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Approximation by Gamma type operators

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In this study, we introduce newly defined Gamma operators which preserve constants and $e^{2\mu\cdot}$, $\mu > 0$ functions. In accordance with this purpose, we focus on their approximation properties such as uniform convergence, rate of convergence, asymptotic formula, and saturation results. Superior properties of introduced operators have been tested both theoretically and numerically in certain senses to highlight the performance of the new constructions of Gamma operators.

KEYWORDS

exponential functions, Gamma-type operators, King approach, modulus of continuity, rate of convergence

MSC CLASSIFICATION

41A25; 41A36

1 | INTRODUCTION

One of the most significant events of the 1950s for the mathematicians studied on approximation theory was that by Korovkin, and independently, by Bohman which introduced one of the most important theorems that provide criteria to check whether a given sequence $(L_n)_{n\geq 1}$ of positive linear operators converge to the identity operator with regard to the uniform norm of the space $C[a,b]$, that is to say, whether it represents or not an approximation process. A considerable amount of literature has been published on approximation theory with the help of this results. One of them was given by King[1] in which a new construction of Bernstein operators preserving test functions $e_i = x^i$, $i = 0, 2$ was introduced. On the other hand, in most recent papers,[2,3] Acar et al have introduced new class of positive linear operators which preserve $e^{2a\cdot}$, $a > 0$ functions in opposition to the classical polynomial-type ones. In these studies, the authors have established that the new family of linear positive operators have superior properties which are proven theoretically. For more detailed information, we refer the readers to previous studies[4-6 and 7-15] and the references therein.

In approximation theory studies, Gamma operators which are introduced by Lupas and Muller[16] have been used extensively. In this study, motivated by Acar et al,[2,3,17] we introduce a refinement of Gamma operators which preserve constants and $e^{2\mu\cdot}$, $\mu > 0$ functions. Numerical experiments are also presented, highlighting the performance of the newly defined Gamma operators in the context of one dimensional approximation.

2 | CONSTRUCTION OF OPERATORS AND RELATED LEMMAS

The classical Gamma operators are defined by

$$G_n(f;x) = \frac{x^{n+1}}{n!}\int_0^\infty e^{-xu}u^n f\left(\frac{n}{u}\right)du, \quad x \in (0, \quad \infty), \quad n \in \mathbb{N},$$

for the function f such that the generalized integral is absolutely convergent.

A new construction of Gamma operators reproducing e_0 and $e^{2\mu\cdot}$, $\mu > 0$, considered here is defined by

$$G_n^{\mu}(f;x) = \frac{\beta^{n+1}(x;n,\mu)}{n!} \int_0^{\infty} e^{-\beta(x;n,\mu)u} u^n f\left(\frac{x^2 u}{n}\right) du, \quad x \in (0,\infty), \quad n \in \mathbb{N}, \tag{1}$$

where

$$\beta(x;n,\mu) = \frac{e^{\frac{2\mu x}{n+1}} 2\mu \frac{x^2}{n}}{e^{\frac{2\mu x}{n+1}} - 1},$$

which is obtained by both the conditions $G_n^{\mu}(e_0;x) = 1$ and $G_n^{\mu}\left(e^{2\mu t};x\right) = e^{2\mu x}$. Indeed,

$$\begin{aligned} G_n^{\mu}(e_0;x) &= \frac{\beta^{n+1}(x;n,\mu)}{n!} \int_0^{\infty} e^{-\beta(x;n,\mu)u} u^n du, \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} G_n^{\mu}\left(e^{2\mu t};x\right) &= \frac{\beta^{n+1}(x;n,\mu)}{n!} \int_0^{\infty} e^{-\left(\beta(x;n,\mu) - 2\mu \frac{x^2}{n}\right)u} u^n du, \\ &= \frac{\beta^{n+1}(x;n,\mu)}{\left(\beta(x;n,\mu) - 2\mu \frac{x^2}{n}\right)^{n+1}} = e^{2\mu x}. \end{aligned}$$

Hence, we have

$$e^{\frac{2\mu x}{n+1}} \left(\beta(x;n,\mu) - 2\mu \frac{x^2}{n}\right) = \beta(x;n,\mu),$$

which presents us explicit form

$$\beta(x;n,\mu) = \frac{e^{\frac{2\mu x}{n+1}} 2\mu \frac{x^2}{n}}{e^{\frac{2\mu x}{n+1}} - 1}. \tag{2}$$

Remark 1. Considering (2), we have the limit:

$$\lim_{n\to\infty} \beta(x;n,\mu) = \lim_{n\to\infty} \frac{e^{\frac{2\mu x}{n+1}} 2\mu \frac{x^2}{n}}{e^{\frac{2\mu x}{n+1}} - 1} = x.$$

Lemma 1. *Let* $\mu > 0, x \in (0,\infty)$. *Then,*

$$\begin{aligned} G_n^{\mu}(e_0;x) &= 1, \\ G_n^{\mu}\left(e^{\mu t};x\right) &= \left(\frac{\beta(x;n,\mu)}{\beta(x;n,\mu) - \frac{x^2\mu}{n}}\right)^{n+1}, \\ G_n^{\mu}\left(e^{2\mu t};x\right) &= e^{2\mu x}. \end{aligned} \tag{3}$$

Proof. Since $G_n^{\mu}(e_0, x) = \mathbf{1}$ and $G_n^{\mu}(e^{2\mu t}, x) = e^{2\mu x}$ were already shown in the construction of the operators, let us only give proof for the fact $G_n^{\mu}(e^{\mu t}, x)$. Indeed,

$$\begin{aligned} G_n^{\mu}\left(e^{\mu t};x\right) &= \frac{\beta^{n+1}(x;n,\mu)}{n!}\int_0^{\infty} e^{-\left(\beta(x;n,\mu)-\frac{x^2\mu}{n}\right)u}u^n du,\\ &= \frac{\beta^{n+1}(x;n,\mu)}{\left(\beta(x;n,\mu)-\frac{x^2\mu}{n}\right)^{n+1}n!}\int_0^{\infty} e^{-t}t^n dt,\\ &= \left(\frac{\beta(x;n,\mu)}{\beta(x;n,\mu)-\frac{x^2\mu}{n}}\right)^{n+1}. \end{aligned}$$

□

Lemma 2. *Let $\mu > 0, x \in (0,\infty)$ and $e_i(x) = x^i, i \in \mathbb{N}$. Then*

$$G_n^{\mu}(e_1;x) = \frac{n+1}{n}\frac{x^2}{\beta(x;n,\mu)}, \tag{4}$$

$$G_n^{\mu}(e_2;x) = \frac{(n+2)(n+1)x^4}{\beta^2(x;n,\mu)n^2}. \tag{5}$$

Furthermore, if we define the central moment operator by $m_n^s(x) := G_n^{\mu}\left((t-x)^s;x\right), s \in \mathbb{N}$, then we get

$$\lim_{n\to\infty} nm_n^1(x) = -x^2\mu, \tag{6}$$

$$\lim_{n\to\infty} nm_n^2(x) = x^2. \tag{7}$$

Proof. Using the definition of the operators G_n^{μ} we get by following calculations that

$$G_n^{\mu}(e_1;x) = \frac{n+1}{n}\frac{x^2}{\beta(x;n,\mu)}.$$

By a similar way, we can also get

$$\begin{aligned} G_n^{\mu}(e_2;x) &= \frac{\beta^{n+1}(x;n,\mu)}{n!}\frac{x^4}{n^2}\int_0^{\infty} e^{-\beta(x;n,\mu)u}u^{n+2}du\\ &= \frac{(n+2)(n+1)}{\beta^2(x;n,\mu)}\frac{x^4}{n^2}. \end{aligned}$$

The equalities (6) and (7) can be obtained as a result of (4) and (5). □

Remark 2. Hence, we immediately have

$$G_n^{\mu}(e_1;x) \to e_1(x),$$

$$G_n^{\mu}(e_2;x) \to e_2(x),$$

as $n \to \infty$.

Lemma 3. *For all $\lambda \in \mathbb{R}$, $\mu > 0, x \in (0,\infty)$,*

$$\lim_{n\to\infty} G_n^{\mu}\left(e^{-\lambda t};x\right) = e^{-\lambda x}$$

holds.

Proof. Thanks to definition of the operators G_n^{μ} we get

$$G_n^{\mu}\left(e^{-\lambda t};x\right) = \frac{\beta^{n+1}(x;n,\mu)}{n!}\int_0^{\infty} e^{-\left(\beta(x;n,\mu)+\lambda\frac{x^2}{n}\right)u}u^n du,$$
$$= \frac{\beta^{n+1}(x;n,\mu)}{\left(\beta(x;n,\mu)+\frac{\lambda x^2}{n}\right)^{n+1}},$$

which follows for all $\lambda \in \mathbb{R}$ that

$$\lim_{n\to\infty} G_n^{\mu}\left(e^{-\lambda t};x\right) = e^{-\lambda x}.$$

□

The remainder of this work is organized as follows: In Section 2, the uniform convergence of newly defined Gamma operators is reviewed, while the rate of convergence is examined. Weighted approximation and pointwise convergence are given in the same section. In Section 3, we discuss some comparisons with numerical and graphical examples. The last section (Sect. 4) presents brief conclusions on advantages of newly constructed operators.

3 | MAIN RESULTS

This section is devoted to uniform convergence of the sequence of linear positive operators $G_n^{\mu}(f)$ to f belonging to weighted space of functions and the rate of convergence of this convergence. The first result is on uniform convergence of the operators $G_n^{\mu}(f)$ for functions belonging to unweighted spaces, that is, the subspace of $C[0,\infty)$, (space of continuous functions on $[0,\infty)$), having the property $\lim_{x\to\infty} f(x)$ exists and is finite. The space of such functions will be denoted by $C_*[0,\infty)$ which is endowed with the uniform norm $\|\cdot\|_{\infty}$. According to result of Boyanov and Veselinov,[18] a method to check uniform convergence of any sequence of positive linear operators for the functions $f \in C_*[0,\infty)$ is as

Theorem 1. *The sequence* $A_n : C_*[0,\infty) \to C_*[0,\infty)$ *of positive linear operators satisfies the conditions*

$$\lim_{n\to\infty} A_n\left(e^{-kt};x\right) = e^{-kx}, k = 0,1,2,$$

uniformly in $[0,\infty)$ *if and only if*

$$\lim_{n\to\infty} A_n(f;x) = f(x) \tag{8}$$

uniformly in $[0,\infty)$, *for all* $f \in C_*[0,\infty)$.

To use the above theorem, we first give the following remark.

Remark 3. Let $n \in \mathbb{N}$, $\mu > 0$ be fixed. $x, u \in [0,\infty)$,

$$K(x,u) := \frac{1}{n!}\beta^{n+1}(x;n,\mu)\,u^n e^{-\beta(x;n,\mu)u}.$$

Then

$$G_n^{\mu}(f;x) = \int_0^{\infty} K(x,u)\, f\left(\frac{x^2u}{n}\right)du, \quad G_n^{\mu}(e_0;x) = 1.$$

$\beta^{n+1}(x;n,\mu)$ vanishes for $x = 0$ and has polynomial growth at infinity; the same is obviously true for u^n. Considering these two functions and the exponent of $e^{-\beta(x;n,\mu)u}$, we conclude that $K(x,u)$ is bounded, ie, $\exists C = C(n,\mu)$ such that $K(x,u) \le C$, $x,u \ge 0$. Let $f \in C[0,\infty)$, $f \ne \mathbf{0}$, $\lim_{x\to\infty} f(x) = 0$. Fix $\varepsilon > 0$. Then $\exists M > 0$ such that $|f(x)| \le \frac{\varepsilon}{2}$, $\forall x \ge M$. Let $a := \frac{\varepsilon}{2C\|f\|_{\infty}}$ and $x \ge \left(\frac{nM}{a}\right)^{1/2}$. We have

$$\int_a^{\infty} K(x,u)\,du \le \int_0^{\infty} K(x,u)\,du = 1,$$

and $\frac{x^2u}{n} \geq \frac{x^2a}{n} \geq M$, hence $\left|f\left(\frac{x^2u}{n}\right)\right| \leq \frac{\varepsilon}{2}, \forall u \geq a$. Now,

$$\begin{aligned}\left|G_n^{\mu}(f;x)\right| &\leq \int_0^{\infty} K(x,u)\left|f\left(\frac{x^2u}{n}\right)\right| du \\ &\leq \int_0^{a} K(x,u)\,\|f\|_{\infty} du + \int_a^{\infty} K(x,u)\frac{\varepsilon}{2} du \\ &\leq aC\|f\|_{\infty} + \frac{\varepsilon}{2}\int_a^{\infty} K(x,u)\,du \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

that is, $\left|G_n^{\mu}(f;x)\right| \leq \varepsilon, \forall x \geq \left(\frac{nM}{a}\right)^{1/2}$, ie, $\lim_{x\to\infty} G_n^{\mu}(f;x) = 0$; hence, $G_n^{\mu}(f) \in C^*[0,\infty)$, $G_n^{\mu} : C^*[0,\infty) \to C^*[0,\infty)$. Hence, the following theorem can be immediately obtained.

Theorem 2. *Let $\mu > 0$. For the sequence of operators $G_n^{\mu} : C_*(0,\infty) \to C_*(0,\infty)$, the convergence $G_n^{\mu}(f;x) \to f(x)$ as $n \to \infty$ is uniformly in $(0,\infty)$, for all $f \in C_*(0,\infty)$.*

Proof. Let define $f_k = e^{-kx}$, for $k = 0,1,2$, throughout the paper. Due to Theorem 1, let us check $\sup_{x\in[0,\infty)}\left|G_n^{\mu}(f_\lambda(t);x) - f_\lambda(x)\right|$ for $\lambda = 0,1,2$. Since $G_n^{\mu}(f_0(t);x) = G_n^{\mu}(e_0(t);x) = 1$, obviously

$$\sup_{x\in[0,\infty)}\left|G_n^{\mu}(1;x) - 1\right| = 0.$$

Furthermore, by Lemma 3, we get with serial expansions that

$$\|G_n^{\mu}(f_1) - f_1\|_{\infty} \leq \frac{2}{n}(2\mu+1)e^{-2} + \frac{1}{4n^2}\left[1283(2\mu+1)e^{-4} + 36(\mu+1)e^{-3} + 8e^{-2}\right] + \mathcal{O}(n^{-3}) := \beta_n$$

which is indeed $\mathcal{O}(n^{-1})$. Here, we note that n-th term ($:= a_n$ (say)) of serial expansion at right hand side will be

$$a_n \leq \frac{1}{n^n}\frac{n^n}{e^n}c = \frac{c}{e^n}, c \text{ is a constant.}$$

Finally, we also get by similar manner that

$$\|G_n^{\mu}(f_2) - f_2\| \leq \frac{2(\mu+1)}{n}e^{-2} + \frac{2}{n^2}(\mu+1)\left[16(\mu+1)e^{-4} + \frac{9}{4}e^{-3} + e^{-2}\right] + \mathcal{O}(n^{-3}) := \gamma_n,$$

which is also $\mathcal{O}(n^{-1})$. Since both the sequences (β_n) and (γ_n) tend to zero uniformly in n on $[0,\infty)$, we get the desired result. □

An estimation for the rate of convergence in (8) for a sequence of linear positive operators fulfilling conditions in Theorem 1 was stated by Holhos[19] in terms of modulus of continuity $\omega^*(f;\delta)$ defined, for every $\delta > 0$ and $f \in C_*[0,\infty)$, by

$$\omega^*(f;\delta) = \sup_{x,t\geq 0, |e^{-x}-e^{-t}|\leq\delta} |f(x) - f(t)|$$

which have the property

$$\omega^*(f;\delta) = \omega(\Phi(f);\delta),$$

where $\omega(\cdot;\delta)$ stands for the usual modulus of continuity and $\Phi : C_*[0,\infty) \to C[0,1]$ is an isometric isomorphism defined by setting

$$\Phi(f)(t) = \begin{cases} f(-\ln t), & 0 < t \leq 1 \\ \lim\limits_{x\to\infty} f(x), & t = 0 \end{cases}, \quad \text{for every } f \in C_*[0,\infty).$$

Hence, the above mentioned quantitative result is given in Holhos[19] as

Theorem 3. *Let $A_n : C_* [0, \infty) \to C_* [0, \infty)$ be a sequence of positive linear operators and set*

$$\begin{aligned} \|A_n(e_0) - e_0\|_\infty &= \alpha_n, \\ \|A_n(f_1) - f_1\|_\infty &= \beta_n, \\ \|A_n(f_2) - f_2\|_\infty &= \gamma_n. \end{aligned}$$

Under the hypothesis that all sequences α_n, β_n, γ_n vanish at infinity, the following estimate holds:

$$\|A_n(f) - f\|_\infty \leq \|f\|_\infty \alpha_n + (2 + \alpha_n)\, \omega^* \left(f; \sqrt{\alpha_n + 2\beta_n + \gamma_n}\right),$$

for every $f \in C_* [0, \infty)$.

As a consequence of Theorem 3, since $\left\|G_n^\mu(e_0) - e_0\right\|_\infty = 0$, we have the following corollary.

Corollary 1. *For $f \in C_* (0, \infty)$, the inequality*

$$\left\|G_n^\mu(f) - f\right\|_\infty \leq 2\omega^* \left(f; \sqrt{2\beta_n + \gamma_n}\right), \tag{9}$$

where β_n and γ_n are the same as indicated in the proof of Theorem 2. Here, we note that the inequality (9) states also uniform convergence $G_n^\mu(f) \to f$ since β_n and γ_n uniformly converge to zero. Furthermore, since both β_n and γ_n are $\mathcal{O}(n^{-1})$, the rate of uniform convergence (9) is $1/\sqrt{n}$.

When we consider weighted spaces of functions, uniform approximation and rate of convergence can be also described. Let $\varphi(x) = 1 + e^{2\mu x}$, $x \in \mathbb{R}^+$. We shall consider the class of functions:

$$\begin{aligned} B_\varphi\left(\mathbb{R}^+\right) &= \left\{f : \mathbb{R}^+ \to \mathbb{R} \,|\, |f(x)| \leq M_f \varphi(x), x \geq 0\right\}, \\ C_\varphi\left(\mathbb{R}^+\right) &= C\left(\mathbb{R}^+\right) \cap B_\varphi\left(\mathbb{R}^+\right), \\ C_\varphi^k\left(\mathbb{R}^+\right) &= \left\{f \in C_\varphi\left(\mathbb{R}^+\right) : \lim_{x\to\infty} \frac{f(x)}{\varphi(x)} = k_f \text{ exists and it is finite}\right\}, \end{aligned}$$

where M_f and k_f are constants depending only on f. The above all three spaces of functions are normed spaces with the norm

$$\|f\|_\varphi = \sup_{x\in\mathbb{R}^+} \frac{|f(x)|}{\varphi(x)}.$$

Remark 4. For the functions $f \in C_\varphi\left(\mathbb{R}^+\right)$, the definition of the operators $G_n^\mu(f)$ immediately yields

$$\left\|G_n^\mu(f)\right\|_\varphi \leq \|f\|_\varphi$$

which means that the sequence of the operators G_n^μ is an approximation process from $C_\varphi\left(\mathbb{R}^+\right)$ to $C_\varphi\left(\mathbb{R}^+\right)$.

Considering the above spaces of functions, the weighted approximation of sequences of linear positive operators acting on unbounded intervals is presented in Gadziev.[20] Similar way direct us to the following theorem.

Theorem 4. *For each function $f \in C_\varphi^k\left(\mathbb{R}^+\right)$,*

$$\lim_{n\to\infty} \left\|G_n^\mu(f) - f\right\|_\varphi = 0$$

holds.

Proof. According to establishment presented in Gadziev,[20] provisions of the conditions

$$\lim_{n\to\infty} \left\|G_n^\mu\left(e^{k\mu\cdot}\right) - e^{k\mu\cdot}\right\|_\varphi = 0, \quad k = 0, 1, 2$$

are sufficient to prove uniform convergence of corresponding operators. Since $G_n^\mu(1;x) = 1$ and $G_n^\mu\left(e^{2\mu t};x\right) = e^{2\mu x}$, the condition follows immediately for $k = 0$ and $k = 2$. To finalize the proof, let us show provision for $k = 1$. By basic calculation one can easily obtain the following expansion:

$$\frac{G_n^\mu(e^{\mu \cdot}) - e^{\mu \cdot}}{1 + e^{2\mu x}} = -\frac{1}{2n}\frac{e^{\mu x}\mu^2 x^2}{1 + e^{2\mu x}} + \frac{1}{8n^2}\frac{e^{\mu x}\mu^2 x^2(\mu^2 x^2 + 4)}{1 + e^{2\mu x}} + \mathcal{O}(n^{-3}),$$

which yields

$$\lim_{n\to\infty}\left\|G_n^\mu(e^{\mu \cdot}) - e^{\mu \cdot}\right\|_\varphi = \lim_{n\to\infty}\sup_{x\in\mathbb{R}^+}\frac{\left|\left(\frac{\beta(x;n,\mu)}{\beta(x;n,\mu)-\frac{x^2\mu}{n}}\right)^{n+1} - e^{\mu x}\right|}{\varphi(x)} = 0.$$

To be more precise, one can explicitly demonstrate that although the numerators of the above expansion are polynomial, the denominators have exponential growth. For this reason, it can be said that this expression is uniformly bounded. Thus, the proof is completed. □

To describe rate of pointwise convergence in weighted spaces, let us obtain Voronovskaya theorem for the operators G_n^μ as follows:

Theorem 5. *Let $f \in C_\varphi\left(\mathbb{R}^+\right)$ and f is two times differentiable functions at any $x \in \mathbb{R}^+$ and f'' is continuous at $x \in \mathbb{R}^+$, then we have*

$$\lim_{n\to\infty} 2n\left[G_n^\mu(f;x) - f(x)\right] = -2x^2\mu f'(x) + x^2 f''(x). \tag{10}$$

Proof. By the Taylor's formula, there exists ξ lying between x and t such that

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)(t-x)^2}{2} + h(t,x)(t-x)^2, \tag{11}$$

where

$$h(t,x) = \frac{f''(\xi) - f''(x)}{2}$$

which is continuous function which vanishes as $t \to x$. When we apply the operators G_n^μ to both sides of equality (11), we have

$$G_n^\mu(f;x) - f(x) = f'(x)m_n^1(x) + \frac{f''(x)}{2}m_n^2(x) + G_n^\mu\left(h(t,x)(t-x)^2;x\right)$$

and

$$n\left[G_n^\mu(f;x) - f(x)\right] = f'(x)nm_n^1(x) + \frac{f''(x)}{2}nm_n^2(x) + nG_n^\mu\left(h(t,x)(t-x)^2;x\right).$$

On the other hand, by Lemma 2, we get

$$\lim_{n\to\infty} nm_n^1(x) = -x^2\mu,$$
$$\lim_{n\to\infty} nm_n^2(x) = x^2.$$

Furthermore, according to inequality[3, p1402]

$$\left|h(t,x)(t-x)^2\right| \le \varepsilon m_n^2(x) + \frac{M}{\delta^2}m_n^4(x)$$

and since $m_n^4(x) = \mathcal{O}\left(n^{-2}\right)$, we conclude that

$$\lim_{n\to\infty} nG_n^\mu\left(h(t,x)(t-x)^2;x\right) = 0.$$

Hence, we have the desired result. □

4 | COMPARISONS AND NUMERICAL AND GRAPHICAL EXAMPLES

In this section, firstly, we compare the approximation properties of newly defined Gamma operators G_n^{μ} with the classical correspondents to show that the new ones superior properties. Then, in order to verify the theoretical results of introduced operators, we present some numerical experiments.

Theorem 6. *Let $f \in C^2(0,\infty)$. Assume that there exists $n_0 \in \mathbb{N}$ such that*

$$f(x) \le G_n^{\mu}(f;x) \le G_n(f;x) \text{ for all } n \ge n_0, x \in (0,\infty). \tag{12}$$

Then,

$$f''(x) \ge 2\mu f'(x) \ge 0. \tag{13}$$

Conversely, if inequality (13) holds at a given point $x \in (0,\infty)$, then there exists $n_0 \in \mathbb{N}$ such that

$$f(x) \le G_n^{\mu}(f;x) \le G_n(f;x) \quad \text{for all } n \ge n_0.$$

Proof. By the inequality (12), we have

$$0 \le 2n\left(G_n^{\mu}(f;x) - f(x)\right) \le 2n\left(G_n(f;x) - f(x)\right).$$

Then using (10), we get

$$0 \le -2\mu f'(x) + f''(x) \le f''(x).$$

Thus, we immediately have the inequality (13).

For the converse part, let us assume that the inequality (13) holds which yields that

$$0 \le -2\mu f'(x) + f''(x) \le f''(x)$$

which direct us to inequality (12). □

4.1 | Numerical experiments

Let us now describe the numerical experiments which were performed in the MATLAB. Here, we give numerical results for both classical Gamma operators and newly defined one in order to compare them. Additionally, we present the convergence behavior of new Gamma operator for different values of n to show that it convergence when n is increasing.

In Figure 1, we graph the results of classical Gamma operators, newly defined Gamma operators, and target function. Obviously, proposed operator shows better convergence behavior than the its classic corresponds to the target functions.

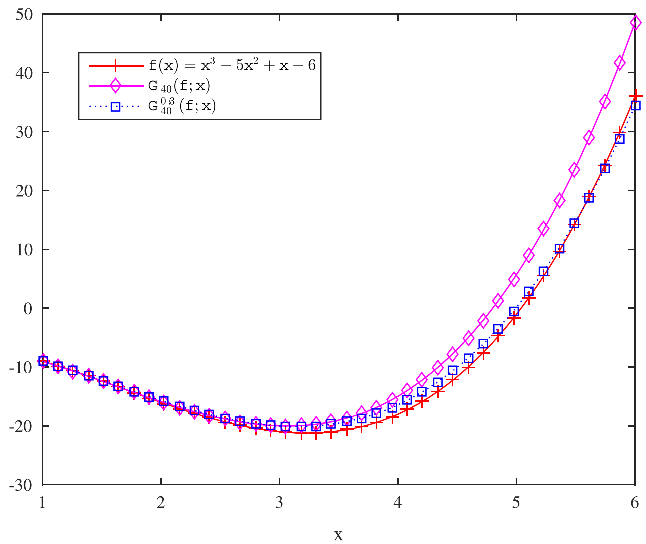

FIGURE 1 $f(x) = x^3 - 5x^2 + x - 6$, classical Gamma operator, and newly defined Gamma operator versus x with $\mu = 0.3$: Exact function (red-plus), classical Gamma operator (purple-diamond), and newly defined Gamma operator (blue-square) on equally spaced evaluation grid [Colour figure can be viewed at wileyonlinelibrary.com]

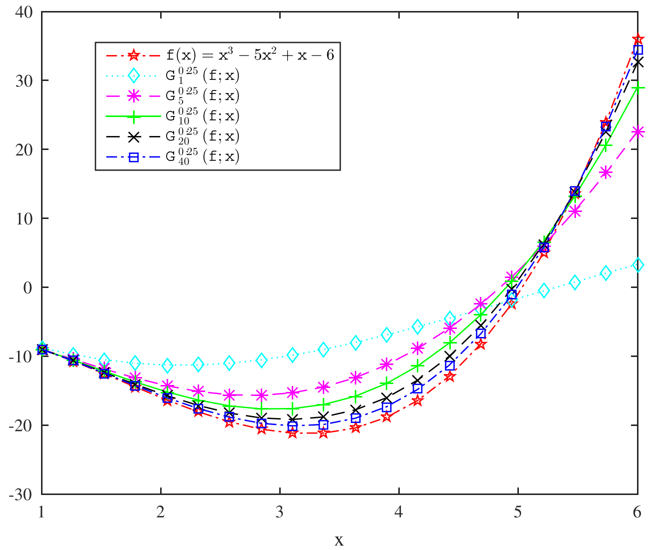

FIGURE 2 $G_n^{\mu}(f;x)$ versus x with $\mu = 0.3$ for $f(x) = x^3 - 5x^2 + x - 6$ and different value of n, say $n = 1, 5, 10, 20, 40$ on equally spaced evaluation grid [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 Root mean square errors for classical Gamma operators and newly defined Gamma operators with $\mu = 0.3$ for different value of n, test function $f(x) = x^3 - 5x^2 + x - 6$, on an equally spaced 5000 evaluation grid

n	RMS Error for $G_n(f;x)$	RMS Error for $G_n^{\mu}(f;x)$
1	5.064254e+00	1.519713e-01
5	1.453463e+00	7.507534e-02
10	7.560564e-01	4.639801e-02
20	5.502076e-01	2.654623e-02
30	9.495942e-02	9.862906e-03
40	6.939498e-02	4.435551e-03

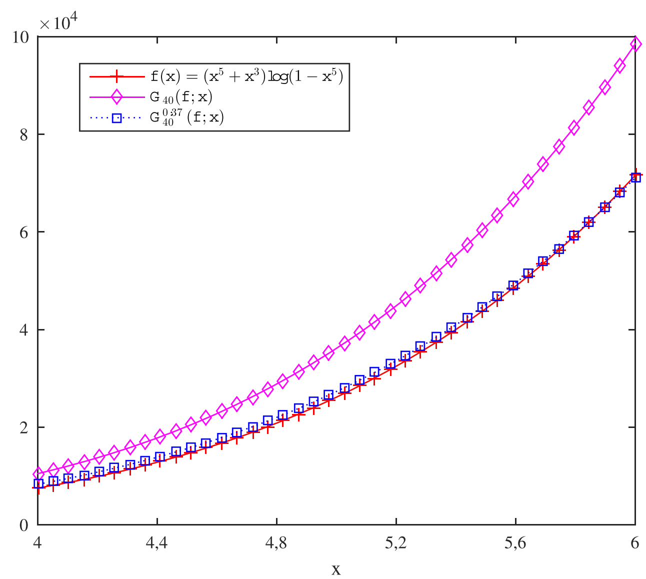

FIGURE 3 $f(x) = (x^5 + x^3)\log(1 - x^5)$, classical Gamma operator, and newly defined Gamma operator versus x with $\mu = 0.37$: Exact function (red-plus), classical Gamma operator (purple-diamond), and newly defined Gamma operator (blue-square) on equally spaced evaluation grid [Colour figure can be viewed at wileyonlinelibrary.com]

Figure 2 shows convergence plots for the proposed method for $n = 1, 5, 10, 20, 40$ with $\mu = 0.3$ for all levels. Clearly, the observed increase in convergence rate of approximations of introduced operators could be attributed to increasing of n as might be expected.

Here, *RMS Error for* $G_n(f;x)$ and *RMS Error for* $G_n^{\mu}(f;x)$ are root mean square (L-norm) errors for $G_n(f;x)$ and $G_n^{\mu}(f;x)$, respectively. We consider the problem with a fixed dimension and investigate the error behavior with respect to the different values of n and $\mu = 0.3$. We see in Table 1 that, as expected, the new construction of Gamma operators achieves better convergence when compared with classical Gamma operators with the same level of n. The error is tested on a 5000 uniform grid.

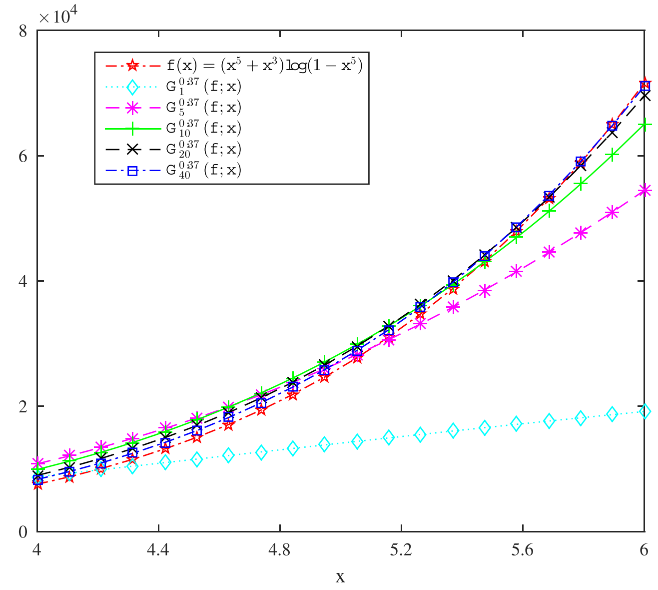

FIGURE 4 $G_n^{\mu}(f;x)$ versus x with $\mu = 0.37$ for $f(x) = (x^5 + x^3)\log(1 - x^5)$ and different value of n, say $n = 1, 5, 10, 20, 40$ on equally spaced evaluation grid [Colour figure can be viewed at wileyonlinelibrary.com]

Now, we consider another test function $f(x) : [4, 6] \to \mathbb{R}$, given by

$$f(x) = (x^5 + x^3)\log(1 - x^5),$$

with $\mu = 0.37$. Similarly, Figure 3 has also confirmed that the newly defined Gamma operators are generally superior to the standard Gamma operators in terms of convergence rate. On the other hand, Figure 4, as expected, confirms that the bigger n gives better convergence.

5 | CONCLUSION

The newly defined Gamma operators which preserve constants and $e^{2\mu\cdot}$, $\mu > 0$ functions are introduced and numerically tested. One of the most important positive aspects of new proposed algorithm is that it can yield better approximation behavior in comparison to its classical correspondents for large classes of functions. Numerical examples also suggest that introduced technique provide better approximation accuracy.

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CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

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