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On classifying simplicial modules

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ABSTRACT

In this paper, we prove that the classifying functors $\overline{W}$ and $Diag \circ N$ are simplicially homotopy equivalent. Using this equivalency, we define classifying simplicial modules and give an alternative definition of (co)homology of crossed modules of algebras.

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Introduction

Given any simplicial group G, there is a reduced simplicial set, traditionally denoted $\overline{W}(G)$ and called the classifying space or classifying complex of G, which is a model for delooping of G and such that the functor $\overline{W}(-)$ is right adjoint to the standart simplicial loop space construction G. In [13], Kan introduced the classifying simplicial set $\overline{W}(G)$. This simplicial set consists of the followings:

- underlying sets are

$$\overline{W}(G)_n := G_n \times G_{n-1} \times \cdots \times G_0$$

- face maps are given by

$$d_i(g_n, g_{n-1}, \ldots, g_0) := \begin{cases} (d_i(g_n), d_{i-1}(g_{n-1}), \ldots, d_0(g_{n-i}).g_{n-i-1}, g_{n-i-2}, \ldots g_0) & \text{if } i < n \\ (d_n(g_n), d_{n-1}(g_{n-1}), \ldots, d_1(g_1)) & \text{if } i = n \end{cases}$$

- degeneracy maps are given by

$$s_i(g_n, g_{n-1}, \ldots, g_0) := (s_i(g_n), s_{i-1}(g_{n-1}), \ldots, s_0(g_{n-i}), e, g_{n-i-1}, \ldots g_0)$$

where e denotes the neutral element.

Additionally, the bisimplicial set $N(G)$ is obtained by applying the nerve functor for groups dimensionwise. From here, we can use the diagonal functor and give a simplicial set $(Diag \circ N)(G)$. A well known result, see Cegarra and Remedies [5], the classifying simplicial sets $\overline{W}(G)$ and $(Diag \circ N)(G)$ are weakly homotopy equivalent. In [16], Thomas proved that the Kan classifying simplicial sets $\overline{W}(G)$ and $(Diag \circ N)(G)$ are simplicially homotopy equivalent.

In this work, we give an analogue of Thomas result for simplicial algebras. That is, the main result of this work is to prove that the classifying simplicial R-modules $\overline{W}(\mathbf{E})$ and $(Diag \circ N)(\mathbf{E})$ are simplicially

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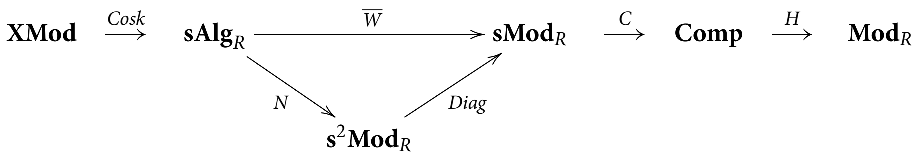
homotopy equivalent. Here, $\overline{W}(\mathbf{E})$ is defined by using tensor product operation in terms of underlying modules and using addition operation in terms of face maps as the modules are additive groups. For the simplicially homotopy equivalency, we show that $\overline{W}(\mathbf{E})$ is a strong simplicial deformation retract of $(Diag \circ N)(\mathbf{E})$.

So, we can say that the following diagram is commutative up to simplicial homotopy equivalence.



Here these notions meanings are as follows:

$$\begin{aligned} \mathbf{XMod} &: \text{ the category of crossed modules of algebras} \\ \boldsymbol{sA}\,\mathrm{lg}_R &: \text{ the category of simplicial } R\text{-algebras} \\ \boldsymbol{sMod}_R &: \text{ the category of simplicial } R\text{-modules} \\ \boldsymbol{s}^2\boldsymbol{Mod}_R &: \text{ the category of bisimplicial } R\text{-modules} \\ \boldsymbol{Comp} &: \text{ the category of chain complexes} \\ \boldsymbol{Mod}_R &: \text{ the category of } R\text{-modules} \end{aligned}$$

From the above commutative diagram, we obtain that the homology of a simplicial algebra and also crossed modules of algebras are obtained by composition of the functors in the upper row. Similarly for cohomology.

In order to get our goals, we organize the paper as follows;

In Section 1, we recall some needed definitions.

In Section 2, we define the analogues of Kan classifying functor $\overline{W}$ from the simplicial algebras to simplicial modules and show that the classifying simplicial R-modules $\overline{W}(\mathbf{E})$ and $(Diag \circ N)(\mathbf{E})$ are simplicially homotopy equivalent.

In Section 3, we give the definitions of classifying simplicial modules of simplicial algebras and crossed modules of algebras. Using the equivalency given in Section 2, we get an alternative definitions of (co)homology of simplicial algebras and crossed modules of algebras.

1. Preliminaries

We recall some standart definitions will be used in this work. For details, see [1, 3, 6, 11, 14].

Simplicial algebras

Suppose that R is a commutative ring with identity. A *simplicial (commutative) algebra* $\mathbf{E}$ consists of a family of algebras $\{E_n\}$ together with face and degeneracy maps $d_i = d_i^n : E_n \to E_{n-1},\ \ 0 \le i \le n,\ (n \neq 0)$ and $s_i = s_i^n : E_n \to E_{n+1},\ \ 0 \le i \le n$, satisfying the usual simplicial identities. It can be completely described as a functor $\Delta^{op} \to \mathbf{Alg}_{\mathrm{R}}$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and increasing maps. We have a category of simplicial algebras the above definitions and it is denoted by $\mathbf{sAlg}_{\mathrm{R}}$.

Consider the product $\Delta \times \Delta$ whose objects are pairs $([p],[q])$ and whose maps are pairs of weakly increasing maps. A functor $(\Delta \times \Delta)^{op} \to \mathbf{Alg}_{\mathrm{R}}$ is called a *bisimplicial algebra* with value in $\mathbf{Alg}_{\mathrm{R}}$. To give this functor is equivalent with giving for each (p, q) algebra $E_{p,q}$ and homomorphisms

$$\begin{aligned} d_i^h &: E_{p,q} \to E_{p-1,q} \\ s_i^h &: E_{p,q} \to E_{p+1,q} \qquad i : 0, \dots, p \\ d_j^v &: E_{p,q} \to E_{p,q-1} \\ s_j^v &: E_{p,q} \to E_{p,q+1} \qquad j : 0, \dots, q \end{aligned}$$

such that the maps d_i^h, s_i^h commute with d_j^v, s_j^v and d_i^h, s_i^h (resp. d_j^v, s_j^v) satisfy the usual simplicial identities. Here d_i^h, s_i^h denote the horizontal operators and d_j^v, s_j^v denote the vertical operators. We have a category of bisimplicial algebras from the above definitions and it is denoted by $\mathbf{s^2Alg}_{\mathrm{R}}$.

Here, we use the notation $d_{\lfloor j,i \rfloor} := d_j d_{j-1} \dots d_i$ resp. $s_{\lceil i,j \rceil} := s_i s_{i+1} \dots s_j$ for composites of faces resp. degeneracies.Moreover, in the rest of the paper, we use some specified brackets such that $\lfloor a, b \rfloor := (z \in \mathbb{Z} \mid a \geq z \geq b)$ for the descending interval and $\lceil a, b \rceil := (z \in \mathbb{Z} \mid a \leq z \leq b)$ for the ascending interval.

The Moore complex of a simplicial R-algebra

A *complex* M of R-algebras is a sequence

$$M = (\cdots \longrightarrow M_2 \xrightarrow{\partial} M_1 \xrightarrow{\partial} M_0)$$

of R-algebras M_n for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and algebra morphism ∂ such that $\partial\partial = 0$. So, we have a category of complexes of R-algebras denoted by $C(\mathbf{Alg}_{\mathrm{R}})$.

Given any complex of R-algebras M, we have $Z_n(M) := \ker(M_n \xrightarrow{\partial} M_{n-1})$ and $B_n(M) := \mathrm{Im}(M_{n+1} \xrightarrow{\partial} M_n)$ for $n \in \mathbb{N}_0$, where $M_{-1} = \{0\}$. If $B_n(M)$ is an ideal in $Z_n(M)$, written $B_n(M) \trianglelefteq Z_n(M)$, for all $n \in \mathbb{N}_0$, we can define the homology of M such that $H_n(M) := Z_n(M)/B_n(M)$ for $n \in \mathbb{N}_0$.

For any simplicial R-algebra $\mathbf{E}$, the *Moore complex* $N(\mathbf{E})$ of $\mathbf{E}$ is the normal chain complex of defined by

$$(N\mathbf{E})_n := \bigcap_{i=0}^{n-1} \ker d_i^n \leq E_n$$

for $n \in \mathbb{N}_0$ and differentials $\partial_n : (N\boldsymbol{E})_n \longrightarrow (N\boldsymbol{E})_{n-1}$ induced from d_n^n by restriction. In particular, $N_0(\mathbf{E}) = E_0$.

The n^{th} homotopy module $\pi_n(\boldsymbol{E})$ of $\boldsymbol{E}$ is the n^{th} homology module of the Moore complex of $\boldsymbol{E}$, i.e.,

Monoidal category

A monoidal category is a category $\mathcal{C}$ consists of
- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ called the tensor product or monoidal product,
- an object I called the unit object or identity object,
- three natural isomorphisms subject to certain coherence conditions expressing the fact that the tensor operation

  and satisfying following conditions:
- is associative: there is a natural isomorphism α, called associator, with components $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.
- has I as left and right identity: there are two natural isomorphisms λ and ρ, respectively called left and right unitor, with components $\lambda_A : I \otimes A \cong A$ and $\rho_A : A \otimes I \cong A$.

where coherence conditions known as commutativity of pentagon and triangle diagrams.

Note: Any category with finite products can be regarded as monoidal with the product as the monoidal product and the terminal object as the unit. Such a category is called a *cartesian monoidal category*.

Examples:

(i) $\mathcal{C}$ =**Set** (the category of sets) is a cartesian monoidal category with the cartesian product and one-element sets serving as the unit.

(ii) $\mathcal{C}$ = **Cat** (the category of small categories) is a cartesian monoidal category with the product category and the category with one object and only its identity morphism serving as the unit.

(iii) $\mathcal{C}$ = $\mathbf{Mod}_R$ (the category of R-modules) is a monoidal category with the tensor product of modules $\otimes_R$ serving as the monoidal product and the ring R serving as the unit.

(iv) $\mathcal{C}$ =$\mathbf{Alg}_R$ (the category of R-algebras) is a monoidal category with the tensor product of algebras as the monoidal product and R as the unit.

The category of monoid objects

A monoid object, in a monoidal category $(\mathcal{C}, \otimes, I)$, is an object E in $\mathcal{C}$ together with two morphisms

$$\begin{array}{llcll} \mu : & E\otimes E & \longrightarrow & E & \text{called multiplication,} \\ \eta : & I & \longrightarrow & E & \text{called unit,} \end{array}$$

such that the following diagrams commutative:

(i)

$$\begin{array}{ccccc} (E\otimes E)\otimes E & \xrightarrow{\alpha} & E\otimes(E\otimes E) & \xrightarrow{1\otimes\mu} & E\otimes E \\ \downarrow{\scriptstyle \mu\otimes 1} & & & & \downarrow{\scriptstyle \mu} \\ E\otimes E & & \xrightarrow[\mu]{\hspace{6em}} & & E \end{array}$$

(ii)

$$\begin{array}{ccccc} I\otimes E & \xrightarrow{\eta\otimes 1} & E\otimes E & \xleftarrow{1\otimes\eta} & E\otimes I \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \rho} & \\ & & E & & \end{array}$$

In the above notions, I is the unit element and α, λ and ρ are respectively the associativity, the left identity and right identity of the monoidal category $\mathcal{C}$.

Examples:

(a) A monoid object in $\mathcal{C}$ = **Sets** is a monoid.

(b) A monoid object in $\mathcal{C}$ = $\mathbf{Mod}_R$ is an R-algebra.

(c) A monoid object in $\mathcal{C}$ =$\mathbf{sMod}_R$ is a simplicial R-algebra.

(d) A monoid object in $\mathcal{C}$ =**Cat** is a small category $\mathcal{C}$ with two functors $\mu : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$ and $\eta : \mathcal{C}^{\otimes 0} \longrightarrow \mathcal{C}$ where $\mathcal{C}^{\otimes 0}$ is a category with one object and only its identity morphism.

A *monoid morphism* from C to D in $\mathcal{C}$ is a morphism $C \xrightarrow{\varphi} D$ such that the following diagrams commutative ($\mu^C\varphi = (\varphi\otimes\varphi)\mu^D$ and $\eta^C\varphi = \eta^D$):

$$\begin{array}{ccc} C\otimes C & \xrightarrow{\mu^C} & C \\ \downarrow{\scriptstyle \varphi\otimes\varphi} & & \downarrow{\scriptstyle \varphi} \\ D\otimes D & \xrightarrow[\mu^D]{} & D \end{array}$$

and

$$\begin{array}{ccc} I & \xrightarrow{\eta^C} & C \\ & \searrow^{\eta^D} & \downarrow \varphi \\ & & D \end{array}$$

where C and D are monoid objects.

So, we get a category of monoid objects from the above definitions and it is denoted by $\mathbf{Mon}(\mathcal{C})$.

The nerve of a monoid object

Let E be a monoid object . The nerve of E is the simplicial object $N(E)$ given by $N_r(E) := E^{\otimes r}$ for all $r \in \mathbb{N}_0$ with the faces $N_r(E) \xrightarrow{d_t} N_{r-1}(E)$ and degeneracies $N_r(E) \xrightarrow{s_t} N_{r+1}(E)$ are

$$d_t = \begin{cases} (pr_m)_{m\in\lfloor r-1,1\rfloor} & \text{if } t = 0 \\ (pr_m)_{m\in\lfloor r-1,t+1\rfloor} \cup ((pr_t \otimes pr_{t-1})\mu) \cup (pr_m)_{m\in\lfloor t-2,0\rfloor} & \text{if } t \in [1, r-1] \\ (pr_m)_{m\in\lfloor r-2,0\rfloor} & \text{if } t = r \end{cases}$$

for $t \in [0, r]$, $r \in \mathbb{N}$, $r \geq 2$, resp.

$$d_0 = d_1 = 0 \text{ for } r = 1$$

and

$$s_t = (pr_m)_{m\in\lfloor r-1,t\rfloor} \cup (1\eta) \cup (pr_m)_{m\in\lfloor t-1,0\rfloor}$$

for $t \in [0, r]$, $r \in \mathbb{N}_0$, where $E^{\otimes r} \xrightarrow{0} I$ denotes the unique morphism to the unit object under the augmentation map from E to I.

Remark. Since any R-algebra is a monoid object in $\mathbf{Mod}_R$, we can define the nerve of an algebra as a functor $\mathbf{Mon}(\mathbf{Mod}_R) \xrightarrow{N} \mathbf{sMod}_R$. It can be applied dimensionwise to a simplicial R-algebra as a functor $\mathbf{Mon}(\mathbf{sMod}_R) \xrightarrow{N} \mathbf{s}^2\mathbf{Mod}_R$.

From bisimplicial modules to simplicial modules (Diag functor)

Let $\mathcal{C}$ be a category. The functor

$$\mathbf{s}^2\mathcal{C} \xrightarrow{Diag} \mathbf{s}\mathcal{C}$$

is an induced functor $Diag := \mathbf{Cat}(\Delta, \mathcal{C})$ obtained by applying the hom-functor $\mathbf{Cat}(-, \mathcal{C})$ to the diagonal morphism

$$\Delta^{op} \xrightarrow{\Delta} \Delta^{op} \times \Delta^{op}$$

in $\mathbf{Cat}$.

Let X be a bisimplicial object in $\mathcal{C}$. $Diag(X)$ is called its *diagonal simplicial object*.

$$\begin{array}{ccc} & & \Delta^{op} \times \Delta^{op} \\ & \nearrow^{\Delta} & \downarrow X \\ \Delta^{op} & \xrightarrow[Diag(X)]{} & \mathcal{C} \end{array}$$

So, diagonal simplicial module $Diag(X)$ has entries $Diag_n(X) := X_{n,n}$ for $n \in \mathbb{N}_0$, while the faces are $d_k := d^h_k d^v_k = d^v_k d^h_k$ and degeneracies are $s_k := s^h_k s^v_k = s^v_k s^h_k$, respectively.

Diagonal of the nerve of the simplicial algebra (Diag ∘ N functor)

By using the well known definitions of nerve and diagonal functors, we can get the diagonal of the nerve of the simplicial algebra as follows:

Let $\mathbf{E}$ be simplicial R-algebra. The diagonal of the nerve of $\mathbf{E}$ is a simplicial R-module with $Diag_r N(\mathbf{E}) = E_r^{\otimes r} = \underbrace{E_r \otimes E_r \cdots \otimes E_r}_{r-th}$ and the faces and degeneracies are as follows

$$d_t((e_m)_{m\in\lfloor r-1,0\rfloor}) = (x_m)_{m\in\lfloor r-2,0\rfloor}$$

$$x_m := \begin{cases} d_t(e_{m+1}) & \text{if } m \in \lfloor r-2, t\rfloor \\ d_t(e_t) + d_t(e_{t-1}) & \text{if } m = t-1 \\ d_t(e_m) & \text{if } m \in \lfloor t-2, 0\rfloor \end{cases}$$

and

$$s_t((e_m)_{m\in\lfloor r-1,0\rfloor}) = (z_m)_{m\in\lfloor r,0\rfloor}$$

$$z_m := \begin{cases} s_t(e_{m-1}) & \text{if } m \in \lfloor r, t+1\rfloor \\ 0 & \text{if } m = t \\ s_t(e_m) & \text{if } m \in \lfloor t-1, 0\rfloor, \end{cases}$$

$(e_m)_{m\in\lfloor r-1,0\rfloor} = e_{r-1} \otimes e_{r-2} \otimes \cdots \otimes e_0$ and $(x_m)_{m\in\lfloor r-2,0\rfloor} = x_{r-2} \otimes x_{r-3} \otimes \cdots \otimes x_0$.

Notions from simplicial homotopy theory

Given any simplicial R-modules $\mathbf{X}$ and $\mathbf{Y}$, if there exists a simplicial module homomorphism $\mathbf{X} \times \Delta^1 \xrightarrow{H} \mathbf{Y}$ such that $ins_0 H = f$ and $ins_1 H = g$ where ins_0 and ins_1 are composite morphisms:

$$\mathbf{X} \xrightarrow{\cong} \mathbf{X} \times \Delta^0 \xrightarrow{id\times d^1} \mathbf{X} \times \Delta^1 \text{ resp. } \mathbf{X} \xrightarrow{\cong} \mathbf{X} \times \Delta^0 \xrightarrow{id\times d^0} \mathbf{X} \times \Delta^1$$

resp., simplicial module homomorphisms $f, g \in \mathbf{sMod}_R(\mathbf{X}, \mathbf{Y})$ are said to be *simplicially homotopic*, written $f \sim g$. Also, H is defined as *simplicial homotopy* from f to g.

If there exist simplicial module homomorphisms $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ and $\mathbf{Y} \xrightarrow{g} \mathbf{X}$ such that $fg \sim id_X$ and $gf \sim id_Y$, the simplicial R-modules $\mathbf{X}, \mathbf{Y}$ are called *simplicially homotopy equivalent*. Then we can denote $\mathbf{X} \simeq \mathbf{Y}$, f and g are called *simplicial homotopy equivalences*.

Finally, we suppose given a dimensionwise injective simplicial module homomorphism $Y \xrightarrow{i} X$. Y is a *simplicial deformation retract* of X if there is a simplicial module homomorphism $X \xrightarrow{r} Y$ such that $ir = id_Y$ and $ri \sim id_X$. Also, r is said to be a *simplicial deformation retraction*. If there is a homotopy $ri \xrightarrow{H} id_X$ which is constant along i, then we define Y is a *strong simplicial deformation retract* of X and r is a *strong simplicial deformation retraction.*

Crossed modules (of R-algebras)

In this paper, all algebras will be commutative. Also, we accept that k is a commutative ring, R is a k-algebra with identity.

A crossed module, (C, R, ∂), (or shortly crossed R-module) consists of an R-algebra C and a morphism $\partial : C \longrightarrow R$ with an action R on C, satisfying the following conditions:

($CM1$)

$$\partial(r \cdot c) = r\partial(c)$$

for all $r \in R$ and $c \in C$.

$(CM2)$

$$\partial(c)\cdot c' = cc'$$

for all $c, c' \in C$.

A morphism, $(\alpha,\beta) : V \longrightarrow V'$ of crossed modules consists of morphisms $\alpha : C \longrightarrow C'$ and $\beta : R \longrightarrow R'$ such that

$$\begin{array}{ll}(i) & \partial\beta = \alpha\partial' \\ (ii) & \alpha(r\cdot c) = \alpha(c)\cdot\beta(r)\end{array}$$

for all $c \in C, r \in R$ where $V = (C, R, \partial)$ and $V' = (C', R', \partial')$ are crossed modules.

So, we have a category of crossed modules from the above definitions denoted by **XMod**.

Examples:

(a) Let A be an R-algebra and I is an ideal of A. Then (I, A, i) is a crossed module with the multiplication action of A on I. Conversely, we induce an ideal from a given crossed module. Indeed, for a given crossed module (C, R, ∂), $\partial(C)$ is an ideal of R.

(b) Let M be an R-module. Then $(M, R, 0)$ is a crossed module. Conversely, for a crossed module (C, R, ∂), one can get that $\ker\partial$ is an $R/\partial(C)$-module.

For the other details about crossed modules of algebras you can see [17–19]. Now, we give the relation between the category of crossed modules and the category of simplicial algebras in the following theorem proved in [2]. By using this theorem and coskeleton functor, we get the classifying simplicial modules of crossed modules in the third section.

Theorem 1. [2] *The category of crossed modules is equivalent to the category of simplicial algebras with Moore complex of length 1.*

2. Simplicially homotopy equivalent classifying simplicial modules

The analogues of Kan classifying functor $\overline{W}$

In this section, we define the analogue of Kan classifying functor, $\overline{W}$, from the category of simplicial algebra to category of simplicial module. Then, we show that the functors $\overline{W}$ and $Diag \circ N$ are simplicial homotopy equivalent by using Thomas [16] method. Also, we obtain that the simplicial R-module $\overline{W}(\mathbf{E})$ is a strong simplicial deformation retract of $(Diag \circ N)(\mathbf{E})$ where $\mathbf{E}$ is a simplicial R-algebra.

Suppose that $\mathbf{E}$ is a simplicial R-algebra. The reduced simplicial R-module $\overline{W}(\mathbf{E})$ given by

$$\overline{W}_r(\mathbf{E}) := \bigotimes_{k\in\lfloor r-1,0\rfloor} E_k = E_{r-1}\otimes E_{r-2}\otimes\cdots\otimes E_0 \text{ for every } r\in\mathbb{N}_0$$

with the following faces and degeneracies:

$$d_t((e_m)_{m\in\lfloor r-1,0\rfloor}) = (f_m)_{m\in\lfloor r-2,0\rfloor}$$

where

$$f_m := \begin{cases} d_t(e_{m+1}) & \text{if } m\in\lfloor r-2,t\rfloor \\ d_t(e_t)+e_{t-1} & \text{if } m = t-1 \\ e_m & \text{if } m\in\lfloor t-2,0\rfloor \end{cases}$$

and

$$s_t((e_m)_{m\in\lfloor r-1,0\rfloor}) = (h_m)_{m\in\lfloor r,0\rfloor}$$

where

$$h_m := \begin{cases} s_t(e_{m-1}) & \text{if } m\in\lfloor r,t+1\rfloor \\ 0 & \text{if } m = t \\ e_m & \text{if } m\in\lfloor t-1,0\rfloor. \end{cases}$$

Here, we use the addition notation for the group operation because of Abelian group.

For following calculations, the composite of morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ is denoted by $X \xrightarrow{fg} Z$. The composite of functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} \mathcal{E}$ is denoted by $\mathcal{C} \xrightarrow{G \circ F} \mathcal{E}$.

Proposition 2. ***(a)***

$$D : Diag \circ N \longrightarrow \overline{W}$$

is a natural transformation such that $(D_{\mathbf{E}})_r = \bigotimes_{m \in \lfloor r-1,0 \rfloor} d_{\lfloor r,m+1 \rfloor} : E_r^{\otimes r} \longrightarrow \bigotimes_{m \in \lfloor r-1,0 \rfloor} E_m$ *for all* $r \in \mathbb{N}_0$, $\mathbf{E} \in Obs\mathbf{Alg}_R$.
(b) *D is a retraction and*

$$S : \overline{W} \longrightarrow Diag \circ N$$

is a coretraction such that $S_{\mathbf{E}}$ *is recursively given by* $(S_{\mathbf{E}})_r : \bigotimes_{m \in \lfloor r-1,0 \rfloor} E_m \longrightarrow E_r^{\otimes r}$, $(e_m)_{m \in \lfloor r-1,0 \rfloor} \longmapsto (y_m)_{m \in \lfloor r-1,0 \rfloor}$ *with*

$$y_m := \sum_{k \in \lfloor r-1,m \rfloor} d_{\lfloor k,m+1 \rfloor} s_{\lceil m,r-1 \rceil}(e_k) - \sum_{k \in \lceil m+1,r-1 \rceil} d_{\lfloor k,m+1 \rfloor} s_{\lceil m,k-1 \rceil}(y_k) \in E_r$$

for each $m \in \lfloor r-1,0 \rfloor$, $r \in \mathbb{N}_0$, $\mathbf{E} \in Obs\mathbf{Alg}_R$.

Proof. **(a)** Since the morphisms $(D_{\mathbf{E}})_r$ for $r \in \mathbb{N}_0$ are compatible with the faces and degeneracies of $\mathbf{E}$, (see [12], for details.)

$$(Diag \circ N)(\mathbf{E}) \xrightarrow{D_{\mathbf{E}}} \overline{W}(\mathbf{E})$$

is a simplicial algebra homomorphism. Also, $(D_{\mathbf{E}})_r = \bigotimes_{m \in \lfloor r-1,0 \rfloor} d_{\lfloor r,m+1 \rfloor}$ is a module homomorphism since all faces are module homomorphisms.

Additionally, given any simplicial algebra homomorphism $\varphi : \mathbf{E} \longrightarrow \mathbf{F}$

$$\begin{aligned}
(D_{\mathbf{E}})_r(\overline{W}_r\varphi) &= \left(\bigotimes_{m \in \lfloor r-1,0 \rfloor} d_{\lfloor r,m+1 \rfloor}\right)\left(\bigotimes_{m \in \lfloor r-1,0 \rfloor} \varphi_m\right) \\
&= \bigotimes_{m \in \lfloor r-1,0 \rfloor} d_{\lfloor r,m+1 \rfloor}\varphi_m \\
&= \bigotimes_{m \in \lfloor r-1,0 \rfloor} \varphi_r d_{\lfloor r,m+1 \rfloor} \\
&= \varphi_r^{\otimes r}\left(\bigotimes_{m \in \lfloor r-1,0 \rfloor} d_{\lfloor r,m+1 \rfloor}\right) \\
&= (Diag_r N\varphi)(D_{\mathbf{F}})_r
\end{aligned}$$

for all $r \in \mathbb{N}_0$. So, we get a commutative diagram that is D is a natural transformation.

$$\begin{array}{ccc}
(Diag \circ N)(\mathbf{E}) & \xrightarrow{D_{\mathbf{E}}} & \overline{W}(\mathbf{E}) \\
{\scriptstyle (Diag \circ N)\varphi}\downarrow & & \downarrow{\scriptstyle \overline{W}\varphi} \\
(Diag \circ N)(\mathbf{F}) & \xrightarrow[D_{\mathbf{F}}]{} & \overline{W}(\mathbf{F})
\end{array}$$

(b) Since the morphisms $(S_{\mathbf{E}})_r$ for $r \in \mathbb{N}_0$ are compatible with the faces and degeneracies of $\mathbf{E}$, (see [12], for details.)

$$\overline{W}(\mathbf{E}) \xrightarrow{S_{\mathbf{E}}} (Diag \circ N)(\mathbf{E}),$$

is a simplicial algebra homomorphism. Also, it is clear that $(S_{\mathbf{E}})_r$ are module homomorphisms from the definition.

Finally, we prove that $D_{\mathbf{E}}$ is a retraction with coretraction $S_{\mathbf{E}}$, that is,

$$(S_{\mathbf{E}})_r(D_{\mathbf{E}})_r = id_{\overline{W}_r(\mathbf{E})}$$

for all $r \in \mathbb{N}_0$.

Suppose that $(y_m)_{m\in\lfloor r-1,0\rfloor}$ denote the image of an element $(e_m)_{m\in\lfloor r-1,0\rfloor} \in \bigotimes_{m\in\lfloor r-1,0\rfloor} E_m$ under $(S_{\mathbf{E}})_r$, where

$$y_m := \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor} s_{\lceil m,r-1\rceil}(e_k) - \sum_{k\in\lceil m+1,r-1\rceil} d_{\lfloor k,m+1\rfloor} s_{\lceil m,k-1\rceil}(y_k)$$

So, we have

$$\begin{aligned}(S_{\mathbf{E}})_r(D_{\mathbf{E}})_r((e_m)_{m\in\lfloor r-1,0\rfloor}) &= (D_{\mathbf{E}})_r((y_m)_{m\in\lfloor r-1,0\rfloor})\\ &= \left(\bigotimes_{m\in\lfloor r-1,0\rfloor} d_{\lfloor r,m+1\rfloor}\right)((y_m)_{m\in\lfloor r-1,0\rfloor})\\ &= (d_{\lfloor r,m+1\rfloor}(y_m))_{m\in\lfloor r-1,0\rfloor}.\end{aligned}$$

Induction on $m \in \lfloor r-1,0\rfloor$ shows that

$$\begin{aligned}d_{\lfloor r,m+1\rfloor}(y_m) &= d_{\lfloor r,m+1\rfloor}\left(\begin{array}{c}\sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor} s_{\lceil m,r-1\rceil}(e_k) - \\ \sum_{k\in\lceil m+1,r-1\rceil} d_{\lfloor k,m+1\rfloor} s_{\lceil m,k-1\rceil}(y_k)\end{array}\right)\\ &= \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor} s_{\lceil m,r-1\rceil} d_{\lfloor r,m+1\rfloor}(e_k) - \\ &\quad \sum_{k\in\lceil m+1,r-1\rceil} d_{\lfloor k,m+1\rfloor} s_{\lceil m,k-1\rceil} d_{\lfloor r,m+1\rfloor}(y_k)\\ &= \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor}(e_k) - \sum_{k\in\lceil m+1,r-1\rceil} d_{\lfloor r,m+1\rfloor}(y_k)\\ &= \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor}(e_k) - \sum_{k\in\lceil m+1,r-1\rceil} d_{\lfloor k,m+1\rfloor}(e_k)\\ &= e_m.\end{aligned}$$

This implies that $(S_{\mathbf{E}})_r(D_{\mathbf{E}})_r = id_{\overline{W}_r(\mathbf{E})}$ for all $r \in \mathbb{N}_0$. □

Theorem 3. *Given any simplicial R-algebra* $\mathbf{E}$*, the simplicial R-module* $\overline{W}(\mathbf{E})$ *is a strong simplicial deformation retract of* $(Diag \circ N)(\mathbf{E})$ *with a strong simplicial deformation retraction given by*

$$D_{\mathbf{E}} : (Diag \circ N)(\mathbf{E}) \longrightarrow \overline{W}(\mathbf{E}).$$

Proof. We have the coretraction S. Now, we show that $D_{\mathbf{E}}S_{\mathbf{E}} \sim id_{(Diag\circ N)(\mathbf{E})}$ via a simplicial homotopy constant along $S_{\mathbf{E}}$. A simplicial homotopy H from $D_{\mathbf{E}}S_{\mathbf{E}}$ to $id_{(Diag\circ N)(\mathbf{E})}$ is given by

$$H_r(\mathbf{E}) : Diag_r N(\mathbf{E}) \times \Delta_r^1 \longrightarrow Diag_r N(\mathbf{E}),\ ((e_{r,m})_{m\in\lfloor r-1,0\rfloor}, \tau^{r+1-t}) \longmapsto (y_m^{(r+1-t)})_{m\in\lfloor r-1,0\rfloor}$$

for all $r \in \mathbb{N}_0$, where $t \in [0, r+1]$ and, defined by descending recursion,

$$y_m^{(r+1-t)} := \begin{cases} e_{r,m} & \text{if } m \in \lfloor r-1, t-1 \rfloor \cap \mathbb{N}_0 \\ \sum\limits_{k \in \lfloor t-2, m \rfloor} d_{\lfloor t-1, m+1 \rfloor} s_{\lceil m, t-2 \rceil}(e_{r,k}) - \\ \sum\limits_{k \in \lceil m+1, t-2 \rceil} d_{\lfloor k, m+1 \rfloor} s_{\lceil m, k-1 \rceil}(y_k^{(r+1-t)}) & \text{if } m \in \lfloor t-2, 0 \rfloor \end{cases}$$

We abbreviate $\widetilde{y_m} := y_m^{(r+1-t)}$ for the respective index $t \in [0, r]$ under consideration to facilities the following calculations.

Here, $H_r(\mathbf{E})$ yields a simplicial algebra homomorphism

$$H : (Diag \circ N)(\mathbf{E}) \times \Delta^1 \longrightarrow (Diag \circ N)(\mathbf{E}).$$

This can be checked from [12]. Now, we prove that $ins_0 H = D_{\mathbf{E}} S_{\mathbf{E}}$ and $ins_1 H = id_{(Diag \circ N)(\mathbf{E})}$. For $r \in \mathbb{N}_0$, $t \in [0, r+1]$, $(e_{r,m})_{m \in \lfloor r-1, 0 \rfloor} \in (Diag \circ N)_r(\mathbf{E})$, we have

$$\begin{aligned} (D_{\mathbf{E}})_r (S_{\mathbf{E}})_r ((e_{r,m})_{m \in \lfloor r-1, 0 \rfloor}) &= (S_{\mathbf{E}})_r (d_{\lfloor r, m+1 \rfloor}(e_{r,m}))_{m \in \lfloor r-1, 0 \rfloor} \\ &= (y_m)_{m \in \lfloor r-1, 0 \rfloor} \end{aligned}$$

with

$$\begin{aligned} y_m \; : &= \sum_{k \in \lfloor r-1, m \rfloor} d_{\lfloor r, k+1 \rfloor} d_{\lfloor k, m+1 \rfloor} s_{\lceil m, r-1 \rceil}(e_{r,k}) - \sum_{k \in \lceil m+1, r-1 \rceil} d_{\lfloor k, m+1 \rfloor} s_{\lceil m, k-1 \rceil}(y_k) \\ &= \sum_{k \in \lfloor r-1, m \rfloor} d_{\lfloor r, m+1 \rfloor} s_{\lceil m, r-1 \rceil}(e_{r,k}) - \sum_{k \in \lceil m+1, r-1 \rceil} d_{\lfloor k, m+1 \rfloor} s_{\lceil m, k-1 \rceil}(y_k) \end{aligned}$$

for all $m \in \lfloor r-1, 0 \rfloor$, and

$$H_r((e_{r,m})_{m \in \lfloor r-1, 0 \rfloor}, \tau^{r+1-t}) = ((y_m^{(r+1-t)})_{m \in \lfloor r-1, 0 \rfloor})$$

with

$$y_m^{(r+1-t)} := \begin{cases} e_{r,m} & \text{if } m \in \lfloor r-1, t-1 \rfloor \cap \mathbb{N}_0 \\ \sum\limits_{k \in \lfloor t-2, m \rfloor} d_{\lfloor t-1, m+1 \rfloor} s_{\lceil m, t-2 \rceil}(e_{r,k}) - \\ \sum\limits_{k \in \lceil m+1, t-2 \rceil} d_{\lfloor k, m+1 \rfloor} s_{\lceil m, k-1 \rceil}(y_k^{(r+1-t)}) & \text{if } m \in \lfloor t-2, 0 \rfloor. \end{cases}$$

But by descending induction on $m \in \lfloor r-1, 0 \rfloor$, we get

$$\begin{aligned} y_m &= \sum_{k \in \lfloor r-1, m \rfloor} d_{\lfloor r, m+1 \rfloor} s_{\lceil m, r-1 \rceil}(e_{r,k}) - \sum_{k \in \lceil m+1, r-1 \rceil} d_{\lfloor k, m+1 \rfloor} s_{\lceil m, k-1 \rceil}(y_k) \\ &= \sum_{k \in \lfloor r-1, m \rfloor} d_{\lfloor r, m+1 \rfloor} s_{\lceil m, r-1 \rceil}(e_{r,k}) - \sum_{k \in \lceil m+1, r-1 \rceil} d_{\lfloor k, m+1 \rfloor} s_{\lceil m, k-1 \rceil}(y_k^{(0)}) \\ &= y_m^{(0)}. \end{aligned}$$

Hence,

$$\begin{aligned} (ins_0)_r H_r((e_{r,m})_{m \in \lfloor r-1, 0 \rfloor}) &= H_r((e_{r,m})_{m \in \lfloor r-1, 0 \rfloor}, \tau^0) \\ &= ((y_m^{(0)}))_{m \in \lfloor r-1, 0 \rfloor} \\ &= (y_m)_{m \in \lfloor r-1, 0 \rfloor} \\ &= (D_{\mathbf{E}})_r (S_{\mathbf{E}})_r ((e_{r,m})_{m \in \lfloor r-1, 0 \rfloor}) \end{aligned}$$

and

$$\begin{aligned}(ins_1)_r H_r((e_{r,m})_{m\in\lfloor r-1,0\rfloor}) &= H_r((e_{r,m})_{m\in\lfloor r-1,0\rfloor},\tau^{r+1})\\ &= (y_m^{(r+1)})_{m\in\lfloor r-1,0\rfloor} = (e_{r,m})_{m\in\lfloor r-1,0\rfloor}\\ &= id_{(Diago\circ N)(\mathbf{E})}((e_{r,m})_{m\in\lfloor r-1,0\rfloor}).\end{aligned}$$

Now, we show that H is constant along $S_{\mathbf{E}}$. For $(e_m)_{m\in\lfloor r-1,0\rfloor}\in\overline{W_r}(\mathbf{E})$, we have

$$\begin{aligned}H_r((S_{\mathbf{E}})_r(e_m)_{m\in\lfloor r-1,0\rfloor},\tau^{r+1-t}) &= H_r((y_m)_{m\in\lfloor r-1,0\rfloor},\tau^{r+1-t})\\ &= (y_m^{(r+1-t)})_{m\in\lfloor r-1,0\rfloor},\end{aligned}$$

where

$$y_m := \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor}s_{\lceil m,r-1\rceil}(e_k) - \sum_{k\in\lceil m+1,r-1\rceil} d_{\lfloor k,m+1\rfloor}s_{\lceil m,k-1\rceil}(y_k)$$

and

$$y_m^{(r+1-t)} := \begin{cases} y_m & \text{if } m\in\lfloor r-1,t-1\rfloor\cap\mathbb{N}_0\\ \sum\limits_{k\in\lfloor t-2,m\rfloor} d_{\lfloor t-1,m+1\rfloor}s_{\lceil m,t-2\rceil}(y_k)- \sum\limits_{k\in\lceil m+1,t-2\rceil} d_{\lfloor k,m+1\rfloor}s_{\lceil m,k-1\rceil}(y_k^{(r+1-t)}) & \text{if } m\in\lfloor t-2,0\rfloor.\end{cases}$$

We also show that $y_m^{(r+1-t)}=y_m$ for all $m\in\lfloor r-1,0\rfloor$, $t\in[0,r+1]$. For $t\in\{r+1,0\}$, this follows since H is a simplicial homotopy from $D_{\mathbf{E}}S_{\mathbf{E}}$ to $id_{(Diago\circ N)(\mathbf{E})}$ and since $S_{\mathbf{E}}D_{\mathbf{E}}S_{\mathbf{E}}=S_{\mathbf{E}}$. So we may assume that $t\in\lfloor r,1\rfloor$ and have to show that $y_m^{(r+1-t)}=y_m$ for every $m\in\lfloor t-2,0\rfloor$. But we have

$$\begin{aligned}d_{\lfloor t-1,m+1\rfloor}s_{\lceil m,t-2\rceil}(y_m) &= d_{\lfloor t-1,m+1\rfloor}s_{\lceil m,t-2\rceil}\Bigg(\sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor}s_{\lceil m,r-1\rceil}(e_k)-\\ &\qquad \sum_{k\in\lceil m+1,r-1\rceil} d_{\lfloor k,m+1\rfloor}s_{\lceil m,k-1\rceil}(y_k)\Bigg)\\ &= \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor}s_{\lceil m,r-1\rceil}d_{\lfloor t-1,m+1\rfloor}s_{\lceil m,t-2\rceil}(e_k) -\\ &\qquad \sum_{k\in\lceil m+1,r-1\rceil} d_{\lfloor k,m+1\rfloor}s_{\lceil m,k-1\rceil}d_{\lfloor t-1,m+1\rfloor}s_{\lceil m,t-2\rceil}(y_k)\\ &= \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor}s_{\lceil m,t-1\rceil}s_{\lceil t,r-1\rceil}d_{\lfloor t-1,m+1\rfloor}s_{\lceil m,t-2\rceil}(e_k) -\\ &\qquad \sum_{k\in\lceil t,r-1\rceil} d_{\lfloor k,m+1\rfloor}s_{\lceil m,t-1\rceil}s_{\lceil t,k-1\rceil}d_{\lfloor t-1,m+1\rfloor}s_{\lceil m,t-2\rceil}(y_k) -\\ &\qquad \sum_{k\in\lceil m+1,t-1\rceil} d_{\lfloor k,m+1\rfloor}s_{\lceil m,k-1\rceil}d_{\lfloor t-1,m+1\rfloor}s_{\lceil m,t-2\rceil}(y_k)\\ &= \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor}s_{\lceil m,t-1\rceil}d_{\lfloor t-1,m+1\rfloor}s_{\lceil m+1,m+r-t\rceil}s_{\lceil m,t-2\rceil}(e_k) -\\ &\qquad \sum_{k\in\lceil t,r-1\rceil} d_{\lfloor k,m+1\rfloor}s_{\lceil m,t-1\rceil}d_{\lfloor t-1,m+1\rfloor}s_{\lceil m+1,m+k-t\rceil}s_{\lceil m,t-2\rceil}(y_k) -\\ &\qquad \sum_{k\in\lceil m+1,t-1\rceil} d_{\lfloor k,m+1\rfloor}d_{\lfloor m+t-1-k,m+1\rfloor}s_{\lceil m,k-1\rceil}d_{\lfloor k,m+1\rfloor}s_{\lceil m,t-2\rceil}(y_k)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor} s_m s_{\lceil m+1,m+r-t\rceil} s_{\lceil m,t-2\rceil}(e_k) - \\
&\quad \sum_{k\in\lceil t,r-1\rceil} d_{\lfloor k,m+1\rfloor} s_m s_{\lceil m+1,m+k-t\rceil} s_{\lceil m,t-2\rceil}(y_k) - \\
&\quad \sum_{k\in\lceil m+1,t-1\rceil} d_{\lfloor k,m+1\rfloor} d_{\lfloor m+t-1-k,m+1\rfloor} s_{\lceil m,t-2\rceil}(y_k) \\
&= \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor} s_{\lceil m,r-1\rceil}(e_k) - \sum_{k\in\lceil m+1,t-1\rceil} d_{\lfloor t-1,m+1\rfloor} s_{\lceil m,t-2\rceil}(y_k) - \\
&\quad \sum_{k\in\lceil t,r-1\rceil} d_{\lfloor k,m+1\rfloor} s_{\lceil m,k-1\rceil}(y_k)
\end{aligned}
$$

and this implies, by induction on $m \in \lfloor t-2, 0\rfloor$ that

$$
\begin{aligned}
y_m^{(r+1-t)} &= \sum_{k\in\lfloor t-2,m\rfloor} d_{\lfloor t-1,m+1\rfloor} s_{\lceil m,t-2\rceil}(y_k) - \sum_{k\in\lceil m+1,t-2\rceil} d_{\lfloor k,m+1\rfloor} s_{\lceil m,k-1\rceil}(y_k^{(r+1-t)}) \\
&= \sum_{k\in\lfloor t-2,m\rfloor} d_{\lfloor t-1,m+1\rfloor} s_{\lceil m,t-2\rceil}(y_k) - \sum_{k\in\lceil m+1,t-2\rceil} d_{\lfloor k,m+1\rfloor} s_{\lceil m,k-1\rceil}(y_k) \\
&= \sum_{k\in\lfloor t-2,m+1\rfloor} d_{\lfloor t-1,m+1\rfloor} s_{\lceil m,t-2\rceil}(y_k) + \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor} s_{\lceil m,r-1\rceil}(e_k) \\
&\quad - \sum_{k\in\lceil m+1,t-1\rceil} d_{\lfloor t-1,m+1\rfloor} s_{\lceil m,t-2\rceil}(y_k) - \sum_{k\in\lceil t,r-1\rceil} d_{\lfloor k,m+1\rfloor} s_{\lceil m,k-1\rceil}(y_k) \\
&\quad - \sum_{k\in\lceil m+1,t-2\rceil} d_{\lfloor k,m+1\rfloor} s_{\lceil m,k-1\rceil}(y_k) \\
&= \sum_{k\in\lfloor r-1,m\rfloor} d_{\lfloor k,m+1\rfloor} s_{\lceil m,r-1\rceil}(e_k) - \sum_{k\in\lceil m+1,r-1\rceil} d_{\lfloor k,m+1\rfloor} s_{\lceil m,k-1\rceil}(y_k) \\
&= y_m
\end{aligned}
$$

for all $m \in \lfloor t-2, 0\rfloor$. □

3. Classifying simplicial modules of crossed modules of algebras

The classical (co)homology of any group G is defined to be the (co)homology of the classifying space $B(G)$. Whitehead [20] has given an algebraic model for (path-connected) CW-space having trivial homotopy groups π_n for $n \geqq 3$. This model is called a crossed module.

In [7], Ellis considered the (co)homology of the space $B(M \xrightarrow{\partial} G)$ as the "right" (co)homology of the crossed module $\partial : M \to G$. Moreover the alternative (co)homology theory have been worked in several papers (see [4, 10, 15]). He also give some computer implemantations for computing homology of a simplicial group and finite crossed module of reasonably small order in [8, 9].

In this last section, we give an alternative definition of (co)homology of simplicial algebras and crossed modules of algebras by using (co)homology of their classifying simplicial R-modules.

For crossed R-module $V = (C \xrightarrow{\partial} R)$, we construct its (co)homology in two steps:

(i) We associate to V a simplicial R-algebra, its coskeleton $Cosk(V)$,
(ii) We construct a classifying simplicial R-module $B(\mathbf{E})$ for any simplicial R-algebra $\mathbf{E}$. Therefore we can define $B(V) = (B \circ Cosk)(V)$.

Firstly, we recall the definitions of classifying (bi)simplicial module and (co)homology of a simplicial R-algebra.

Let $\mathbf{E}$ be a simplicial R-algebra. $B^{(2)}(\mathbf{E}) := N(\mathbf{E})$ is called the *classifying bisimplicial module* of $\mathbf{E}$ and $B(\mathbf{E}) := (Diag \circ N)(\mathbf{E})$ is called the *classifying simplicial module* of $\mathbf{E}$.

Let $\mathbf{E}$ be a simplicial R-algebra, A be a commutative ring, M be an A-module and $n \in \mathbb{N}_0$ be a non-negative integer. The *n-th homology module of* $\mathbf{E}$ with coefficients in M over A is defined to be the n-th homology module of its classifying simplicial module, that is

$$H_n(\mathbf{E}, M; A) := H_n(B(\mathbf{E}), M; A).$$

Dually, we let

$$H^n(\mathbf{E}, M; A) := H^n(B(\mathbf{E}), M; A)$$

be the *n-th cohomology module* of $\mathbf{E}$ with coefficients in M over A.
Also, we abbreviate

$$\begin{aligned} H_n(\mathbf{E}, A) &:= H_n(\mathbf{E}, A; A) \\ H_n(\mathbf{E}, M) &:= H_n(\mathbf{E}, M; \mathbb{Z}) \\ H_n(\mathbf{E}) &:= H_n(\mathbf{E}, \mathbb{Z}; \mathbb{Z}), \end{aligned}$$

and

$$\begin{aligned} H^n(\mathbf{E}, A) &:= H^n(\mathbf{E}, A; A) \\ H^n(\mathbf{E}, M) &:= H^n(\mathbf{E}, M; \mathbb{Z}) \\ H^n(\mathbf{E}) &:= H^n(\mathbf{E}, \mathbb{Z}; \mathbb{Z}). \end{aligned}$$

Corollary 4. *Let* $\mathbf{E}$ *be a simplicial R-algebra, A be a commutative ring and M be a A-module. Homology and cohomology of* $\mathbf{E}$ *can be computed as*

$$H_n(\mathbf{E}, M; A) \cong H_n(\overline{W}(\mathbf{E}), M; A)$$

and

$$H^n(\mathbf{E}, M; A) \cong H^n_n(\overline{W}(\mathbf{E}), M; A)$$

for all $n \in \mathbb{N}_0$.

Proof. Since $B(\mathbf{E}) := (Diag \circ N)(\mathbf{E})$ and $(Diag \circ N)(\mathbf{E}) \simeq \overline{W}(\mathbf{E})$, we have

$$\begin{aligned} H_n(\mathbf{E}, M; A) &= H_n(B(\mathbf{E}), M; A) \\ &\cong H_n(\overline{W}(\mathbf{E}), M; A) \end{aligned}$$

and

$$\begin{aligned} H^n(\mathbf{E}, M; A) &= H^n(B(\mathbf{E}), M; A) \\ &\cong H^n(\overline{W}(\mathbf{E}), M; A). \end{aligned}$$ □

We know that a crossed module $V = (C, R, \mu)$ can be associated to a simplicial R-algebra by its coskeleton, $Cosk(V)$. So, we can give following definitions for crossed modules by similar way.

We define $B(V) := B(Cosk(V))$ and $B^{(2)}(V) := B^{(2)}(Cosk(V))$ to be *the classifying simplicial module* and *the classifying bisimplicial module of a crossed module* $V \in Ob\mathbf{XMod}$, respectively.

Let $V = (C, R, \mu)$ be a crossed module, A be a commutative ring, M be an A-module and $n \in \mathbb{N}_0$ be a non-negative integer. The *n-th homology module of* V with coefficients in M over A is defined to be the n-th homology module of its classifying simplicial module, that is

$$H_n(V, M; A) := H_n(B(V), M; A).$$

The *n-th cohomology module of* V with coefficients in M over A is defined to be

$$H^n(V, M; A) := H^n(B(V), M; A).$$

We abbreviate

$$\begin{aligned} H_n(V,A) &:= H_n(V,A;A) \\ H_n(V,M) &:= H_n(V,M;\mathbb{Z}) \\ H_n(V) &:= H_n(V,\mathbb{Z};\mathbb{Z}), \end{aligned}$$

and

$$\begin{aligned} H^n(V,A) &:= H^n(V,A;A) \\ H^n(V,M) &:= H^n(V,M;\mathbb{Z}) \\ H^n(V) &:= H^n(V,\mathbb{Z};\mathbb{Z}). \end{aligned}$$

Corollary 5. *Let V be a crossed module, A be commutative ring, M be an A-module. Then*

$$H_n(V,M;A) = H_n(Cosk(V),M;A)$$

and

$$H^n(V,M;A) = H^n(Cosk(V),M;A)$$

for all $n \in \mathbb{N}_0$.

Proof. We have

$$H_n(V,M;A) = H_n(B(V),M;A) = H_n(B(Cosk(V)),M;A) = H_n(Cosk(V),M;A)$$

and analogously,

$$H^n(V,M;A) = H^n(B(V),M;A) = H^n(B(Cosk(V)),M;A) = H^n(Cosk(V),M;A)$$

for all $n \in \mathbb{N}_0$. □

Corollary 6. *Also, we can give the above homology and cohomology as follows from the Corollary 4*

$$H_n(V,M;A) = H_n(Cosk(V),M;A) \cong H_n(\overline{W}(Cosk(V)),M;A)$$

and

$$H^n(V,M;A) = H^n(Cosk(V),M;A) \cong H^n(\overline{W}(Cosk(V)),M;A).$$

References

[1] Arvasi, Z., Porter, T. (1997). Higher dimensional peiffer elements in simplicial commutative algebras. *Theory Appl. Categories* 3:1–23.

[2] Arvasi, Z., Porter, T. (1998). Freeness conditions for 2-crossed modules of commutative algebras. *Appl. Categorical Struct.* 6:455–471.

[3] Brown, R., Loday, J. L. (1987). Van-Kampen theorems for diagram of spaces. *Topology* 26:311–335.

[4] Carrasco, P., Cegarra, A. M., Grandjean, A. R. (2002). (Co)Homology of crossed modules. *J. Pure Appl. Algebra* 168:147–176.

[5] Cegarra, A. M., Remedios, J. (2005). The relationship between the diagonal and the bar constructions on a bisimplicial set. *Topol. Appl.* 153(1):21–51.

[6] Curtis, E. B. (1971). Simplicial homotopy theory. *Adv. Math.* 6:107–209.

[7] Ellis, G. J. (1992). Homology of 2-types. *J. London Math. Soc.* 46:1–27.

[8] Ellis, G. J., Luyen, L. V. (2012). Computational homology of n-types. *J. Symb. Comput.* 47:1309–1317.

[9] Ellis, G. J., Luyen, L. V. (2014). Homotopy 2-types of low order. *J. Exp. Math.* 23:383–389.

[10] Grandjean, A. R., Ladra, M., Pirashvili, T. (2000). CCG-homology of crossed modules via classifying spaces. *J. Algebra* 229:660–665.

[11] Goerss, P. G., Jardine, J. F. (1999). *Simplicial Homotopy Theory*. Progress in Mathematics, Vol. 174. Basel-Boston-Berlin: Birkhauser.

[12] Ilgaz, E. (2018) (Co)Homology of crossed modules of algebras. PhD. Eskişehir Osmangazi University, Turkey.

[13] Kan, D. M. (1958) On homotopy theory and css groups. *Ann. Math.* 68:38–53.
[14] May, J. P. (1967). *Simplicial Objects in Algebraic Topology*. Mathematic Studies, Vol. 11. Princeton: Van Nostrand.
[15] Paoli, S. (2003) (Co)Homology of crossed modules with coefficients in a π_1-module. *Homol. Homotopy Appl.* 5(1):261–296.
[16] Thomas, S. (2008). The functors $\overline{W}$ and $Diag \circ Nerve$ are simplicially homotopy equivalent. *J. Homotopy Relat. Struct.* 3(1):359–378.
[17] Porter, T. (1986). Homology of commutative algebras and an invariant of Simis and Vasconceles. *J. Algebra* 99: 458–465.
[18] Porter, T. (1987). Some categorical results in the theory of crossed modules in commutative algebras. *J. Algebra*, 109:415–429.
[19] Porter, T. (2011). The crossed menagerie. http://ncatlab.org/timporter/files/menagerie11.pdf.
[20] Whitehead, J. H. C. (1949). Combinatorial homotopy I-II. *Bull. Amer. Math. Soc.* 55:213–245, 453–496.