbb{E}^3$ .

## KEYWORDS

soft continuous function, soft element, soft rotation, soft set, soft topology

## MSC CLASSIFICATION

54A05; 54A10

## 1 | INTRODUCTION

In 1999, Molodtsov<sup>1</sup> introduced the concept of soft sets, and after that, several applications settling many suitable issues in various areas have used the soft sets. Maji et al<sup>2</sup> and Ali et al<sup>3</sup> looked into soft set theory in detail. The topological structures on the soft sets began to be studied after Shabir and Naz.<sup>4</sup> Many authors<sup>5-21</sup> studied on the soft topology and its properties, which are based on the notion of soft point. Later, Das and Samanta<sup>22,23</sup> brought forward the concepts of soft elements, soft real sets, soft real numbers and the elementary union ( $\epsilon$ -union), intersection ( $\epsilon$ -intersection), and complement ( $\epsilon$ -complement) operations over the soft sets. Samanta et al and several authors examined some mathematical structures by using the notion of soft element.<sup>23-29</sup> Also, the soft set theory has been associated and applied to data science and various mathematical structures.<sup>30-37</sup> In addition, many authors have integrated the soft sets, fuzzy sets, rough sets, and neutrosophic sets.<sup>38-42</sup>

Our main goal in this article is to introduce elementary soft topological space ( $\epsilon$ -soft topological space) by using the  $\epsilon$ -union and  $\epsilon$ -intersection operations that is different from Shabir and Naz<sup>4</sup> and Hazra et al.<sup>7</sup> Firstly, we give some new properties of the  $\epsilon$ -operations over the soft sets and define a topology, members of which are collections of the soft elements. We also investigate the relations between this topology, other soft topologies, and  $\epsilon$ -soft topology. We give some topological concepts like open set, closed set, neighbourhood, interior elements, closure element, basis, and local basis and then examine basic properties of them in  $\epsilon$ -soft topological space. Lastly, we describe soft function over the soft sets and soft continuous function over the  $\epsilon$ -soft topological spaces based on the soft elements and present an application of the soft functions as soft set approach to the rotation which may gain a different point of view to motion in  $\mathbb{E}^3$ .

## 2 | PRELIMINARIES

Let  $X$  be a universal set,  $\mathcal{P}(X)$  be the power set of  $X$ , and  $A$  be a set of parameters. A soft set over  $X$ , denoted by  $(F, A)$ , is a pair, where  $F : A \rightarrow \mathcal{P}(X)$ ,<sup>1</sup>

$$(F, A) = \{(\lambda, F(\lambda)) : \lambda \in A, F(\lambda) \subset X\}.$$

Throughout this paper, we use only  $F$  instead of  $(F, A)$ , for simplicity.

Let  $F$  and  $G$  be two soft sets over a universe  $X$  with parameter set  $A$ .  $\Phi$  and  $\tilde{X}$  are called null soft set and absolute soft set, if for all  $\lambda \in A$ ,  $\Phi(\lambda) = \emptyset$ , and  $\tilde{X}(\lambda) = X$ , respectively.  $F$  is called a soft subset of  $G$  and a soft upper set of  $G$  if for all  $\lambda \in A$ ,  $F(\lambda) \subseteq G(\lambda)$ , and  $F(\lambda) \supseteq G(\lambda)$ , denoted by  $F \subseteq G$  and  $F \supseteq G$ , respectively.  $F$  and  $G$  are called equal if  $F$  is a soft subset of  $G$  and  $G$  is a soft subset of  $F$ . The intersection  $H = F \tilde{\cap} G$ , the union  $H = F \tilde{\cup} G$ , and the difference  $H = F \tilde{-} G$  of  $F$  and  $G$  over  $X$  are defined as  $H(\lambda) = F(\lambda) \cap G(\lambda)$ ,  $H(\lambda) = F(\lambda) \cup G(\lambda)$ , and  $H(\lambda) = F(\lambda) - G(\lambda)$  for all  $\lambda \in A$ , respectively. The complement of  $F$  is denoted as  $F^C$ , where  $F^C : A \rightarrow \mathcal{P}(X)$  given by  $F^C(\lambda) = X \setminus F(\lambda)$  for all  $\lambda \in A$ . Then,  $\tilde{X}^C = \Phi$  and  $\Phi^C = \tilde{X}$ . Also,  $(F \tilde{\cup} G)^C = F^C \tilde{\cap} G^C$  and  $(F \tilde{\cap} G)^C = F^C \tilde{\cup} G^C$ . For detailed information, see Majji et al<sup>2</sup> and Ali et al.<sup>3</sup>

**Definition 1** (Shabir and Naz<sup>4</sup>). Let  $\tau$  be a collection of soft sets over  $X$  with parameter set  $A$ . Then,  $\tau$  is called a soft topology on  $X$ , and  $(X, \tau, A)$  is called a soft topological space over  $X$  if it satisfies the followings:

1.  $\tilde{X}, \Phi \in \tau$ .
2. If  $F, G \in \tau$ , then  $F \tilde{\cap} G \in \tau$ .
3. If  $F_i \in \tau, \forall i \in I$ , then  $\bigcup_{i \in I} F_i \in \tau$ .

Let  $(X, \tau, A)$  be a soft topological space. Then, the family  $\tau_\lambda = \{F(\lambda) : F \in \tau\}, \forall \lambda \in A$ , forms a topology on  $X$ .<sup>4,8</sup>

**Definition 2** (Hazra et al<sup>7</sup>). Let  $\tau$  be the collection of soft sets over  $X$  with parameter set  $A$ . Define  $\tau_\lambda = \{F(\lambda) : F \in \tau\}, \forall \lambda \in A$ . Then,  $\tau$  is said to be a topology of soft subsets over  $\tilde{X}$ , if  $\tau_\lambda$  is a crisp topology on  $X, \forall \lambda \in A$ .

**Definition 3** (Das and Samanta<sup>22,23</sup>). A function  $e : A \rightarrow X$  is called a soft element of  $X$ , which is expressed by  $e \in \tilde{X}$ . If  $e(\lambda) \in F(\lambda)$  for any  $\lambda \in A$ , then the soft element  $e$  is called belonging to the soft set  $F \tilde{\subset} \tilde{X}$ . So, for a soft set  $F \tilde{\subset} \tilde{X}$ , according to the parameters set  $A$ , we can describe  $F(\lambda) = \{e(\lambda) : e \in F\}$ . If for all  $\lambda \in A, F(\lambda)$  is unit set, then  $F$  is called a unit soft set. Thus, every unit, soft set can be determined with a soft element.

Throughout this paper,  $S(\tilde{X})$  represents a collection of those soft sets  $F$  over  $X$  for which  $F(\lambda) \neq \emptyset, \forall \lambda \in A$  and  $\Phi$ .  $SE(F)$  indicates the collection of all soft elements of  $F \in S(\tilde{X})$ . Also,  $\tilde{x}$  shows soft element of a soft set; in addition to this,  $\bar{x}$  shows a specific soft element such that  $\bar{x}(\lambda) = x, \forall \lambda \in A$ .

**Proposition 1** (Das and Samanta<sup>23</sup>). Let  $F, G \in S(\tilde{X})$ .

1. We have  $F \tilde{\subseteq} G$  iff every soft element of  $F$  is also a soft elements of  $G$ .
2. Any collection of the soft elements of a soft set produces a soft subset of this soft set. The soft set produced by a collection  $\mathfrak{B}$  of the soft elements is expressed by  $SS(\mathfrak{B})$ .
3.  $SS(SE(F)) = F$ . But  $SE(SS(\mathfrak{B})) \supset \mathfrak{B}$  for a collection  $\mathfrak{B}$ , ie,  $SE(SS(\mathfrak{B})) \neq \mathfrak{B}$  generally.

*Remark 1.* (Das and Samanta<sup>23</sup>).  $\tilde{x} \in F \tilde{\cup} G$  does not necessarily imply that either  $\tilde{x} \in F$  or  $\tilde{x} \in G$ . Also, the intersection of two soft sets or complement of a soft set of  $S(\tilde{X})$  is not necessarily a member of  $S(\tilde{X})$ .

**Definition 4.** Let  $F, G \in S(\tilde{X})$ .

1.  $F \uplus G$  denotes the  $\epsilon$ -union of  $F$  and  $G$ , that is defined by  $F \uplus G = SS(\mathfrak{B})$ , where

$$\mathfrak{B} = \{\tilde{x} \in \tilde{X} : \tilde{x} \in F \text{ or } \tilde{x} \in G\},$$

ie,  $F \uplus G = SS(SE(F) \cup SE(G))$ .

2.  $F \pitchfork G$  denotes the  $\epsilon$ -intersection of  $F$  and  $G$ , that is defined by  $F \pitchfork G = SS(\mathfrak{B})$ , where

$$\mathfrak{B} = \{\tilde{x} \in \tilde{X} : \tilde{x} \in F \text{ and } \tilde{x} \in G\},$$

ie,  $F \pitchfork G = SS(SE(F) \cap SE(G))$ . If the two soft sets have no soft elements in common, then  $F \pitchfork G = \Phi$ .

3.  $F^{\mathbb{C}}$  denotes the  $\epsilon$ -complement of  $F$ , that is defined by  $F^{\mathbb{C}} = SS(\mathfrak{B})$ , where

$$\mathfrak{B} = \{\tilde{x} \in \tilde{X} : \tilde{x} \notin F^{\mathbb{C}}\},$$

$$\text{ie, } F^{\mathbb{C}} = SS(SE(F^{\mathbb{C}})).$$

*Remark 2.* (Das and Samanta<sup>23</sup>). It can be easily verified that  $F \uplus G$ ,  $F \frown G$ , and  $F^{\mathbb{C}}$  are members of  $S(\tilde{X})$ , if  $F, G \in S(\tilde{X})$ .

### 3 | SOME PROPERTIES OF $\epsilon$ -OPERATIONS

In this section, we prove some new properties of  $\epsilon$ -union,  $\epsilon$ -intersection, and  $\epsilon$ -complement operations on soft sets. We use those properties in next sections.

**Proposition 2.** Let  $F, G \in S(\tilde{X})$ .

1.  $F \uplus G = F \dot{\cup} G$ .
2.  $F \frown G \check{C} F \tilde{\cap} G$  and if  $F \tilde{\cap} G \in S(\tilde{X})$ , then  $F \frown G = F \tilde{\cap} G$ .
3.  $F^{\mathbb{C}} \check{C} F^{\mathbb{C}}$  and if  $F^{\mathbb{C}} \in S(\tilde{X})$ , then  $F^{\mathbb{C}} = F^{\mathbb{C}}$ .
4.  $F \uplus F^{\mathbb{C}} \check{C} \tilde{X}$  and if  $F^{\mathbb{C}} \in S(\tilde{X})$ , then  $F \uplus F^{\mathbb{C}} = \tilde{X}$ .
5.  $F \frown F^{\mathbb{C}} = \Phi$ .
6.  $(F \uplus G)^{\mathbb{C}} \check{C} F^{\mathbb{C}} \frown G^{\mathbb{C}}$  and  $(F \frown G)^{\mathbb{C}} \check{C} F^{\mathbb{C}} \uplus G^{\mathbb{C}}$ .
7. If  $F \tilde{\cap} G, F^{\mathbb{C}} \tilde{\cap} G^{\mathbb{C}}, F^{\mathbb{C}}, G^{\mathbb{C}} \in S(\tilde{X})$ , then  $(F \uplus G)^{\mathbb{C}} = F^{\mathbb{C}} \frown G^{\mathbb{C}}, (F \frown G)^{\mathbb{C}} = F^{\mathbb{C}} \uplus G^{\mathbb{C}}$ .
8. If  $F_i = SS(\mathfrak{B}_i), i \in I$ , then  $\bigcup_{i \in I} F_i = SS\left(\bigcup_{i \in I} \mathfrak{B}_i\right)$ .
9. If  $F_i = SS(\mathfrak{B}_i), i \in I$ , then  $\bigcap_{i \in I} F_i \check{C} SS\left(\bigcap_{i \in I} \mathfrak{B}_i\right)$ .

*Proof.* The proofs of the equalities 1, 5, and 8 can be found in Das and Samanta.<sup>23</sup>

1. If  $F(\lambda) \cap G(\lambda) = \emptyset, \forall \lambda \in A$ , then  $F \tilde{\cap} G \in S(\tilde{X})$  and  $F \frown G = F \tilde{\cap} G = \Phi$ .  
If  $F(\lambda) \cap G(\lambda) \neq \emptyset, \forall \lambda \in A$ , then  $F \tilde{\cap} G \in S(\tilde{X})$  and  $F \frown G \neq \Phi$ . So  $F \frown G = F \tilde{\cap} G$ .  
If  $F(\lambda) \cap G(\lambda) = \emptyset$  for some  $\lambda \in A$  and  $F(\lambda') \cap G(\lambda') \neq \emptyset$  for  $\lambda' \in A \setminus \{\lambda\}$ , then  $F \frown G = \Phi$ , but  $F \tilde{\cap} G \neq \Phi$  and  $F \tilde{\cap} G \notin S(\tilde{X})$ . Hence,  $F \frown G \check{C} F \tilde{\cap} G$ .
2. If  $F^{\mathbb{C}}(\lambda) = \emptyset, \forall \lambda \in A$ , then  $F^{\mathbb{C}} \in S(\tilde{X})$  and  $F^{\mathbb{C}} = F^{\mathbb{C}} = \Phi$ .  
If  $F^{\mathbb{C}}(\lambda) \neq \emptyset, \forall \lambda \in A$ , then  $F^{\mathbb{C}} \neq \Phi$  and  $F^{\mathbb{C}} \neq \Phi$  implies that  $F^{\mathbb{C}} \in S(\tilde{X})$  and  $F^{\mathbb{C}} = F^{\mathbb{C}}$ .  
If  $F^{\mathbb{C}}(\lambda) = \emptyset$  for some  $\lambda \in A$  and  $F^{\mathbb{C}}(\lambda') \neq \emptyset$  for  $\lambda' \in A \setminus \{\lambda\}$ , then  $F^{\mathbb{C}} = \Phi$  and  $F^{\mathbb{C}} \neq \emptyset$ . So  $F^{\mathbb{C}} \notin S(\tilde{X})$  and  $F^{\mathbb{C}} \check{C} F^{\mathbb{C}}$ .
3. The proof follows from 3.
4. The proof follows from 1, 2, and 3.
- 5.

$$\begin{aligned} (F \uplus G)^{\mathbb{C}} &= SS\{\tilde{x} \in \tilde{X} : \tilde{x} \in (F \uplus G)^{\mathbb{C}}\} \\ &= SS\{\tilde{x} \in \tilde{X} : \tilde{x} \in (F \dot{\cup} G)^{\mathbb{C}}\} \\ &= SS\{\tilde{x} \in \tilde{X} : \tilde{x} \in F^{\mathbb{C}} \tilde{\cap} G^{\mathbb{C}}\} \\ &= SS\{\tilde{x} \in \tilde{X} : \tilde{x} \in F^{\mathbb{C}} \frown G^{\mathbb{C}}\} \quad (\text{with } F^{\mathbb{C}} \tilde{\cap} G^{\mathbb{C}} \in S(\tilde{X})) \\ &= SS\{\tilde{x} \in \tilde{X} : \tilde{x} \in F^{\mathbb{C}} \text{ and } \tilde{x} \in G^{\mathbb{C}}\} \\ &= SS(SE(F^{\mathbb{C}}) \cap SE(G^{\mathbb{C}})). \end{aligned} \tag{1}$$

$$\begin{aligned} F^{\mathbb{C}} \frown G^{\mathbb{C}} &= SS\{\tilde{x} \in \tilde{X} : \tilde{x} \in F^{\mathbb{C}} \text{ and } \tilde{x} \in G^{\mathbb{C}}\} \\ &= SS\{\tilde{x} \in \tilde{X} : \tilde{x} \in F^{\mathbb{C}} \text{ and } \tilde{x} \in G^{\mathbb{C}}\} \quad (\text{with } F^{\mathbb{C}} \in S(\tilde{X}) \text{ and } G^{\mathbb{C}} \in S(\tilde{X})) \\ &= SS(SE(F^{\mathbb{C}}) \cap SE(G^{\mathbb{C}})). \end{aligned} \tag{2}$$

Result follows from (1) and (2).

$$\begin{aligned}
 (F \pitchfork G)^{\mathbb{C}} &= SS\{\tilde{x}\tilde{\in}\tilde{X} : \tilde{x}\tilde{\in}(F \pitchfork G)^{\mathbb{C}}\} \\
 &= SS\{\tilde{x}\tilde{\in}\tilde{X} : \tilde{x}\tilde{\in}(F\tilde{\cap}G)^{\mathbb{C}}\} \text{ (with } F\tilde{\cap}G \in S(\tilde{X})\text{)} \\
 &= SS\{\tilde{x}\tilde{\in}\tilde{X} : \tilde{x}\tilde{\in}F^{\mathbb{C}}\tilde{\cup}G^{\mathbb{C}}\} \\
 &= SS\{\tilde{x}\tilde{\in}\tilde{X} : \tilde{x}\tilde{\in}F^{\mathbb{C}} \cup G^{\mathbb{C}}\} \\
 &= SS\{\tilde{x}\tilde{\in}\tilde{X} : \tilde{x}\tilde{\in}F^{\mathbb{C}} \text{ or } \tilde{x}\tilde{\in}G^{\mathbb{C}}\} \\
 &= SS(SE(F^{\mathbb{C}}) \cup SE(G^{\mathbb{C}})).
 \end{aligned} \tag{3}$$

$$\begin{aligned}
 F^{\mathbb{C}} \cup G^{\mathbb{C}} &= SS\{\tilde{x}\tilde{\in}\tilde{X} : \tilde{x}\tilde{\in}F^{\mathbb{C}} \text{ or } \tilde{x}\tilde{\in}G^{\mathbb{C}}\} \\
 &= SS\{\tilde{x}\tilde{\in}\tilde{X} : \tilde{x}\tilde{\in}F^{\mathbb{C}} \text{ or } \tilde{x}\tilde{\in}G^{\mathbb{C}}\} \text{ (with } F^{\mathbb{C}} \in S(\tilde{X}) \text{ and } G^{\mathbb{C}} \in S(\tilde{X})\text{)} \\
 &= SS(SE(F^{\mathbb{C}}) \cup SE(G^{\mathbb{C}})).
 \end{aligned} \tag{4}$$

Result follows from (3) and (4).

6. Since we have  $\mathfrak{B}_i \subset SE(F_i), \forall i \in I, \bigcap_{i \in I} \mathfrak{B}_i \subset \bigcap_{i \in I} SE(F_i)$ , and so

$$SS\left(\bigcap_{i \in I} \mathfrak{B}_i\right) \check{\subset} SS\left(\bigcap_{i \in I} SE(F_i)\right)$$

implies that

$$SS\left(\bigcap_{i \in I} \mathfrak{B}_i\right) \check{\subset} \pitchfork_{i \in I} F_i. \quad \square$$

*Remark 3.* By using  $\epsilon$ -operations, the De Morgan's laws are not provided in general. For example, let  $X = \{a, b, c, d\}$  and  $A = \{\lambda, \mu\}$ . We consider the following soft sets and their  $\epsilon$ -complements

$$\begin{aligned}
 F &= \{(\lambda, \{a, b\}), (\mu, X)\}, & F^{\mathbb{C}} &= \Phi, \\
 G &= \{(\lambda, \{c, d\}), (\mu, \{a, b\})\}, & G^{\mathbb{C}} &= \{(\lambda, \{a, b\}), (\mu, \{c, d\})\}.
 \end{aligned}$$

Then,  $(F \cup G)^{\mathbb{C}} = \Phi$  and  $F^{\mathbb{C}} \pitchfork G^{\mathbb{C}} = \Phi$ . But  $(F \pitchfork G)^{\mathbb{C}} = \tilde{X}$  and  $F^{\mathbb{C}} \cup G^{\mathbb{C}} = G^{\mathbb{C}}$ .

So, from Proposition 2, if  $F\tilde{\cap}G, F^{\mathbb{C}}\tilde{\cap}G^{\mathbb{C}}, F^{\mathbb{C}}, G^{\mathbb{C}} \in S(\tilde{X})$ , then

$$(F \cup G)^{\mathbb{C}} = F^{\mathbb{C}} \pitchfork G^{\mathbb{C}} \quad \text{and} \quad (F \pitchfork G)^{\mathbb{C}} = F^{\mathbb{C}} \cup G^{\mathbb{C}}.$$

*Remark 4.* Let  $F, G \in S(\tilde{X})$ .  $F \pitchfork G = \Phi$  cannot imply  $F\check{\subset}G^{\mathbb{C}}$  and  $G\check{\subset}F^{\mathbb{C}}$  in general. For example, let  $X = \{a, b, c, d\}$  and  $A = \{\lambda, \mu\}$ . We consider the following soft sets and their  $\epsilon$ -complements:

$$\begin{aligned}
 F &= \{(\lambda, \{a, b\}), (\mu, \{a, c\})\}, & F^{\mathbb{C}} &= \{(\lambda, \{c, d\}), (\mu, \{b, d\})\}, \\
 G &= \{(\lambda, \{c, d\}), (\mu, \{b, c\})\}, & G^{\mathbb{C}} &= \{(\lambda, \{a, b\}), (\mu, \{a, d\})\}, \\
 H &= \{(\lambda, \{c, d\}), (\mu, \{a, b, d\})\}, & H^{\mathbb{C}} &= \{(\lambda, \{a, b\}), (\mu, \{c\})\}.
 \end{aligned}$$

Although  $F \pitchfork G = \Phi$  and  $F \pitchfork H = \Phi$ ,  $F\check{\subset}G^{\mathbb{C}}$  or  $G\check{\subset}F^{\mathbb{C}}$  and  $F\check{\subset}H^{\mathbb{C}}$  or  $H\check{\subset}F^{\mathbb{C}}$  but  $H^{\mathbb{C}}\check{\subset}F$  and  $F^{\mathbb{C}}\check{\subset}H$ . Also,  $G \pitchfork H \neq \Phi$  but  $G^{\mathbb{C}} \pitchfork H^{\mathbb{C}} = \Phi$ .

**Proposition 3.** Let  $F, G \in S(\tilde{X})$ . If  $F\tilde{\cap}G \in S(\tilde{X})$  and  $F \pitchfork G = \Phi$ , then  $F\check{\subset}G^{\mathbb{C}}$  and  $G\check{\subset}F^{\mathbb{C}}$ .

*Proof.* From Proposition 2 (2), it follows that  $F \pitchfork G = F\tilde{\cap}G$ . Therefore,  $F \pitchfork G = \Phi$  implies that  $F\tilde{\cap}G = \Phi$ . Since  $F\check{\subset}G^{\mathbb{C}}$  and  $G\check{\subset}F^{\mathbb{C}}$ ,  $SE(F) \subset SE(G^{\mathbb{C}})$  and  $SE(G) \subset SE(F^{\mathbb{C}})$ . Hence,  $SS(SE(F)) \check{\subset} SS(SE(G^{\mathbb{C}}))$  and  $SS(SE(G)) \check{\subset} SS(SE(F^{\mathbb{C}}))$ . So  $F\check{\subset}G^{\mathbb{C}}$  and  $G\check{\subset}F^{\mathbb{C}}$ .  $\square$

**Proposition 4.** Let  $F, G \in S(\tilde{X})$ , then

1.  $SE(F \pitchfork G) = SE(F) \cap SE(G)$ ,
2.  $SE(F \pitchfork G) \supset SE(F) \cup SE(G)$ .

*Proof.* From Definition 4, if  $\tilde{x} \in SE(F \pitchfork G)$ , then  $\tilde{x} \in SE(F) \cap SE(G)$ . Hence,  $SE(F \pitchfork G) \subset SE(F) \cap SE(G)$ . Also,  $SE(F) \cap SE(G) \subset SE(F \pitchfork G)$  and  $SE(F) \cup SE(G) \subset SE(F \pitchfork G)$  are found as a result of Proposition 1.  $\square$

*Remark 5.* In general, the  $\epsilon$ -union and  $\epsilon$ -intersection operations are not distributive. For example, if we consider the soft sets in Remark 4, then we have

$$(F \pitchfork G) \pitchfork H = H, \quad (F \pitchfork H) \pitchfork (G \pitchfork H) = \{(\lambda, \{c, d\}), (\mu, X)\}$$

and

$$(F \pitchfork H) \pitchfork G = G, \quad (F \pitchfork G) \pitchfork (H \pitchfork G) = \{(\lambda, \{c, d\}), (\mu, \{b\})\}.$$

**Proposition 5.** Let  $F, G, H \in S(\tilde{X})$ , then

1.  $(F \pitchfork G) \pitchfork H \check{\subset} (F \pitchfork H) \pitchfork (G \pitchfork H)$ ,
2.  $(F \pitchfork G) \pitchfork H \check{\supset} (F \pitchfork H) \pitchfork (G \pitchfork H)$ .

*Proof.*

$$\begin{aligned} 1. \quad & (F \pitchfork G) \pitchfork H = SS(SE(F \pitchfork G) \cup SE(H)) \\ & = SS((SE(F) \cap SE(G)) \cup SE(H)) \quad (\text{From Proposition 4 (1)}) \\ & = SS((SE(F) \cup SE(H)) \cap (SE(G) \cup SE(H))) \\ & \check{\subset} SS(SE(F \pitchfork H) \cap SE(G \pitchfork H)) \quad (\text{From Proposition 4 (2)}) \\ & = (F \pitchfork H) \pitchfork (G \pitchfork H). \end{aligned}$$

$$\begin{aligned} 2. \quad & (F \pitchfork G) \pitchfork H = SS(SE(F \pitchfork G) \cap SE(H)) \\ & \check{\supset} SS((SE(F) \cup SE(G)) \cap SE(H)) \quad (\text{From Proposition 4 (2)}) \\ & = SS((SE(F) \cap SE(H)) \cup (SE(G) \cap SE(H))) \\ & = SS(SE(F \pitchfork H) \cup SE(G \pitchfork H)) \quad (\text{From Proposition 4 (1)}) \\ & = (F \pitchfork H) \pitchfork (G \pitchfork H). \end{aligned} \quad \square$$

*Remark 6.* In the above proposition, if  $F \check{\pitchfork} G \in S(\tilde{X})$ , then  $(F \pitchfork G) \pitchfork H = (F \pitchfork H) \pitchfork (G \pitchfork H)$ . In addition, if  $F \check{\pitchfork} H, G \check{\pitchfork} H \in S(\tilde{X})$ , then  $(F \pitchfork G) \pitchfork H = (F \pitchfork H) \pitchfork (G \pitchfork H)$ .

## 4 | $\epsilon$ -SOFT TOPOLOGY

**Definition 5.** Let  $\mathcal{T}^* \subset \mathcal{P}(SE(\tilde{X}))$  be a family of the collections of soft elements over universal set  $X$  with parameter set  $A$ . Then,  $\mathcal{T}^*$  is a topology on  $\tilde{X}$ , if it satisfies the following conditions:

1.  $SE(\tilde{X}), \emptyset \in \mathcal{T}^*$ .
2. If  $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{T}^*$ , then  $\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \mathcal{T}^*$ .
3. If  $\mathfrak{B}_i \in \mathcal{T}^*, \forall i \in I$ , then  $\bigcup_{i \in I} \mathfrak{B}_i \in \mathcal{T}^*$ .

The topology  $\mathcal{T}^*$  is called collections of soft elements (CSE) topology, and the triplet  $(\tilde{X}, \mathcal{T}^*, A)$  is called CSE topological space. Each member of  $\mathcal{T}^*$  is called an open collection, and a collection, whose complement is an open collection, is called a closed collection.

*Remark 7.* All (classical) topological concepts are also provided for CSE topology.

**Example 1.** Let  $(X, \mathcal{T}^*, A)$  be a CSE topological space. Then, the family

$$\mathcal{T}_\lambda^* = \{\mathfrak{B}_\lambda \subset X : \tilde{x}(\lambda) \in \mathfrak{B}_\lambda, \forall \tilde{x} \in \mathfrak{B} \in \mathcal{T}^*\},$$

$\forall \lambda \in A$ , produces a topology on  $X$ .

**Example 2.** In soft metric space  $(\tilde{X}, d, A)$ , the family of all open ball with centre at  $\tilde{x} \in \tilde{X}$ ,

$$\beta = \{B(\tilde{x}, \tilde{\epsilon}) : \tilde{x} \in \tilde{X}, \tilde{\epsilon} > \tilde{0}\},$$

is a basis for a CSE topology on  $\tilde{X}$ . The topology generated by this basis, ie, CSE topology generated by soft metric  $d$ , is called metric CSE topology on  $\tilde{X}$  (for detailed information about the soft metric space, see Das and Samanta<sup>23</sup>).

Now, we define a soft topology using the  $\epsilon$ -operations. Although  $\epsilon$ -operations are not distributive in general, it can be defined such a topology.

**Definition 6.** Let  $\mathcal{T} \subset S(\tilde{X})$  be a family of soft sets over  $X$  with parameter set  $A$ .  $\mathcal{T}$  is a topology on  $\tilde{X}$  according to the  $\epsilon$ -operations if it meets the following conditions.

1.  $\tilde{X}, \Phi \in \mathcal{T}$ .
2. If  $U, V \in \mathcal{T}$ , then  $U \cap V \in \mathcal{T}$ .
3. If  $U_i \in \mathcal{T}, \forall i \in I$ , then  $\bigcup_{i \in I} U_i \in \mathcal{T}$ .

This topology is called  $\epsilon$ -soft topology, and the triplet  $(\tilde{X}, \mathcal{T}, A)$  is called a  $\epsilon$ -soft topological space.

**Example 3.**  $\mathcal{T} = \{\Phi, \tilde{X}\}$ , which is called indiscrete soft topology on  $X$ , is also an  $\epsilon$ -soft topology on  $\tilde{X}$ .

**Example 4.**  $\mathcal{T} = S(\tilde{X})$ , which is called discrete  $\epsilon$ -soft topology on  $\tilde{X}$ , is an  $\epsilon$ -soft topology on  $\tilde{X}$ .

**Example 5.** Let  $X = \{a, b, c\}$ ,  $A = \{\lambda, \mu\}$ ,  $\mathcal{T}_1 = \{\Phi, \tilde{X}, F_1, F_2\}$ ,  $\mathcal{T}_2 = \{\Phi, \tilde{X}, F_2, F_3\}$ ,  $\mathcal{T}_3 = \{\Phi, \tilde{X}, F_1, F_3\}$ , and  $\mathcal{T}_4 = \{\Phi, \tilde{X}, F_1, F_2, F_3\}$ , where  $F_1, F_2$ , and  $F_3$  are soft sets over  $X$ , defined as follows:

$$F_1 = \{(\lambda, \{a, b\}), (\mu, \{a, c\})\}, \quad F_2 = \{(\lambda, \{c\}), (\mu, \{b\})\}, \quad F_3 = \{(\lambda, \{c\}), (\mu, \{b, c\})\}.$$

Then,  $\mathcal{T}_1$  is both an  $\epsilon$ -soft topology and a soft topology as Definitions 1 and 2 on  $\tilde{X}$ .  $\mathcal{T}_2$  is neither an  $\epsilon$ -soft topology nor a soft topology as Definitions 1 and 2 on  $\tilde{X}$ . Though  $\mathcal{T}_3$  is an  $\epsilon$ -soft topology on  $\tilde{X}$ , it is not a soft topology as Definitions 1 and 2 on  $\tilde{X}$ . Indeed,  $F_1 \tilde{\cap} F_3 = \{(\mu, \{c\})\} \notin \mathcal{T}_3$  and  $\tau_\mu = \{\emptyset, X, \{a, c\}, \{b, c\}\}$  is not a topology on  $\tilde{X}$ . Finally,  $\mathcal{T}_4$  is  $\epsilon$ -soft topology and soft topology as Definition 2 on  $\tilde{X}$ , but it is not a soft topology as Definition 1.

*Remark 8.* Let  $\mathcal{T} \subset S(\tilde{X})$ . Then, according to Example 5, the fact that  $\mathcal{T}$  is an  $\epsilon$ -soft topology does not require it to be a soft topology as Definitions 1 and 2 on  $\tilde{X}$ . Also, in Example 4,  $S(\tilde{X})$  is a discrete  $\epsilon$ -soft topology and a soft topology as Definition 2 on  $\tilde{X}$ , but it is not a soft topology as Definition 1 on  $\tilde{X}$ . In Example 3, indiscrete soft topology on  $\tilde{X}$  is both  $\epsilon$ -soft topology and soft topology as Definitions 1 and 2 on  $\tilde{X}$ .

**Example 6.** It ensures that the collection  $\mathcal{T}$  of all soft open sets in  $(\tilde{X}, d, A)$  satisfying the condition (M5) establishes a topology on  $\tilde{X}$  called  $\epsilon$ -soft metric topology according to  $\epsilon$ -operations (for the condition (M5) and others, see Das and Samanta<sup>23</sup>).

*Remark 9.* From Proposition 2,

$$\bigcup_{i \in I} U_i = \tilde{\bigcup}_{i \in I} U_i$$

and  $U \cap V = U \tilde{\cap} V$ , for  $U \cap V \neq \Phi$ . Thus, an  $\epsilon$ -soft topology  $\mathcal{T}$  is a soft topology as Definition 1 on  $\tilde{X}$  when  $U \cap V \neq \Phi$ , for all non-null soft sets  $U, V \in S(\tilde{X})$ .

**Proposition 6.** Let  $(\tilde{X}, \mathcal{T}, A)$  be an  $\epsilon$ -soft topological space. If for  $U, V \in \mathcal{T}$ ,  $U \tilde{\cap} V \in S(\tilde{X})$ , then the family  $\mathcal{T}_\lambda = \{U(\lambda) : U \in \tau\}$ ,  $\forall \lambda \in A$ , gives a crisp topology on  $\tilde{X}$ .

*Proof.* If for  $U, V \in \mathcal{T}$ ,  $U \tilde{\cap} V \in S(\tilde{X})$ , then  $U \tilde{\cap} V = U \cap V$ . Hence,  $\mathcal{T}$  is a soft topology. From Definition 1,  $\mathcal{T}_\lambda = \{U(\lambda) : U \in \tau\}$ ,  $\forall \lambda \in A$ , gives a crisp topology on  $\tilde{X}$ .  $\square$

The relationship between  $\epsilon$ -soft topology and soft topology as Definition 2 is given by the following proposition.

**Proposition 7.** Let  $(X, \tau, A)$  be soft topological space as Definition 2. Then,  $\mathcal{T} = \{U \in S(\tilde{X}) : U(\lambda) \in \tau_\lambda, \forall \lambda \in A\}$  is a  $\epsilon$ -soft topology on  $\tilde{X}$ .

*Proof.*

1. Since  $\Phi, \tilde{X} \in \tau$ , it can be easily seen that  $\Phi, \tilde{X} \in \mathcal{T}$ .
2. Let  $U, V \in \mathcal{T}$ . If  $U \cap V = \Phi$ , then obviously  $U \cap V \in \mathcal{T}$ . If  $U \cap V \neq \Phi$ , then  $U \cap V = U \tilde{\cap} V$ , and so  $(U \cap V)(\lambda) = (U \tilde{\cap} V)(\lambda) = U(\lambda) \cap V(\lambda) \in \tau_\lambda$ , for all  $\lambda \in A$ . Thus,  $U \cap V \in \mathcal{T}$ .
3. Let  $U_i \in \mathcal{T}$ ,  $\forall i \in I$ . Since  $\bigcup_{i \in I} U_i = \tilde{\bigcup}_{i \in I} U_i$ , obviously  $\bigcup_{i \in I} U_i \in \mathcal{T}$ .

Thus,  $\mathcal{T}$  is an  $\epsilon$ -soft topology on  $\tilde{X}$ .  $\square$

The relationship between  $\epsilon$ -soft topology and CSE topology is given by the following proposition.

**Proposition 8.** Let  $(\tilde{X}, \mathcal{T}^*, A)$  be a CSE topological space. Then, the family  $\mathcal{T} = \{U \in S(\tilde{X}) : SE(U) \in \mathcal{T}^*\}$  is an  $\epsilon$ -soft topology on  $\tilde{X}$ .

*Proof.*

1. Since  $\emptyset \in \mathcal{T}^*$  and  $SE(\tilde{X}) \in \mathcal{T}^*$ ,  $SS(\emptyset) = \Phi$  and  $SS(SE(\tilde{X})) = \tilde{X}$ . Hence,  $\Phi \in \mathcal{T}$  and  $\tilde{X} \in \mathcal{T}$ .
2. Let  $U, V \in \mathcal{T}$ . Then,

$$U \cap V = SS(SE(U) \cap SE(V)).$$

Since  $\mathcal{T}^*$  is a topology,  $SE(U) \cap SE(V) \in \mathcal{T}^*$ . Hence,  $U \cap V \in \mathcal{T}$ .

3. Let  $U_i \in \mathcal{T}$ ,  $\forall i \in I$ . Then,

$$\bigcup_{i \in I} U_i = SS\left(\bigcup_{i \in I} SE(U_i)\right).$$

Since  $\mathcal{T}^*$  is a topology,  $\bigcup_{i \in I} SE(U_i) \in \mathcal{T}^*$ . Hence,  $\bigcup_{i \in I} U_i \in \mathcal{T}$ .  $\square$

**Definition 7.** Let  $(\tilde{X}, \mathcal{T}, A)$  be an  $\epsilon$ -soft topological space. A soft open set is defined as a member of  $\mathcal{T}$ , and  $K \in S(\tilde{X})$  is called a soft closed set if  $K^C \in S(\tilde{X})$  and  $K^C \in \mathcal{T}$ .

**Proposition 9.** For any  $\epsilon$ -soft topological space  $(\tilde{X}, \mathcal{T}, A)$ ,

1.  $\tilde{X}$  and  $\Phi$  are soft closed.
2. Let  $\{K_i : i \in I\}$  be a family of soft closed sets in  $(\tilde{X}, \mathcal{T}, A)$ . Then,  $K = \cap\{K_i : i \in I\}$  is soft closed.
3. Let  $K_1$  and  $K_2$  be soft closed sets in  $(\tilde{X}, \mathcal{T}, A)$  such that  $K_1^C \cap K_2^C \neq \Phi$ . Then,  $K_1 \cup K_2$  is soft closed.

*Proof.*

1. Since  $\Phi^C = \tilde{X}$  and  $\tilde{X}^C = \Phi$  are soft open in  $(\tilde{X}, \mathcal{T}, A)$ , it follows that  $\Phi$  and  $\tilde{X}$  are soft closed.
2. If  $K_i = \Phi$ ,  $\forall i \in I$ , or  $K_j = \Phi$ ,  $\exists j \in I$ , then obviously  $K$  is soft closed. If  $K_i \neq \Phi$ ,  $\forall i \in I$ , and  $K = \Phi$ , then obviously  $K$  is soft closed. If  $K \neq \Phi$ , then  $K^C = (\cap\{K_i : i \in I\})^C = \cup\{K_i^C : i \in I\}$  from Remark 3.  $K^C$  is soft open, since arbitrary  $\epsilon$ -union of soft open sets is soft open. Hence,  $K = \cap\{K_i : i \in I\}$  is soft closed.
3. If one or both of  $K_1$  and  $K_2$  are the null soft set  $\Phi$ , then obviously  $K_1 \cup K_2$  is soft closed. Let  $K_1 \neq \Phi$  and  $K_2 \neq \Phi$ . Since  $K_1$  and  $K_2$  are soft closed sets,  $K_1^C, K_2^C \in S(\tilde{X})$ , and  $K_1^C, K_2^C \in \mathcal{T}$ . For  $K_1^C \cap K_2^C \neq \Phi$ ,  $(K_1 \cup K_2)^C = K_1^C \cap K_2^C$  from Remark 3. Since  $\mathcal{T}$  is an  $\epsilon$ -soft topology and  $K_1^C, K_2^C \in \mathcal{T}$ ,  $K_1^C \cap K_2^C = (K_1 \cup K_2)^C \in \mathcal{T}$ . Hence,  $K_1 \cup K_2$  is soft closed.  $\square$

**Definition 8.** A soft set  $N \in S(\tilde{X})$  in  $(\tilde{X}, \mathcal{T}, A)$  is called a soft neighbourhood of soft element  $\tilde{x} \in \tilde{X}$ , if there exists a  $U \in \mathcal{T}$  such that  $\tilde{x} \in U \tilde{C} N$ .  $\mathcal{N}(\tilde{x})$  indicates the family of all neighbourhoods of the soft element  $\tilde{x}$ .

A soft set  $N \in S(\tilde{X})$  is called a soft neighbourhood of soft set  $F$  if there exists a  $U \in \mathcal{T}$  such that  $F \tilde{C} U \tilde{C} N$ .

**Proposition 10.** Let  $(\tilde{X}, \mathcal{T}, A)$  be an  $\epsilon$ -soft topological space and  $\tilde{x} \in \tilde{X}$ . Then,

1.  $\forall N \in \mathcal{N}(\tilde{x}), \tilde{x} \in N$ .
2. For  $N, N' \in \mathcal{N}(\tilde{x}), N \cap N' \in \mathcal{N}(\tilde{x})$ .
3. If  $N \in \mathcal{N}(\tilde{x})$  and  $N \tilde{C} N', N' \in \mathcal{N}(\tilde{x})$ .
4.  $\forall N \in \mathcal{N}(\tilde{x}),$  there exists  $N' \in \mathcal{N}(\tilde{x})$  such that  $\forall \tilde{y} \in N', N \in \mathcal{N}(\tilde{y})$ .

*Proof.* We prove only the property 2.

If  $N, N' \in \mathcal{N}(\tilde{x})$ , there exists  $U \in \mathcal{T}$  and  $U' \in \mathcal{T}$  such that  $\tilde{x} \in U \tilde{C} N$  and  $\tilde{x} \in U' \tilde{C} N'$ , respectively. Then,  $\tilde{x} \in SE(U) \subset SE(N)$  and  $\tilde{x} \in SE(U') \subset SE(N')$ . Since  $\tilde{x} \in SE(U) \cap SE(U') \subset SE(N) \cap SE(N')$ ,  $\tilde{x} \in SS(SE(U)) \cap SS(SE(U')) \tilde{C} SS(SE(N)) \cap SS(SE(N'))$  or  $\tilde{x} \in U \cap U' \tilde{C} N \cap N'$ . Thus,  $N \cap N' \in \mathcal{N}(\tilde{x})$ .  $\square$

**Corollary 1.** For any  $(\tilde{X}, \mathcal{T}, A)$  and for all  $F_i \in \mathcal{T}, \forall i \in I, \tilde{\cap}_{i \in I} F_i \in S(\tilde{X})$ . If  $F$  is a soft neighbourhood of soft element  $\tilde{x} \in \tilde{X}$  in  $(\tilde{X}, \mathcal{T}, A)$ , then  $F(\lambda)$  is a neighbourhood of  $\tilde{x}(\lambda)$  in  $(X, \mathcal{T}_\lambda), \forall \lambda \in A$ .

**Definition 9.** For any  $(\tilde{X}, \mathcal{T}, A)$  and  $F \in S(\tilde{X})$ , soft element  $\tilde{x} \in F$  is said to be a soft interior element of  $F$ , if there exists a  $U \in \mathcal{T}$  such that  $\tilde{x} \in U \tilde{C} F$ .  $IntF$  and  $F^\circ = SS(IntF)$  indicate the collection of all soft interior elements of  $F$  and the soft set produced by the collection of all soft interior elements of  $F$  called soft interior of  $F$ , respectively.

**Proposition 11.** For any  $\epsilon$ -soft topological space  $(\tilde{X}, \mathcal{T}, A)$  and  $F \in S(\tilde{X})$ ,

1.  $F^\circ$  is the  $\epsilon$ -union of soft open subsets of  $F$ .
2.  $F^\circ$  is the largest soft open subset of  $F$ .
3.  $F$  is a soft open set  $\iff F^\circ = F$ .

*Proof.* We prove only the property 1.

Let  $G = \Psi\{U \in S(\tilde{X}) : U \tilde{C} F \text{ and } U \in \mathcal{T}\}$  be the  $\epsilon$ -union of soft open subsets of  $F$ . We must prove that  $F^\circ = G$ . Let  $\tilde{x} \in IntF$ . Then, there exists  $U \in \mathcal{T}$  such that  $\tilde{x} \in U \tilde{C} F$ . Hence,  $U \tilde{C} G$  and  $\tilde{x} \in SE(G)$ . So  $IntF \subset SE(G)$ , and thus,  $SS(IntF) \tilde{C} SS(SE(G))$  or

$$F^\circ \tilde{C} G. \quad (5)$$

Conversely, let  $\tilde{x} \in G$ . Then,  $\tilde{x} \in SE(G)$ , and there exists  $U \in \mathcal{T}$  such that  $\tilde{x} \in U \tilde{C} F$ . Hence,  $\tilde{x} \in IntF$ . So,  $SE(G) \subset IntF$ , and thus,  $SS(SE(G)) \subset SS(IntF)$  or

$$G \tilde{C} F^\circ. \quad (6)$$

The equality follows from (5) and (6).  $\square$

**Proposition 12.** For any  $\epsilon$ -soft topological space  $(\tilde{X}, \mathcal{T}, A)$  and  $F, G \in S(\tilde{X})$ ,

1.  $\tilde{X}^\circ = \tilde{X}$  and  $\Phi^\circ = \Phi$ .
2. If  $F \tilde{C} G$ , then  $F^\circ \tilde{C} G^\circ$ .
3.  $(F^\circ)^\circ = F^\circ$ .
4.  $F^\circ \cap G^\circ = (F \cap G)^\circ$ .
5.  $F^\circ \cup G^\circ \tilde{C} (F \cup G)^\circ$ .

*Proof.* We prove only the property 4.

From the property 2,  $(F \cap G)^\circ \tilde{C} F^\circ$  and  $(F \cap G)^\circ \tilde{C} G^\circ$ , since  $F \cap G \tilde{C} F$  and  $F \cap G \tilde{C} G$ , respectively. Hence,

$$(F \cap G)^\circ \tilde{C} F^\circ \cap G^\circ. \quad (7)$$

Conversely,  $F^\circ \cap G^\circ \tilde{C} F \cap G$ , since  $F^\circ \tilde{C} F$  and  $G^\circ \tilde{C} G$  by Proposition 11 (2). Also,  $(F \cap G)^\circ$  is the largest soft open set, which is covered by  $F \cap G$ , from Proposition 11 (2). Hence,

$$F^\circ \cap G^\circ \tilde{C} (F \cap G)^\circ. \quad (8)$$

The property follows from (7) and (8).  $\square$

**Definition 10.** For any  $(\tilde{X}, \mathcal{T}, A)$  and  $F \in S(\tilde{X})$ , a soft element  $\tilde{x} \in \tilde{X}$  is called a soft closure element of  $F$ , if  $F \cap N \neq \Phi$ , for each  $N \in \mathcal{N}(\tilde{x})$ .  $clF$  and  $\bar{F} = SS(clF)$  indicate the collection of all soft closure elements of  $F$  and the soft set produced by the collection of all soft closure elements of  $F$  called soft closure of  $F$ , respectively.

The soft element  $\tilde{x} \in \tilde{X}$  is not a soft closure element of  $F$ , if there is at least one  $N \in \mathcal{N}(\tilde{x})$  such that  $F \cap N = \Phi$  with  $F \cap N \in S(\tilde{X})$ .

**Proposition 13.** For any  $\epsilon$ -soft topological space  $(\tilde{X}, \mathcal{T}, A)$  and  $F \in S(\tilde{X})$ ,

1.  $\bar{F}$  is the  $\epsilon$ -intersection of soft closed supersets of  $F$ .
2.  $\bar{F}$  is the smallest soft closed superset of  $F$ .
3.  $F$  is a soft closed set  $\iff \bar{F} = F$ .

*Proof.* We prove only the property 1.

Let  $G = \cap \{K \in S(\tilde{X}) : F \check{C} K \text{ and } K \text{ soft closed}\}$  be the  $\epsilon$ -intersection of soft closed superset of  $F$ . We show that  $\bar{F} = G$ . Let  $\tilde{x} \in G$  and assume that  $\tilde{x} \notin \bar{F}$ . Then, there is at least one  $N \in \mathcal{N}(\tilde{x})$  such that  $F \cap N = \Phi$  with  $F \cap N \in S(\tilde{X})$ . Also, there exists  $U \in \mathcal{T}$  such that  $\tilde{x} \in U \check{C} N$ , since  $N \in \mathcal{N}(\tilde{x})$ . Hence,  $F \cap U = \Phi$  with  $F \cap U \in S(\tilde{X})$ . Now, we have  $U^C \in S(\tilde{X})$  and by Proposition 3,  $F \check{C} U^C$ . Thus,  $U^C = U^C$  implies that  $(U^C)^C = U \in \mathcal{T}$ , ie,  $U^C$  is soft closed, but  $\tilde{x} \notin U^C$ . This is contradiction with  $\tilde{x} \in G$ . Thus,  $\tilde{x} \in \bar{F}$  and

$$G \check{C} \bar{F}. \quad (9)$$

On the other hand, let  $\tilde{x} \in clF$ , then  $\tilde{x} \in \bar{F}$ . We assume that  $\tilde{x} \notin G$ . Then, there exists a soft closed set  $K$  such that  $F \check{C} K$  and  $\tilde{x} \notin K$ . Since  $\tilde{x} \in K^C$  and  $K^C \in \mathcal{T}$ ,  $K^C \in \mathcal{N}(\tilde{x})$ . Also,  $F \cap K^C = \Phi$  and  $F \cap K^C (= \Phi) \in S(\tilde{X})$ , since  $F \check{C} K$ . Hence,  $\tilde{x}$  is not a soft closure element of  $F$ . This contradicts with  $\tilde{x} \in clF$  or  $\tilde{x} \in \bar{F}$ . Thus,  $\tilde{x} \in G$  and

$$\bar{F} \check{C} G. \quad (10)$$

The proof follows from (9) and (10). □

**Proposition 14.** For any  $\epsilon$ -soft topological space  $(\tilde{X}, \mathcal{T}, A)$  and  $F, G \in S(\tilde{X})$ ,

1.  $\bar{\bar{X}} = \bar{X}$  and  $\bar{\Phi} = \Phi$ .
2. If  $F \check{C} G$ , then  $\bar{F} \check{C} \bar{G}$ .
3.  $(\bar{F})^C = \bar{F^C}$ .
4.  $\bar{F} \cup \bar{G} = \overline{(F \cup G)}$ , if  $\bar{F^C} \cap \bar{G^C} \neq \Phi$ .
5.  $\bar{F} \cap \bar{G} \check{C} \overline{(F \cap G)}$ .

*Proof.* We prove only the property 4.

From the property 2, we get  $\bar{F} \check{C} \overline{(F \cup G)}$  and  $\bar{G} \check{C} \overline{(F \cup G)}$ , since  $F \check{C} F \cup G$  and  $G \check{C} F \cup G$ . Hence,

$$\bar{F} \cup \bar{G} \check{C} \overline{(F \cup G)}. \quad (11)$$

Conversely,  $F \cup G \check{C} \bar{F} \cup \bar{G}$ , since  $F \check{C} \bar{F}$  and  $G \check{C} \bar{G}$  from Proposition 13 (2). Now, let  $\bar{F^C} \cap \bar{G^C} \neq \Phi$ . Then,  $\bar{F} \cup \bar{G}$  is soft closed from Proposition 9. Also,  $\overline{(F \cup G)}$  is the smallest soft closed set covering  $F \cup G$  from Proposition 13. Hence,

$$F \cup G \check{C} \overline{(F \cup G)} \check{C} \bar{F} \cup \bar{G}. \quad (12)$$

The proof follows from (11) and (12). □

## 5 | BASIS FOR $\epsilon$ -SOFT TOPOLOGY

**Definition 11.** Let  $\mathcal{B} \subset S(\tilde{X})$ . Then,  $\mathcal{B}$  is called a soft basis for an  $\epsilon$ -soft topology on  $\tilde{X}$  if the following two conditions are satisfied:

- (B1) For all  $\tilde{x} \in \tilde{X}$ , there is at least one soft set  $B \in \mathcal{B}$  such that  $\tilde{x} \in B$ ,
- (B2) If  $\tilde{x} \in \tilde{X}$  and  $\tilde{x} \in B_1 \cap B_2$  for  $B_1, B_2 \in \mathcal{B}$ , then there exists  $B_3 \in \mathcal{B}$  such that  $\tilde{x} \in B_3 \check{C} B_1 \cap B_2$ .

**Example 7.** Let  $X = \{x, y, z\}$ ,  $A = \{\lambda, \mu\}$ , and  $B = \{B_1, B_2, B_3, B_4, B_5\}$ , where

$$\begin{aligned} B_1 &= \{(\lambda, \{x\}), (\mu, \{x\})\}, & B_4 &= \{(\lambda, \{x, y\}), (\mu, X)\}, \\ B_2 &= \{(\lambda, \{x\}), (\mu, \{y\})\}, & B_5 &= \{(\lambda, \{x, z\}), (\mu, X)\}. \\ B_3 &= \{(\lambda, \{x\}), (\mu, \{z\})\}. \end{aligned}$$

Then,  $B$  is a soft basis for an  $\epsilon$ -soft topology on  $\tilde{X}$ .

Indeed,  $SE(\tilde{X}) = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7, \tilde{x}_8, \tilde{x}_9\}$ , where

$$\begin{aligned} \tilde{x}_1 &= \{(\lambda, x), (\mu, x)\}, & \tilde{x}_4 &= \{(\lambda, y), (\mu, x)\}, & \tilde{x}_7 &= \{(\lambda, z), (\mu, x)\}, \\ \tilde{x}_2 &= \{(\lambda, x), (\mu, y)\}, & \tilde{x}_5 &= \{(\lambda, y), (\mu, y)\}, & \tilde{x}_8 &= \{(\lambda, z), (\mu, y)\}, \\ \tilde{x}_3 &= \{(\lambda, x), (\mu, z)\}, & \tilde{x}_6 &= \{(\lambda, y), (\mu, z)\}, & \tilde{x}_9 &= \{(\lambda, z), (\mu, z)\}. \end{aligned}$$

(B1) We have  $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6 \in B_4$  and  $\tilde{x}_7, \tilde{x}_8, \tilde{x}_9 \in B_5$ .

(B2) It is clear that

$$\begin{aligned} B_1 \cap B_4 &= B_1 \in \mathcal{B}, & B_2 \cap B_4 &= B_2 \in \mathcal{B}, & B_3 \cap B_4 &= B_3 \in \mathcal{B}, \\ B_1 \cap B_5 &= B_1 \in \mathcal{B}, & B_2 \cap B_5 &= B_2 \in \mathcal{B}, & B_3 \cap B_5 &= B_3 \in \mathcal{B}, \end{aligned}$$

and  $B_4 \cap B_5 = \{(\lambda, \{x\}), (\mu, X)\} = B_6 \notin \mathcal{B}$ . For the soft set  $B_6$ ,  $\tilde{x}_1 \in B_1 \tilde{C} B_6$ ,  $\tilde{x}_2 \in B_2 \tilde{C} B_6$ ,  $\tilde{x}_3 \in B_3 \tilde{C} B_6$ .

Thus,  $B$  is a soft basis.

*Remark 10.* If  $B$  is a soft basis for an  $\epsilon$ -soft topology on  $\tilde{X}$ , for  $B_1, B_2 \in \mathcal{B}$ ,  $B_1 \cap B_2$  does not have to belong to the soft basis  $\mathcal{B}$ . For example, in Example 7,  $B_4, B_5 \in \mathcal{B}$  but  $B_4 \cap B_5 = B_6 \notin \mathcal{B}$ .

**Proposition 15.** Let  $\mathcal{B} \subset S(\tilde{X})$  be a soft basis. Then,

$$\mathcal{T}_B = \{U \in S(\tilde{X}) : \forall \tilde{x} \in U, \exists B \in \mathcal{B}, \text{ such that } \tilde{x} \in B \tilde{C} U\}$$

is an  $\epsilon$ -soft topology on  $\tilde{X}$ .

*Proof.*

1. If  $U = \Phi$ , for all  $\tilde{x} \in \tilde{X}$ ,  $\tilde{x} \notin U$ . So  $\Phi \in \mathcal{T}_B$ . If  $U = \tilde{X}$ , since  $B \tilde{C} U$  for all  $B \in \mathcal{B}$ ,  $\tilde{X} \in \mathcal{T}_B$ .
2. Let  $U_1, U_2 \in \mathcal{T}_B$  and  $\tilde{x} \in U_1 \cap U_2$ . Then,  $\tilde{x} \in U_1$  and  $\tilde{x} \in U_2$ . Since  $U_1, U_2 \in \mathcal{T}_B$ , there exists  $B_1, B_2 \in \mathcal{B}$  such that  $\tilde{x} \in B_1 \tilde{C} U_1$  and  $\tilde{x} \in B_2 \tilde{C} U_2$ . Hence,  $\tilde{x} \in B_1 \cap B_2$ . By Definition 11, there exists  $B_3 \in \mathcal{B}$  such that  $\tilde{x} \in B_3 \tilde{C} B_1 \cap B_2$ . Thus,  $\tilde{x} \in B_3 \tilde{C} B_1 \cap B_2 \tilde{C} U_1 \cap U_2$ , and so  $U_1 \cap U_2 \in \mathcal{T}_B$ .
3. Let  $U_i \in \mathcal{T}_B, \forall i \in I$ . For all  $\tilde{x} \in \tilde{X}$ , there exists  $B_i \in \mathcal{B}$  such that  $\tilde{x} \in B_i \tilde{C} U_i, \forall i \in I$ . Hence,  $\tilde{x} \in \bigcup_{i \in I} U_i$  and  $\tilde{x} \in B_i \tilde{C} \bigcup_{i \in I} U_i$ .

Thus,  $\bigcup_{i \in I} U_i \in \mathcal{T}_B$ . □

**Definition 12.** Suppose that  $\mathcal{B} \subset S(\tilde{X})$  is a soft basis and  $\mathcal{T}_B$  is an  $\epsilon$ -soft topology given in Proposition 15. Then, the  $\epsilon$ -soft topology  $\mathcal{T}_B$  is called the  $\epsilon$ -soft topology generated by the soft basis  $\mathcal{B}$ .

According to this definition, since  $\mathcal{B} \tilde{C} \mathcal{T}_B$ , the members of  $\mathcal{B}$  consist of soft open sets.

**Proposition 16.** Let  $(\tilde{X}, \mathcal{T}, A)$  be  $\epsilon$ -soft topological space and  $\mathcal{B}$  be a soft basis for  $\mathcal{T}$  (ie,  $\mathcal{T} = \mathcal{T}_B$ ). Then, every soft open set could be stated as the  $\epsilon$ -union of some members of  $\mathcal{B}$ .

*Proof.* Let  $U \in \mathcal{T}$ . By Proposition 15, for all  $\tilde{x} \in U$ , there exists  $B_{\tilde{x}} \in \mathcal{B}$  such that  $\tilde{x} \in B_{\tilde{x}} \tilde{C} U$ . Then,

$$U = \bigcup_{\tilde{x} \in U} B_{\tilde{x}}. \quad \square$$

**Example 8.** In Example 7, the  $\epsilon$ -soft topology on  $\tilde{X}$  generated by the soft basis  $\mathcal{B}$  is

$$\mathcal{T} = \{\Phi, \tilde{X}, U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8, U_9\},$$

where

$$\begin{aligned} U_1 &= \{(\lambda, \{x\}), (\mu, \{x\})\}, & U_4 &= \{(\lambda, \{x\}), (\mu, \{x, y\})\}, & U_7 &= \{(\lambda, \{x\}), (\mu, X)\}, \\ U_2 &= \{(\lambda, \{x\}), (\mu, \{y\})\}, & U_5 &= \{(\lambda, \{x\}), (\mu, \{x, z\})\}, & U_8 &= \{(\lambda, \{x, y\}), (\mu, X)\}, \\ U_3 &= \{(\lambda, \{x\}), (\mu, \{z\})\}, & U_6 &= \{(\lambda, \{x\}), (\mu, \{y, z\})\}, & U_9 &= \{(\lambda, \{x, z\}), (\mu, X)\}. \end{aligned}$$

**Remark 11.** The converse of Proposition 16 is not true.

In Proposition 15, an  $\epsilon$ -soft topology is defined with the help of the soft basis. Now, let us do the opposite. That is, let us find the soft basis of the given  $\epsilon$ -soft topology.

**Theorem 1.** Let  $(\tilde{X}, \mathcal{T}, A)$  be  $\epsilon$ -soft topological space and  $\mathcal{B} \subset \mathcal{T}$ . If the following condition  $(B_*)$  is satisfied, then  $\mathcal{B}$  is a soft basis for the  $\mathcal{T}$ .

$(B_*)$  For all  $U \in \mathcal{T}$  and all  $\tilde{x} \in U$ , there exists  $B \in \mathcal{B}$  such that  $\tilde{x} \in B \tilde{C} U$ .

*Proof.*

- (B1) Let  $U = \tilde{X}$ . Since  $U \in \mathcal{T}$  and  $\tilde{x} \in U$ , there exists  $B \in \mathcal{B}$  such that  $\tilde{x} \in B \tilde{C} U$  by  $(B_*)$ . Hence, for all  $\tilde{x} \in \tilde{X}$ , there exists  $B \in \mathcal{B}$  such that  $\tilde{x} \in B$ .
- (B2) Let  $\tilde{x} \in B_1 \cap B_2 = U$  for  $\tilde{x} \in \tilde{X}$  and  $B_1, B_2 \in \mathcal{B}$ . Since  $B_1$  and  $B_2$  are soft open sets,  $U$  is a soft open set. Hence, there exists  $B_3 \in \mathcal{B}$  such that  $\tilde{x} \in B_3 \tilde{C} U$  by  $(B_*)$ , and so  $\tilde{x} \in B_3 \tilde{C} B_1 \cap B_2$ .

Thus,  $\mathcal{B}$  is a soft basis. Clearly,  $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$ . □

**Example 9.** Let  $\mathcal{B} = \{SS(B(\tilde{x}, \tilde{\epsilon})) : \tilde{x} \in \tilde{X} \text{ and } \tilde{\epsilon} > \tilde{0}\}$  be a collection of soft open balls in  $(\tilde{X}, d, A)$ . Then, if  $(\tilde{X}, d, A)$  satisfies the  $(M5)$ ,  $\mathcal{B}$  is soft basis for the  $\epsilon$ -soft metric topology  $\mathcal{T}$  in Example 6.

**Definition 13.** Let  $(X, \mathcal{T}, A)$  be  $\epsilon$ -soft topological space and  $\tilde{x} \in \tilde{X}$  be an arbitrary soft element. A collection  $\mathcal{B}_{\tilde{x}}$  of soft open sets containing  $\tilde{x}$  is called a soft local basis of  $\tilde{x}$ , if for each  $U \in \mathcal{T}$  containing  $\tilde{x}$ , there exists  $B_{\tilde{x}} \in \mathcal{B}_{\tilde{x}}$  such that  $\tilde{x} \in B_{\tilde{x}} \tilde{C} U$ .

**Proposition 17.** Let  $\mathcal{B}$  be a soft basis for an  $\epsilon$ -soft topology on  $\tilde{X}$  and  $\tilde{x} \in \tilde{X}$ . The members of the soft basis  $\mathcal{B}$  that contain  $\tilde{x}$  form a soft local basis of  $\tilde{x}$ .

*Proof.* Proof follows from Theorem 1. □

**Proposition 18.** Let  $(X, \mathcal{T}, A)$  be  $\epsilon$ -soft topological space and  $\mathcal{B} \subset \mathcal{T}$ .  $\mathcal{B}$  is a soft basis for  $\mathcal{T}$  if and only if for each  $\tilde{x} \in \tilde{X}$ ,  $\mathcal{B}_{\tilde{x}} = \{B \in \mathcal{B} : \tilde{x} \in B\}$  is a soft local basis for  $\tilde{x}$ .

**Definition 14.** Let  $(\tilde{X}, \mathcal{T}, A)$  be an  $\epsilon$ -soft topological space, and  $S \subset \mathcal{T}$ .  $S$  is called soft subbase for the  $\mathcal{T}$ , if the collection of all finite  $\epsilon$ -intersection of members  $S$  is a soft base for  $\mathcal{T}$ .

## 6 | SOFT FUNCTION AND SOFT CONTINUITY

**Definition 15.** A soft mapping from  $X$  to  $Y$  with parameters set  $A$  is denoted by the mapping  $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$ , where  $X$  and  $Y$  are nonempty sets.

**Example 10.** Let  $\{f_{\lambda} : \lambda \in A\}$  be a family of crisp mappings from  $X$  to  $Y$ . Then,  $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$  is a soft mapping such that  $f(\tilde{x})(\lambda) = f_{\lambda}(\tilde{x}(\lambda))$ ,  $\forall \tilde{x} \in \tilde{X}$ , and  $\forall \lambda \in A$ . So every parametrized family of crisp mappings can be considered as a soft mapping.

**Example 11.**  $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$  is a soft mapping by  $f(\tilde{x})(\lambda) = f^*(\tilde{x}(\lambda))$ ,  $\forall \tilde{x} \in \tilde{X}$  and  $\forall \lambda \in A$ , such that  $f^* : X \rightarrow Y$  is a crisp mapping. Then, the soft mapping defined using the crisp mapping  $f^*$  is called a soft mapping produced by  $f^*$ .

**Remark 12.** The converse of Example 11 is not provided in general, ie, any soft mapping is not just a parametrized mappings family. For example, let  $X = \{a, b\}$ ,  $Y = \{c, d\}$ , and  $A = \{\lambda, \mu\}$ , then  $SE(\tilde{X}) = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\}$  and  $SE(\tilde{Y}) = \{\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4\}$ , where

$$\begin{aligned}\tilde{x}_1 &= \{(\lambda, a), (\mu, a)\} & \tilde{y}_1 &= \{(\lambda, c), (\mu, c)\} \\ \tilde{x}_2 &= \{(\lambda, a), (\mu, b)\} & \tilde{y}_2 &= \{(\lambda, c), (\mu, d)\} \\ \tilde{x}_3 &= \{(\lambda, b), (\mu, a)\} & \tilde{y}_3 &= \{(\lambda, d), (\mu, c)\} \\ \tilde{x}_4 &= \{(\lambda, b), (\mu, b)\} & \tilde{y}_4 &= \{(\lambda, d), (\mu, d)\}\end{aligned}$$

Let  $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$  be a soft mapping defined by  $f(\tilde{x}_1) = \tilde{y}_4$ ,  $f(\tilde{x}_2) = \tilde{y}_1$ ,  $f(\tilde{x}_3) = \tilde{y}_2$ , and  $f(\tilde{x}_4) = \tilde{y}_3$ . If  $f$  is decomposed parameter wise as  $f(\tilde{x})(\lambda) = f_\lambda(\tilde{x}(\lambda))$ ,  $\forall \lambda \in A$ ,  $\forall \tilde{x} \in \tilde{X}$ , then  $f_\lambda : X \rightarrow Y$  is not a well-defined mapping. Clearly,

$$\begin{aligned}f(\tilde{x}_1)(\lambda) = \tilde{y}_4(\lambda) &\Rightarrow f_\lambda(\tilde{x}_1(\lambda)) = f_\lambda(a) = d, \\ f(\tilde{x}_2)(\lambda) = \tilde{y}_1(\lambda) &\Rightarrow f_\lambda(\tilde{x}_2(\lambda)) = f_\lambda(a) = c, \\ f(\tilde{x}_3)(\lambda) = \tilde{y}_2(\lambda) &\Rightarrow f_\lambda(\tilde{x}_3(\lambda)) = f_\lambda(b) = c, \\ f(\tilde{x}_4)(\lambda) = \tilde{y}_3(\lambda) &\Rightarrow f_\lambda(\tilde{x}_4(\lambda)) = f_\lambda(b) = d.\end{aligned}$$

Hence,  $f_\lambda$  cannot be a crisp mapping from  $X$  to  $Y$ . Thus, the soft mapping is more exhaustive than any parametrized family of crisp mappings.

**Theorem 2.** If the soft mapping  $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$  meets the following condition  $(f_*)$  and if  $f_\lambda : X \rightarrow Y$  given by  $f_\lambda(\tilde{x}(\lambda)) = f(\tilde{x})(\lambda)$ ,  $\forall \lambda \in A$  and  $\forall \tilde{x} \in \tilde{X}$ , then  $f_\lambda$  is a mapping from  $X$  to  $Y$ .

$(f_*) \{f(\tilde{x})(\lambda) : \tilde{x}(\lambda) = x\}$  is a unit set set,  $\forall \lambda \in A$ ,  $\forall \tilde{x} \in \tilde{X}$  and  $\forall x \in X$ .

*Proof.* For all  $\lambda \in A$ ,  $f_\lambda : X \rightarrow Y$  is a regulation that appoints an element of the set  $Y$  corresponding to each element of the set  $X$ . The condition  $(f_*)$  ensures the well-defined property of  $f_\lambda$ ,  $\forall \lambda \in A$ . Hence, a parametrized family of crisp mappings is given by the soft mapping satisfying  $(f_*)$ . Also, a soft mapping satisfying  $(f_*)$  is a particular soft mapping as defined in Majumdar and Samanta,<sup>43</sup> where  $f : A \rightarrow Y^X$ .  $\square$

**Definition 16.** Suppose that a soft mapping  $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$  satisfies the condition  $(f_*)$ . Then,  $f$  is called a soft function.

The soft function  $f$  is injective (if  $\forall \tilde{x}, \tilde{y} \in \tilde{X}$ ,  $f(\tilde{x}) = f(\tilde{y})$  implies  $\tilde{x} = \tilde{y}$ ) and surjective (if  $f(SE(\tilde{X})) = SE(\tilde{Y})$ ) if and only if  $f_\lambda : X \rightarrow Y$ , defined by  $f_\lambda(\tilde{x}(\lambda)) = f(\tilde{x})(\lambda)$ ,  $\forall \lambda \in A$  and  $\forall \tilde{x} \in \tilde{X}$ , is injective and surjective mapping, respectively.

**Definition 17.** Let  $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$  be a soft mapping.

1.  $f(SE(F))$  and  $SS(f(SE(F)))$  denote image of  $F \in S(\tilde{X})$  and soft image of  $F$  under the  $f$  that is the soft set in  $S(\tilde{Y})$ , respectively.
2.  $f^{-1}(SE(F'))$  and  $SS(f^{-1}(SE(F')))$  denote inverse image of  $F' \in S(\tilde{Y})$  and soft inverse image of  $F'$  under the  $f$  that is soft set in  $S(\tilde{X})$ , respectively.

**Remark 13.** Let  $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$  be a soft mapping and  $F \in S(\tilde{X})$ . Then,  $f(SE(F)) \subset SE(SS(f(SE(F))))$ .

**Definition 18.** Let  $(\tilde{X}, \mathcal{T}, A)$  and  $(\tilde{Y}, \mathcal{T}', A)$  be two  $\epsilon$ -soft topological spaces. A soft mapping  $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$  is called soft continuous at  $\tilde{x} \in \tilde{X}$  if for any soft neighbourhood  $N' \in N(f(\tilde{x}))$ , there exists  $N \in \mathcal{N}(\tilde{x})$  such that  $f(SE(N)) \subset SE(N')$ .  $f$  is soft continuous over  $(\tilde{X}, \mathcal{T}, A)$ , if it is soft continuous at all  $\tilde{x} \in \tilde{X}$ .

**Theorem 3.** Let  $(\tilde{X}, \mathcal{T}, A)$  and  $(\tilde{Y}, \mathcal{T}', A)$  be two  $\epsilon$ -soft topological spaces and  $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$  be a soft mapping.  $f$  is soft continuous if and only if for all  $U' \in \mathcal{T}'$ ,  $SS(f^{-1}(SE(U'))) \in \mathcal{T}$ .

*Proof.* Let  $f$  be soft continuous,  $U' \in \mathcal{T}'$  and arbitrary  $\tilde{x} \in f^{-1}(SE(U'))$ . Then,  $f(\tilde{x}) \in SE(U')$  and  $U' \in N(f(\tilde{x}))$ . Since  $f$  is soft continuous, there exists  $N \in \mathcal{N}(\tilde{x})$  such that  $f(SE(N)) \subset SE(U')$ . Then,  $\tilde{x} \in SE(N) \subset f^{-1}(SE(U'))$ , and so  $\tilde{x} \in N \subset SS(f^{-1}(SE(U')))$ . Since  $N \in \mathcal{N}(\tilde{x})$ , there exists  $U \in \mathcal{T}$  such that  $\tilde{x} \in U \subset N \subset SS(f^{-1}(SE(U')))$ . Hence,  $\tilde{x} \in IntSS(f^{-1}(SE(U')))$ , and so  $\tilde{x} \in SS(f^{-1}(SE(U')))$ . Thus,  $SS(f^{-1}(SE(U'))) \in \mathcal{T}$  from Proposition 11.

Conversely, let for all  $U' \in \mathcal{T}'$ ,  $U = SS(f^{-1}(SE(U'))) \in \mathcal{T}$ , arbitrary  $\tilde{x} \in \tilde{X}$  and  $N \in N(f(\tilde{x}))$ . Then, there exists  $U' \in \mathcal{T}'$  such that  $f(\tilde{x}) \in U' \subset N$ , and so  $f(\tilde{x}) \in SE(U') \subset SE(N)$ . Hence,  $\tilde{x} \in f^{-1}(SE(U')) \subset f^{-1}(SE(N))$ , and so  $\tilde{x} \in U \subset SS(f^{-1}(SE(N)))$ . Since  $U \in \mathcal{T}$ ,  $U \in \mathcal{N}(\tilde{x})$ . Thus,

$$f(SE(U)) \subset f(SE(SS(f^{-1}(SE(N)))))$$

and  $f$  is a soft continuous at  $\tilde{x}$  because  $SS(f(SE(SS(f^{-1}(SE(N))))))$  is a soft neighbourhood of  $f(\tilde{x})$  from Proposition 10.  $\square$

**Corollary 2.** Let  $(\tilde{X}, \mathcal{T}, A)$  and  $(\tilde{Y}, \mathcal{T}', A)$  be two  $\epsilon$ -soft topological spaces and finite soft intersection of soft open sets in  $\tilde{X}$  and  $\tilde{Y}$  be in  $S(\tilde{X})$  and  $S(\tilde{Y})$ , respectively. The soft function  $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$  is soft continuous over  $(\tilde{X}, \mathcal{T}, A)$ , if and only if the mapping  $f_\lambda : X \rightarrow Y$ , given by  $f_\lambda(\tilde{x}(\lambda)) = f(\tilde{x})(\lambda)$ ,  $\forall \lambda \in A$  and  $\forall \tilde{x} \in \tilde{X}$ , is continuous over  $(X, \mathcal{T}_\lambda)$ ,  $\forall \lambda \in A$ .

## 6.1 | An application of soft function: A soft set approach to rotation in $\mathbb{E}^3$

Euler angles, which from Euler's rotation theorem express that any rotation can be described by three angles about three predefined axes in three-dimensional Euclidean space,  $\mathbb{E}^3$ , are the most frequently used to define rotation in literature.<sup>44</sup> Several conventions for Euler angles exist according to the axes in which the rotations are actualized in the original coordinate system or in the axes of the rotating coordinate system that switches orientation afterward each rotation. Regardless of the two different arrangements for the rotational axes, there exist 12 eventual series of rotational axes, divided into two groups:<sup>45</sup>

Proper (classic) Euler angles :  $z - x - z, x - y - x, y - z - y, z - y - z, x - z - x, y - x - y$

Tait-Bryan (Cardan) angles :  $x - y - z, y - z - x, z - x - y, x - z - y, z - y - x, y - x - z$ .

Tait-Bryan angles, which are the most common conventions and describe rotation by three angles about three different axes, are often used in aerospace engineering, robotics, computer graphics, and many other engineering applications involving motion.<sup>46-48</sup> Further information can be seen in the literature.<sup>44-49</sup>

In addition to Euler angles, there are various formalisms to express a rotation in  $\mathbb{E}^3$  as a mathematical transformation. Usually, rotations are represented by rotation matrices, quaternions, and axis-angle representations.<sup>45,49</sup> Here, we newly propose to using soft elements and soft sets mentioned in previous sections to represent the rotation or the orientation as follows:

Let a coordinate system in  $\mathbb{E}^3$ ,  $A = \{x_1, x_2, x_3\}$ , be considered as parameter set and  $[-\pi, \pi]$  be a universal set. Then, a rotation or an orientation in  $\mathbb{E}^3$  around three axes can be represented a soft element  $\tilde{x} : A \rightarrow [-\pi, \pi]$  by

$$\tilde{x} = \{(x_1, \alpha), (x_2, \beta), (x_3, \gamma)\},$$

where for all  $x \in A$  and  $\alpha \in [-\pi, \pi]$ ,  $\tilde{x}(x) = \alpha$  implies that  $x$  axis rotates by  $\alpha$  angle. Hence, the set of soft elements considering the above gives a family of rotations around three axes.

A soft set  $F : A \rightarrow P([-\pi, \pi])$ , which is produced by these soft elements, represents a union of the rotations in  $\mathbb{E}^3$  around three axes with respect to specified intervals or sets of angles. We denote this representation as soft rotation.

If for all  $x \in A$ ,  $F(x) = [-\pi, \pi]$ , then  $F$  represents complete rotation in  $\mathbb{E}^3$ . But if for some  $x \in A$ ,  $F(x) \subset [-\pi, \pi]$ , then  $F$  represents a restricted rotation on some axes. Also, for any  $x \in A$ , we indicate that  $F(x) = \emptyset$  has the same meaning as  $F(x) = \{0\}$  in this representation. Thus, with this representation of a rotation in  $\mathbb{E}^3$ , application of algebraic operations, available in soft sets, may gain a different point of view to motion in  $\mathbb{E}^3$ .

For example, let

$$F_1 = \{(x_1, (-\pi, 0)), (x_2, (-\pi/2, \pi]), (x_3, [0, \pi))\}$$

and

$$F_2 = \{(x_1, (-\pi/2, \pi)), (x_2, (0, \pi)), (x_3, [-\pi, \pi))\}$$

be two soft rotations as representations of two different moving bodies. In order for two independent systems to move together or separately from each other, we can obtain the representation of rotation as soft intersection or soft difference of these:

$$F_1 \tilde{\cap} F_2 = \{(x_1, (-\pi/2, 0)), (x_2, (0, \pi)), (x_3, [0, \pi))\}$$

or

$$F_1 \tilde{=} F_2 = \{(x_1, (-\pi, \pi/2]), (x_2, (-\pi/2, 0]), (x, \emptyset)\},$$

$$F_2 \tilde{=} F_1 = \{(x_1, (-\pi, \pi/2]), (x_2, (-\pi/2, 0]), (x_3, \emptyset)\}.$$

Moreover, we propose to using soft mapping to make the rotation and to find the rotated coordinate system as follows:

Let  $F : A \rightarrow P([-\pi, \pi])$  be a soft rotation and  $f_{ijk} : SE(F) \rightarrow SE(\mathbb{R})$  be a soft mapping such that for all  $\tilde{x} \in SE(F)$ ,  $f$  applies rotation by a certain Tait-Bryan angles convention  $ijk$  according to all  $x \in A$  and their corresponding angles. Then, for all  $x \in A$ , we obtain rotated coordinate system by this soft mapping, and here,  $f(\tilde{x}) \in SE(\mathbb{R})$ . Also, we indicate that  $f(SE(F))$  gives family of the rotated coordinate systems and  $SS(f(SE(F)))$  represents union of the rotated coordinate systems around three axes with respect to specified intervals or sets of angels by a certain Tait-Bryan angles convention.

For example,  $f_{321} : SE(F) \rightarrow SE(\mathbb{R})$  is a soft mapping that applies rotation by  $z - y - x$  convention known as yaw-pitch-roll. Then, for all  $\tilde{x} \in SE(F)$ , we express  $f_{321}$  analytically as

$$f_{321}(\tilde{x}) = \{(x_1, x'_1), (x_2, x'_2), (x_3, x'_3)\},$$

where

$$x'_1 = x_1 C_2 C_3 + x_2 (S_1 S_2 C_3 - C_1 S_3) + x_3 (S_1 S_3 + C_1 S_2 C_3),$$

$$x'_2 = x_1 C_2 S_3 + x_2 (S_1 S_2 S_3 + C_1 C_3) + x_3 (C_1 S_2 S_3 - S_1 C_3),$$

$$x'_3 = -x_1 S_2 + x_2 S_1 C_2 + x_3 C_1 C_2$$

such that for  $i = 1, 2, 3$ ,  $S_i$  and  $C_i$  represent  $\sin \tilde{x}(x_i)$  and  $\cos \tilde{x}(x_i)$ , respectively, for simplicity.

## CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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