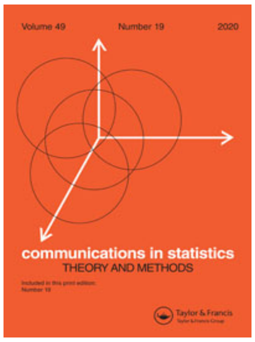

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Plug-in estimators for the mean value and variance functions in delayed renewal processes

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ABSTRACT
In this paper, we deal with the problem of estimating the delayed renewal and variance functions in delayed renewal processes. Two parametric plug-in estimators for these functions are proposed and their unbiasedness, asymptotic unbiasedness and consistency properties are investigated. The asymptotic normality of these estimators are established. Further, a method for the computation of the estimators is given. Finally, the performances of the estimators are evaluated for small sample sizes by a simulation study.

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1. Introduction

Renewal processes have proved to be a powerful tool in stochastic modeling in a wide variety of fields of applied probability and statistics such as reliability theory (Barlow and Proschan 1975; Goldberg 1981), inventory theory (Ross 1970; Moors and Strijbosch 2002), queuing analysis (Kumar and Knezevic 1998), insurance risk analysis (Willmot 2004), product warranty analysis (Frees 1986a; Huang, Liu and Murthy 2007; Chien 2008) and telecommunication (Rodriguez-Dagnino and Takagi 2005). The mean value function and the variance function arise in many of the applications involving renewal processes.

Let $\{X_1,\ X_2,\ \dots\}$ be a sequence of independent and identically distributed positive random variables representing the successive lifetimes with distribution function F. Set $S_0 = 0$ and $S_n = X_1 + \dots + X_n,\ \ n = 1, 2, \dots$. For each $t \geq 0$, define $N(t) = \max\{n : S_n \leq t\}$. The stochastic process $\{N(t),\ t \geq 0\}$ is called an ordinary renewal process. $N(t)$ is the number of renewals in the time interval $(0, t]$. The mean value function (renewal function) and the variance function of the ordinary renewal process $\{N(t),\ t \geq 0\}$ are $M(t) = E(N(t)),\ \ t \geq 0$ and $V(t) = Var(N(t)),\ \ t \geq 0$, respectively. For the renewal function it is well known that

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$$M(t) = \sum_{k=1}^{\infty} F^{k*}(t), \;\; t \geq 0 \tag{1.1}$$

and it satisfies the integral equation (renewal equation)

$$M(t) = F(t) + \int_0^t M(t-x)dF(x), \;\; t \geq 0 \tag{1.2}$$

where $*$ denotes Stieltjes convolution and F^{k*} is the k-fold Stieltjes convolution of F with

$$F^{0*}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t<0 \end{cases}$$

Furthermore, it is easily seen that

$$M * M(t) = \sum_{k=1}^{\infty} kF^{(k+1)*}(t), \;\; t \geq 0 \tag{1.3}$$

For the variance function it is written that

$$V(t) = 2\sum_{k=1}^{\infty} k\mathrm{F}^{k*}(t) - \sum_{k=1}^{\infty} F^{k*}(t)\left(1 + \sum_{k=1}^{\infty} F^{k*}(t)\right), \;\; t \geq 0 \tag{1.4}$$

Since the random variable $N(t)$ for each fixed $t \geq 0$ has finite moments of all orders, $M(t)$ and $V(t)$ are finite for all $t \geq 0$ (Grimmett and Stirzaker 1992).

We continue to assume that $X_1, \; X_2, \; \ldots$ are all independent positive random variables but only $X_2, \; X_3, \ldots$ are identically distributed with distribution function F, while X_1 has possibly a different distribution function G. As previously, let $S_0 = 0$ and $S_n = X_1 + \ldots + X_n, \;\; n = 1, 2, \ldots$. For each $t \geq 0$ define $N_d(t) = \max\{n : S_n \leq t\}$. The stochastic process $\{N_d(t), \;\; t \geq 0\}$ is called a delayed renewal process.

Delayed renewal processes can be used to model the following two types of situations. In the first of these, the initial time to the first renewal has a different distribution than subsequent interrenewal times. For example, the first engine used in a vehicle generally lasts longer than renewals (where the renewal might be the same engine after being reconditioned). In the second situation the ordinary renewal process is not actually from time zero onward, but it is only being interested in the process from time z onward. By defining X_1 as the excess lifetime at z for the original ordinary renewal process, it can be constructed a delayed renewal process with time zero in the delayed renewal process corresponding to time z of the original process. Suppose that the elapsed time backward from z to the last renewal is y. Then it is easily seen that in the constructed delayed renewal process

$$G(x) = \frac{F(x+y) - F(y)}{1 - F(y)}, x \geq 0$$

where F is the interrenewal time distribution function of the original ordinary renewal process.

Let μ be the mean of the distribution F. It is known that the delayed renewal process $\{N_d(t), \;\; t \geq 0\}$ has stationary increments if and only if

$$G(x) = \frac{1}{\mu}\int_0^x (1 - F(y))dy$$

We recognize this distribution as the limiting excess lifetime distribution.

A delayed renewal process for which the first renewal time has the above distribution function is called a stationary (or equilibrium) renewal process. We are attempting to model a renewal process that began indefinitely far in the past, so that the remaining life of the item in service at the origin has the limiting distribution of the excess lifetime in an ordinary renewal process.

The mean value function (delayed renewal function) and the variance function of the delayed renewal process $\{N_d(t),\ t \geq 0\}$ are $M_d(t) = E(N_d(t)),\ t \geq 0$ and $V_d(t) = Var(N_d(t)),\ t \geq 0$, respectively. $M_d(t)$ and $V_d(t)$ are finite for all $t \geq 0$. For the delayed renewal function it is easily seen that

$$M_d(t) = \sum_{k=1}^{\infty} G * F^{(k-1)*}(t),\ \ t \geq 0 \tag{1.5}$$

and it satisfies the integral equation

$$M_d(t) = G(t) + \int_0^t M_d(t-x)dF(x),\ \ t \geq 0 \tag{1.6}$$

From the Equation (1.6), it is clear that

$$M_d(t) = G(t) + G * M(t),\ \ t \geq 0 \tag{1.7}$$

For the delayed variance function it is written that

$$V_d(t) = M_d(t)(1 - M_d(t)) + 2M * M_d(t),\ \ t \geq 0 \tag{1.8}$$

see Soland (1968). By Equations (1.1), (1.3), (1.5) and (1.7), from Equation (1.8) we obtain a new form for $V_d(t)$ as

$$\begin{aligned} V_d(t) = &\sum_{k=1}^{\infty} G * F^{(k-1)*}(t)\left(1 - \sum_{k=1}^{\infty} G * F^{(k-1)*}(t)\right) \\ &+ 2\sum_{k=1}^{\infty} kG * F^{k*}(t),\ \ t \geq 0 \end{aligned} \tag{1.9}$$

If the delayed renewal process $\{N_d(t),\ t \geq 0\}$ is specially an equilibrium renewal process, then it is easily seen that

$$M_d(t) = \frac{t}{\mu},\ \ t \geq 0 \tag{1.10}$$

and

$$V_d(t) = \frac{t}{\mu} - \left(\frac{t}{\mu}\right)^2 + \frac{2}{\mu}\int_0^t M(x)dx,\ \ t \geq 0 \tag{1.11}$$

respectively.

When the distribution functions F and G are known the functions M_d and V_d can be obtained in principle from the above expressions. However, without some special distributions, these functions can not be analytically evaluated. In what follows, some illustrative examples for M_d and V_d are given under some classical distributions.

Example 1: Let F and G be exponential distribution functions with parameters $\beta_1 > 0$ and $\beta_2 > 0$, respectively, i.e., $F(t) = 1 - e^{-t/\beta_1}, t \geq 0$ and $G(t) = 1 - e^{-t/\beta_2}, t \geq 0$, then the functions $M_d(t)$ and $V_d(t)$ are obtained analytically from Equations (1.7) and (1.8) as

$$M_d(t) = \frac{t}{\beta_1} + \left(1 - \frac{\beta_2}{\beta_1}\right)\left(1 - e^{-t/\beta_2}\right), \quad t \geq 0 \tag{1.12}$$

and

$$\begin{aligned} V_d(t) = {} & \left(\frac{t}{\beta_1} + \left(1 - \frac{\beta_2}{\beta_1}\right)\left(1 - e^{-t/\beta_2}\right)\right)\left(1 - \frac{t}{\beta_1} - \left(1 - \frac{\beta_2}{\beta_1}\right)\left(1 - e^{-t/\beta_2}\right)\right) \\ & + 2\left(\frac{t^2}{2\beta_1^2} + \frac{(\beta_1 - \beta_2)\left(t - \beta_2\left(1 - e^{-t/\beta_2}\right)\right)}{\beta_1^2}\right), \quad t \geq 0 \end{aligned} \tag{1.13}$$

Example 2: Let F and G be an exponential distribution function with parameter $\beta_1 > 0$ and gamma distribution function with parameters $\alpha = 2$ and $\beta_2 > 0$, respectively. That is, $F(t) = 1 - e^{-t/\beta_1}$, $t \geq 0$ and $G(t) = 1 - e^{-t/\beta_2}(1 + t/\beta_2)$, $t \geq 0$. It is obvious that $M(t) = t/\beta_1$, $t \geq 0$. Then, from Equations (1.7) and (1.8) we obtain the analytical expressions of $M_d(t)$ and $V_d(t)$ as

$$M_d(t) = 1 - \frac{2\beta_2}{\beta_1} + \frac{t}{\beta_1} - \left(1 - \frac{2\beta_2}{\beta_1} + \frac{t}{\beta_2} - \frac{t}{\beta_1}\right)e^{-t/\beta_2}, \quad t \geq 0 \tag{1.14}$$

and

$$\begin{aligned} V_d(t) = {} & \frac{2\beta_2^2}{\beta_1^2} - \frac{2\beta_2}{\beta_1} + \frac{t}{\beta_1} + \left(1 + \frac{2\beta_2^2}{\beta_1^2} - \frac{2\beta_2}{\beta_1} - \frac{t}{\beta_1} + \frac{t}{\beta_2} - \frac{2\beta_2 t}{\beta_1^2} + \frac{2t^2}{\beta_1\beta_2} - \frac{2t^2}{\beta_1^2}\right)e^{-t/\beta_2} \\ & - \left(1 - \frac{2\beta_2}{\beta_1} + \frac{t}{\beta_2} - \frac{t}{\beta_1}\right)^2 e^{-2t/\beta_2}, \quad t \geq 0 \end{aligned} \tag{1.15}$$

In above examples it has been considered the situation that the distributions F and G do not share common parameters. However, F and G may share common parameters. Assume that $\beta_1 = \beta$ and $\beta_2 = 2\beta$ in Example 1. Then the analytical expressions for the M_d and V_d reduce to

$$M_d(t) = \frac{t}{\beta} - \left(1 - e^{-t/2\beta}\right), t \geq 0 \tag{1.16}$$

$$V_d(t) = \left(\frac{t}{\beta} - \left(1 - e^{-t/2\beta}\right)\right)\left(2 - \frac{t}{\beta} - e^{-t/2\beta}\right) + 2\left(\frac{t^2}{2\beta^2} + \frac{\left(t - 2\beta\left(1 - e^{-t/2\beta}\right)\right)}{\beta}\right), \quad t \geq 0 \tag{1.17}$$

As for Example 2, assume that $\beta_1 = 2\beta$ and $\beta_2 = \beta$. In that case, M_d and V_d are obtained from Equations (1.14) and (1.15) as

$$M_d(t) = \frac{t}{2\beta}\left(1 - e^{-t/\beta}\right), t \geq 0 \tag{1.18}$$

$$V_d(t) = \frac{t}{2\beta} - \frac{1}{2} + \left(\frac{1}{2} + \frac{t^2}{2\beta^2}\right)e^{-t/\beta} - \frac{t^2}{4\beta^2}e^{-t/2\beta}, t \geq 0 \tag{1.19}$$

respectively.

In most cases, F and G are unknown or the functional forms of F and G are known but the parameters of the distributions are unknown. In those cases $M_d(t)$ and $V_d(t)$ must be estimated from data. These functions can be estimated parametrically, if the functional forms of F and G are known and nonparametrically, if the forms of F and G are unknown.

Estimation of the renewal function $M(t)$ and the variance function $V(t)$ in an ordinary renewal process is considered by Frees (1986a, 1986b); Schneider, Lin and O'Cinneide (1990); Grübel and Pitts (1993); Aydoğdu (2005); Markovich and Krieger (2006) and some other authors. In this study, estimation problem of the delayed renewal function $M_d(t)$ and the delayed variance function $V_d(t)$ in a delayed renewal process is considered. Two plug-in estimators for $M_d(t)$ and $V_d(t)$ are proposed in Section 2 and their statistical properties such as unbiasedness, asymptotic unbiasedness, consistency and asymptotic normality are investigated in Section 3. Computation procedures for $M_d(t)$, $V_d(t)$ and the related plug-in estimators are given in Section 4. A simulation study is presented in order to evaluate the small sample performance of the estimators in Section 5.

2. Parametric estimation of the delayed renewal and variance functions

Suppose that the functional forms of the distribution functions F and G are known and let $\theta_1, \theta_2, ..., \theta_r$ and $\gamma_1, \gamma_2, ..., \gamma_s$ be the unknown parameters of $F = F(\theta_1, \theta_2, ..., \theta_r)$ and $G = G(\gamma_1, \gamma_2, ..., \gamma_s)$. Consider two independent random samples $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_m$ from the distribution functions F and G respectively. In practice, it is important how to observe these samples from the underlying delayed renewal process. For this purpose, two different sampling schemes can arise. These are given as follows.

Scheme 1: Let the distribution functions F and G do not share common parameters. Then m independent realizations $\left\{X_1^1, X_2^1, ..., X_{n_1}^1\right\}, ..., \left\{X_1^m, X_2^m, ..., X_{n_m}^m\right\}$ from the delayed renewal process are observed such that $n_i \in \{2, 3, ...\}$, $i = 1, ..., m$. Hence $X_1^1, X_1^2, ..., X_1^m$ are a sample of size m from the distribution G and the remaining observations are a sample of size n from the distribution F, where $n = n_1 + n_2 + ... + n_m - m$.

Scheme 2: Let F and G share common parameters. Then it is enough to observe a single realization of the delayed renewal process, say $\{X_1, X_2, ..., X_n\}$. Thus the common parameters of F and G can be estimated based on the data set $\{X_1, X_2, ..., X_n\}$.

Let $\hat{\theta}_1, \hat{\theta}_2 ..., \hat{\theta}_r$ and $\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_s$ be some estimators of $\theta_1, \theta_2, ..., \theta_r$ and $\gamma_1, \gamma_2, ..., \gamma_s$ based on the samples $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_m$ respectively. $\hat{F}_n = F(\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_r)$ and $\hat{G}_m = G(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_s)$ are the resulting estimators of F and G. Then, we can define two plug-in estimators of the delayed renewal function $M_d(t)$ and the delayed variance function $V_d(t)$ for each fixed t $(t \geq 0)$ as

$$\hat{M}_d(t) = \sum_{k=1}^{\infty} \hat{G}_m * \hat{F}_n^{(k-1)*}(t) \tag{2.1}$$

and

$$\begin{aligned}\hat{V}_d(t) &= \sum_{k=1}^{\infty} \hat{G}_m * \hat{F}_n^{(k-1)*}(t)\left(1 - \sum_{k=1}^{\infty} \hat{G}_m * \hat{F}_n^{(k-1)*}(t)\right) \\ &\quad + 2\sum_{k=1}^{\infty} k\hat{G}_m * \hat{F}_n^{k*}(t)\end{aligned} \tag{2.2}$$

by replacing $F(t)$ and $G(t)$ in the Equations (1.5) and (1.9) with their estimators $\hat{F}_n(t)$ and $\hat{G}_m(t)$, where $\hat{F}_n^{k*}$ is the k-fold Stieltjes convolution of $\hat{F}_n$ and $\hat{F}_n^{0*}(t) = F^{0*}(t)$. In order to obtain some asymptotic results for the estimators $\hat{M}_d(t)$ and $\hat{V}_d(t)$, assume that the sample size m is a function of the sample size n, $m = m(n)$, such that $m \uparrow \infty$ as $n \uparrow \infty$.

If there exist some analytical expressions depending on the parameters of F and G for M_d and V_d, it is clear that the estimators obtained by replacing the parameters with their estimators in the analytical expressions of M_d and V_d equal the estimators defined in Equations (2.1) and (2.2).

Let $\{N_d(t),\ t \geq 0\}$ be an equilibrium renewal process. Then considering the Equations (1.10) and (1.11) for M_d and V_d, the estimators given in Equations (2.1) and (2.2) can be written as

$$\hat{M}_d(t) = \frac{t}{\hat{\mu}},\quad t \geq 0$$

and

$$\hat{V}_d(t) = \frac{t}{\hat{\mu}} - \left(\frac{t}{\hat{\mu}}\right)^2 + \frac{2}{\hat{\mu}}\int_0^t \hat{M}(x)dx,\quad t \geq 0$$

respectively, where $\hat{\mu}$ is the estimator for the mean of the distribution F and $\hat{M}(x)$ is the plug-in estimator for the renewal function $M(x)$ of the ordinary renewal process based on the data set $\{X_1,\ X_2, \ldots, X_n\}$.

Now, let us consider Example 1-2. For Example 1, some estimators of β_1 and β_2 are, respectively,

$$\hat{\beta}_1 = \overline{X}_n \text{ and } \hat{\beta}_2 = \overline{Y}_m$$

where $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $\overline{Y}_m = m^{-1}\sum_{i=1}^m Y_i$. Then, from the expressions (1.12) and (1.13) the estimators $\hat{M}_d(t)$ and $\hat{V}_d(t)$ for each fixed t are obtained as

$$\hat{M}_d(t) = \frac{t}{\overline{X}_n} + \left(1 - \frac{\overline{Y}_m}{\overline{X}_n}\right)\left(1 - e^{-t/\overline{Y}_m}\right), t \geq 0 \tag{2.3}$$

and

$$\begin{aligned}\hat{V}_d(t) &= \left(\frac{t}{\overline{X}_n} + \left(1 - \frac{\overline{Y}_m}{\overline{X}_n}\right)(1 - e^{-t/\overline{Y}_m})\right)\left(1 - \frac{t}{\overline{X}_n} - \left(1 - \frac{\overline{Y}_m}{\overline{X}_n}\right)(1 - e^{-t/\overline{Y}_m})\right) \\ &\quad + 2\left(\frac{t^2}{2\overline{X}_n^2} + \frac{(\overline{X}_n - \overline{Y}_m)(t - \overline{Y}_m(1 - e^{-t/\overline{Y}_m}))}{\overline{X}_n^2}\right),\quad t \geq 0\end{aligned} \tag{2.4}$$

It can be shown that the estimators given in Equations (2.3) and (2.4) are not unbiased for any fixed t though $\overline{X}_n$ and $\overline{Y}_m$ are unbiased for β_1 and β_2, respectively. However, by the assumption $m = m(n)$ such that $m \uparrow \infty$ as $n \uparrow \infty$, it is clear that $\hat{M}_d(t) \underset{n\to\infty}{\overset{a.s.}{\to}} M_d(t)$ and $\hat{V}_d(t) \underset{n\to\infty}{\overset{a.s.}{\to}} V_d(t)$, that is, $\hat{M}_d(t)$ and $\hat{V}_d(t)$ are almost surely consistent estimators of $M_d(t)$ and $V_d(t)$, since $\overline{X}_n$ and $\overline{Y}_m$ are almost surely consistent estimators for β_1 and β_2.

As for Example 2, some estimators of β_1 and β_2 are, respectively,

$$\hat{\beta}_1 = \overline{X}_n \quad \text{and} \quad \hat{\beta}_2 = \overline{Y}_m/2$$

Then, from the expressions Equations (1.14) and (1.15) the estimators $\hat{M}_d(t)$ and $\hat{V}_d(t)$ for each fixed t are obtained as

$$\hat{M}_d(t) = 1 - \frac{\overline{Y}_m}{\overline{X}_n} + \frac{t}{\overline{X}_n} - \left(1 - \frac{\overline{Y}_m}{\overline{X}_n} + \frac{2t}{\overline{Y}_m} - \frac{t}{\overline{X}_n}\right) e^{-2t/\overline{Y}_m}$$

and

$$\hat{V}_d(t) = \frac{\overline{Y}_m^2}{2\overline{X}_n^2} - \frac{\overline{Y}_m}{\overline{X}_n} + \frac{t}{\overline{X}_n} + \left(1 + \frac{\overline{Y}_m^2}{2\overline{X}_n^2} - \frac{\overline{Y}_m}{\overline{X}_n} - \frac{t}{\overline{X}_n} + \frac{2t}{\overline{Y}_m} - \frac{\overline{Y}_m t}{\overline{X}_n^2} + \frac{4t^2}{\overline{X}_n \overline{Y}_m} - \frac{2t^2}{\overline{X}_n^2}\right) e^{-2t/\overline{Y}_m}$$
$$- \left(1 - \frac{\overline{Y}_m}{\overline{X}_n} + \frac{2t}{\overline{Y}_m} - \frac{t}{\overline{X}_n}\right)^2 e^{-4t/\overline{Y}_m}$$

These estimators are not unbiased. Since $\overline{X}_n$ and $\overline{Y}_m/2$ are almost surely consistent estimators for β_1 and β_2, by the assumption $m = m(n)$ such that $m \uparrow \infty$ as $n \uparrow \infty$, it is clear that $\hat{M}_d(t) \underset{n\to\infty}{\overset{a.s.}{\to}} M_d(t)$ and $\hat{V}_d(t) \underset{n\to\infty}{\overset{a.s.}{\to}} V_d(t)$, that is, $\hat{M}_d(t)$ and $\hat{V}_d(t)$ are almost surely consistent estimators of $M_d(t)$ and $V_d(t)$.

In much cases, the functions $M_d(t)$ and $V_d(t)$ don't have analytical forms so do the related plug-in estimators $\hat{M}_d(t)$ and $\hat{V}_d(t)$. In such situations, the estimators $\hat{M}_d(t)$ and $\hat{V}_d(t)$ have to be computed numerically. Computation procedures of these estimators are given in detail in Section 4.

3. Asymptotic properties of $\hat{M}_d(t)$ and $\hat{V}_d(t)$

From the previous section, it is known that the estimators $\hat{M}_d(t)$ and $\hat{V}_d(t)$ for each fixed t are not in general unbiased. In this section, the almost sure consistency, asymptotic unbiasedness and normality of $\hat{M}_d(t)$ and $\hat{V}_d(t)$ are investigated under some conditions.

Let us now give a theorem to establish the almost sure consistency of $\hat{M}_d(t)$ and $\hat{V}_d(t)$ for each fixed t.

Theorem 3.1. *Let $F = F(\theta_1, \theta_2, \ldots, \theta_r)$ and $G = G(\gamma_1, \gamma_2, \ldots, \gamma_s)$ be absolutely continuous distribution functions with probability density functions $f = f(\theta_1, \theta_2, \ldots, \theta_r)$ and $g = g(\gamma_1, \gamma_2, \ldots, \gamma_s)$. Suppose that $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r$ and $\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_s$ are almost surely consistent estimators of $\theta_1, \theta_2, \ldots, \theta_r$ and $\gamma_1, \gamma_2, \ldots, \gamma_s$. If f and g are continuous in each parameter*

θ_i, $i = 1, \ldots, r$ and each parameter γ_i, $i = 1, \ldots, s$ respectively, then for any $a \in R$ and each fixed t, by the assumption $m = m(n)$ such that $m \uparrow \infty$ as $n \uparrow \infty$,

$$\sum_{k=1}^{\infty} k^a \hat{G}_m * \hat{F}_n^{k*}(t) \underset{n\to\infty}{\overset{a.s.}{\to}} \sum_{k=1}^{\infty} k^a G * F^{k*}(t)$$

Proof. Let $\hat{f}_n = f(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r)$ and $\hat{g}_m = g(\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_s)$. By the continuity of f and g, since $\hat{\theta}_i \underset{n\to\infty}{\overset{a.s.}{\to}} \theta_i$, $i = 1, \ldots, r$ and $\hat{\gamma}_i \underset{n\to\infty}{\overset{a.s.}{\to}} \gamma_i$, $i = 1, \ldots, s$,

$$\hat{f}_n(t) \underset{n\to\infty}{\overset{a.s.}{\to}} f(t) \tag{3.1}$$

and

$$\hat{g}_m(t) \underset{n\to\infty}{\overset{a.s.}{\to}} g(t) \tag{3.2}$$

for each fixed t. Let h be the convolution of f and g, that is,

$$h(x) = g * f(x) = \int_0^x f(x-y) g(y) dy, \ \ x \geq 0$$

The function h is the probability density function of the distribution function $G * F$. $\hat{h}_n = h(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r, \hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_s)$ is the resulting estimator of $h = h(\theta_1, \theta_2, \ldots, \theta_r, \gamma_1, \gamma_2, \ldots, \gamma_s)$. By the continuity of f and g, h is continuous in each parameter θ_i, $i = 1, \ldots, r$ and γ_i, $i = 1, \ldots, s$. Hence, for each fixed t

$$\hat{h}_n(t) \underset{n\to\infty}{\overset{a.s.}{\to}} h(t) \tag{3.3}$$

By Scheffe's theorem (Serfling 1980), from Equations (3.1), (3.2) and (3.3) we have

$$\hat{F}_n(t) \underset{n\to\infty}{\overset{a.s.}{\to}} F(t), \ \ \hat{G}_m(t) \underset{n\to\infty}{\overset{a.s.}{\to}} G(t) \ \ \text{and} \ \ \hat{G}_m * \hat{F}_n(t) \underset{n\to\infty}{\overset{a.s.}{\to}} G * F(t)$$

for each fixed t. Thus, it is easily seen by induction that, for $k = 1, 2, \ldots$ and for each fixed t

$$\hat{F}_n^{k*}(t) \underset{n\to\infty}{\overset{a.s.}{\to}} F^{k*}(t) \tag{3.4}$$

and

$$\hat{G}_m * \hat{F}_n^{k*}(t) \underset{n\to\infty}{\overset{a.s.}{\to}} G * F^{k*}(t) \tag{3.5}$$

Let A be the set such that Equations (3.4) and (3.5) hold for all $k \geq 1$ and fix $\omega \in A$ $(P(A) = 1)$. Since F is continuous, there is some $c > 0$ such that $F(c) < 1$. Choose $r \in N$ such that $t \leq rc$. Let $X_1, \ldots, X_r$ be independent and identically distributed random variables with distribution function F. For $S_r = X_1 + \ldots + X_r$, the event $(S_r \leq t)$ implies the event $(S_r \leq rc)$, which implies that the event $(X_1 > c, \ldots, X_r > c)$ has not occurred. Thus, we obtain

$$F^{r*}(t) \leq 1 - \left[1 - F(t/r)\right]^r, \ \ \left[1 - F(t/r)\right]^r > 0 \tag{3.6}$$

Therefore, for $t > 0$ there exists an $r \geq 1$ with $F^{r*}(t) < 1$. Let $\varepsilon = (1 - F^{r*}(t))/2$ for an integer r satisfying Equation (3.6). Then, from Equation (3.4), there exists an $n_0(\omega)$

such that for all $n \geq n_0(\omega)$, we have $\hat{F}_n^{r*}(t) < 1 - \varepsilon$. It is obvious that $\frac{1}{2} \leq 1 - \varepsilon < 1$. For any k, it is seen that

$$\hat{F}_n^{(ir+j)*}(t) = \int_0^t \hat{F}_n^{((i-1)r+j)*}(t-x)d\hat{F}_n^{r*}(x) \leq \hat{F}_n^{((i-1)r+j)*}(t)\hat{F}_n^{r*}(\mathrm{t})$$

for any integers i and j. Direct iteration leads to the relations

$$\hat{F}_n^{(ir+j)*}(t) \leq (\hat{F}_n^{r*}(t))^i \hat{F}_n^{j*}(t), \;\; 0 \leq j \leq \mathrm{r}$$

Then, for $k \geq \mathrm{r}$, it is obtained that

$$\hat{F}_n^{k*}(t) \leq (\hat{F}_n^{r*}(t))^{[k/\mathrm{r}]}$$

where [.] is the greatest integer part. Define $u_{n,n_0(\omega)} = (1-\varepsilon)I(n \geq n_0(\omega)) + I(n<n_0(\omega))$. Then, for $k \geq r$, $k^a\hat{G}_m * \hat{F}_n^{k*}(t) \leq k^a(u_{n,n_0(\omega)})^{[k/\mathrm{r}]}$. For $n > n_0(\omega)$, we have

$$\sum_{k=1}^{\infty} k^a\hat{G}_m * \hat{F}_n^{k*}(t) \leq \sum_{k=1}^{r} k^a\hat{G}_m * \hat{F}_n^{k*}(t) + \sum_{k=\mathrm{r}+1}^{\infty} k^a(u_{n,n_0(\omega)})^{[k/\mathrm{r}]} < \infty$$

From Equation (3.5), for all $k \geq 1$, $k^a\hat{G}_m * \hat{F}_n^{k*}(t) \underset{n\to\infty}{\overset{a.s.}{\to}} k^aG * F^{k*}(t)$. Thus, from Lebesgue's dominated convergence theorem we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} k^a\hat{G}_m * \hat{F}_n^{k*}(t) = \sum_{k=1}^{\infty} k^aG * F^{k*}(t)$$

for each $\omega \in A$. This completes the proof.

From Theorem 3.1 with $a = 0$ we have the following corollary.

Corollary 3.1. *Under the conditions of Theorem 3.1, for each fixed t*

$$\hat{M}_d(t) \underset{n\to\infty}{\overset{a.s.}{\to}} M_d(t) \tag{3.7}$$

that is, $\hat{M}_d(t)$ is almost surely consistent estimator of $M_d(t)$ for each fixed t.

From Theorem 3.1 with $a = 1$ and Corollary 3.1 we establish the consistency of $\hat{V}_d(t)$ by the following corollary.

Corollary 3.2. *Under the conditions of Theorem 3.1, for each fixed t*

$$\hat{V}_d(t) \underset{n\to\infty}{\overset{a.s.}{\to}} V_d(t) \tag{3.8}$$

that is, $\hat{V}_d(t)$ is almost surely consistent estimator of $V_d(t)$ for each fixed t.

As is known the estimators $\hat{M}_d(t)$ and $\hat{V}_d(t)$ are not in general unbiased. Their asymptotic unbiasedness is established by the following theorem.

Theorem 3.2. *Suppose that the conditions of Theorem 3.1 hold. If $F(t) < 1$, then*

$$\mathop{lim}_{n\to\infty} E(\hat{M}_d(t)) = M_d(t)$$

and

$$\lim_{n\to\infty} E\left(\hat{V}_d(t)\right) = V_d(t)$$

that is, $\hat{M}_d(t)$ and $\hat{V}_d(t)$ are asymptotically unbiased estimators of $M_d(t)$ and $V_d(t)$.

Proof. From Equations (1.1) and (1.5), it is easily seen that

$$M(t) \le \frac{F(t)}{1-F(t)} \text{ and } M_d(t) \le \frac{1}{1-F(t)}.$$

Using these inequalities, from the expression (1.8) of $V_d(t)$ we obtain

$$V_d(t) \le \frac{1}{1-F(t)} + \frac{2F(t)}{(1-F(t))^2}$$

Since $F(t) < 1$, $\hat{F}_n(t) < 1$. Then, it is written that, from the above inequalities,

$$\hat{M}_d(t) \le \frac{1}{1-\hat{F}_n(t)} \tag{3.9}$$

and

$$\hat{V}_d(t) \le \frac{1}{1-\hat{F}_n(t)} + \frac{2\hat{F}_n(t)}{(1-\hat{F}_n(t))^2} \tag{3.10}$$

Since $\hat{F}_n(t) < 1$, there exists some $c > 0$ such that $\hat{F}_n(t) < c/(1+c)$. Therefore, from Equations (3.9) and (3.10) we have that $\hat{M}_d(t) \le 1 + c$ and $\hat{V}_d(t) \le (1+c)(1+2c)$. Hence, from these inequalities, Equations (3.7) and (3.8), by the bounded convergence theorem (Serfling 1980) it is obtained that

$$\lim_{n\to\infty} E\left(\hat{M}_d(t)\right) = M_d(t) \text{ and } \lim_{n\to\infty} E\left(\hat{V}_d(t)\right) = V_d(t)$$

Thus, the proof is completed.

Note that the condition "$F(t) < 1$" is satisfied for commonly used distributions such as the exponential, gamma, Weibull and lognormal distributions in renewal theory. If $F(t) < 1$, then $F(t_0) < 1$ for all $t_0 \le t$. Therefore, Theorem 3.2 holds for all $t_0 \le t$ such that $F(t) < 1$.

Further, the asymptotic distributions of the plug-in estimators $\hat{M}_d$ and $\hat{V}_d$ can be obtained, in principle, by means of the Delta method. This property is given in the following theorem.

Theorem 3.3. *Let $\boldsymbol{\delta} = (\boldsymbol{\theta}, \boldsymbol{\gamma})$ and $\hat{\boldsymbol{\delta}} = (\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\gamma}})$ where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_r)$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_s)$, $\boldsymbol{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r)$ and $\boldsymbol{\gamma} = (\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_s)$. Suppose that $\sqrt{n}(\hat{\boldsymbol{\delta}} - \boldsymbol{\delta})$ converges in distribution to a multivariate normal distribution with mean vector zero and positive definite covariance-variance matrix Σ as $n \uparrow \infty$. Then, for a fixed t, $\sqrt{n}(\hat{M}_d(t) - M_d(t))$ converges in distribution to a multivariate normal distribution with mean vector zero and covariance-variance matrix $\Psi(\boldsymbol{\delta})'\Sigma\Psi(\boldsymbol{\delta})$, under suitable regularity conditions. Here $\Psi(\boldsymbol{\delta}) = \frac{\partial}{\partial\delta} M_d$ and " ' " denotes transpose of a matrix.*

Similarly, the asymptotic distribution of $V_d(t)$ can be found in principle. But, if the function $M_d(t)$ does not have analytical expression as a function of $\boldsymbol{\delta}$, computation of the $\Psi(\boldsymbol{\delta})$ will be quite difficult and intractable. In such a case, computation of the asymptotic variance of $V_d(t)$ will be quite difficult, too.

As a simple illustration, let us reconsider the estimator given in Equation (2.3) for Equation (1.12) in Example 1. From central limit theorem, it is well known that $\sqrt{n}\left(\overline{X}_n - \beta_1\right) \sim AN\left(0, \beta_1^2\right)$ and $\sqrt{m}\left(\overline{Y}_m - \beta_2\right) \sim AN\left(0, \beta_2^2\right)$, where AN denotes asymptotically normal and $m = m(n)$ such that $m \uparrow \infty$ as $n \uparrow \infty$. Then, by considering Theorem 3.3, $\sqrt{n}\left(\hat{M}_d(t) - M_d(t)\right) \sim AN\left(0, \sigma^2(t)\right)$ for each fixed $t > 0$. Here,

$$\sigma^2(t) = \left(-\frac{t}{\beta_1} + \frac{\beta_2 - \beta_2 e^{-t/\beta_2}}{\beta_1^2}\right)^2 \beta_1^2 + \left(\frac{1 - e^{-t/\beta_2}}{\beta_1} + \frac{(\beta_1 - \beta_2) t e^{-t/\beta_2}}{\beta_1 \beta_2^2}\right)^2 \beta_2^2$$

4. Computational remarks

The expressions (2.1) and (2.2) are in general cumbersome for the computation of $\hat{M}_d(t)$ and $\hat{V}_d(t)$ without analytical forms for $M_d(t)$ and $V_d(t)$. By Equations (1.6) and (1.8), the estimators $\hat{M}_d(t)$ and $\hat{V}_d(t)$ can be rewritten as

$$\hat{M}_d(t) = \hat{G}_m(t) + \int_0^t \hat{M}_d(t - x) d\hat{F}_n(x) \tag{4.1}$$

and

$$\hat{V}_d(t) = \hat{M}_d(t)\left(1 - \hat{M}_d(t)\right) + 2\hat{M} * \hat{M}_d(t) \tag{4.2}$$

where $\hat{M}(t)$ is given by the renewal integral equation

$$\hat{M}(t) = \hat{F}_n(t) + \int_0^t \hat{M}(t - x) d\hat{F}_n(x) \tag{4.3}$$

Equations (2.3) and (2.4) are easier than Equations (2.1) and (2.2) to compute $\hat{M}_d(t)$ and $\hat{V}_d(t)$. $\hat{M}(t)$ and $\hat{M}_d(t)$ can be calculated by solving the renewal Equations (2.3) and (2.5) via Xie's Riemann-Stieltjes (RS) method (Xie 1989). Furthermore, this method can be easily adapted to the computation of the convolution $\hat{M} * \hat{M}_d(t)$ and so $\hat{V}_d(t)$ in Equation (2.4). We now summarize the RS-method for evaluating the delayed renewal function $M_d(t)$ and adapt this to the computation of the variance function $V_d(t)$. In consequence, by using the same computational procedures in the method mentioned above, $\hat{M}_d(t)$ and $\hat{V}_d(t)$ are computed from the Equations (4.1)–(4.3).

By the definition of the Riemann-Stieltjes integration, an integral of the form $\int_a^b g(t) dh(t)$ can be approximated numerically as

$$\int_a^b g(t) dh(t) \approx \sum_{i=1}^{p} g\left(\frac{t_i + t_{i-1}}{2}\right)(h(t_i) - h(t_{i-1})) \tag{4.4}$$

where $\{t_0, t_1, \ldots, t_p\}$ is a partition of the interval $[a, b]$.

Consider the Equations (1.6) and (1.8). For a given t, let $\{t_0, t_1, \ldots, t_p\}$ be a partition of the interval $[0, t]$ such that $0 = t_0 < t_1 < \ldots < t_p = t$. By using Equation (4.4),

$$M_d(t_i) \approx G(t_i) + \sum_{j=1}^{i} F\left(t_i - \left(\frac{t_j + t_{j-1}}{2}\right)\right)\left(M_d(t_j) - M_d(t_{j-1})\right)$$

Thus, $M_d(t_i)$ can be calculated recursively by

$$\widetilde{M}_d(t_i) = \frac{G(t_i) + U_i - F\left(t_i - \left(\frac{t_i + t_{i-1}}{2}\right)\right)\widetilde{M}_d(t_{i-1})}{1 - F\left(t_i - \left(\frac{t_i + t_{i-1}}{2}\right)\right)}, \quad i = 1, \ldots, p \tag{4.5}$$

where $\widetilde{M}_d(t_0) = 0$ and $U_i = \sum_{j=1}^{i-1} F\left(t_i - \left(\frac{t_j + t_{j-1}}{2}\right)\right)\left(\widetilde{M}_d(t_j) - \widetilde{M}_d(t_{j-1})\right)$

Now, consider the expression (4.2) for the estimator $V_d(t)$. For the partition $\{t_0, t_1, \ldots, t_p\}$ of the interval $[0, t]$, by using Equation (4.4),

$$\int_0^{t_i} M_d(t_i - x)dM(x) \approx \sum_{j=1}^{i} M_d\left(t_i - \left(\frac{t_j + t_{j-1}}{2}\right)\right)\left(M(t_j) - M(t_{j-1})\right)$$

Hence,

$$V_d(t_i) \approx M_d(t_i)(1 - M_d(t_i)) + 2\sum_{j=1}^{i} M_d\left(t_i - \left(\frac{t_j + t_{j-1}}{2}\right)\right)\left(M(t_j) - M(t_{j-1})\right)$$

$V_d(t_i)$ can be calculated iteratively from the following equation.

$$\begin{aligned}\widetilde{V}_d(t_i) &= \widetilde{M}_d(t_i)\left(1 - \widetilde{M}_d(t_i)\right) \\ &\quad + 2\sum_{j=1}^{i} \widetilde{M}_d\left(t_i - \left(\frac{t_j + t_{j-1}}{2}\right)\right)\left(\widetilde{M}(t_j) - \widetilde{M}(t_{j-1})\right), i = 1, \ldots p\end{aligned} \tag{4.6}$$

where $M(t_i)$ can be calculated similarly by the formula

$$\widetilde{M}(t_i) = G(t_i) + \sum_{j=1}^{i} F\left(t_i - \left(\frac{t_j + t_{j-1}}{2}\right)\right)\left(\widetilde{M}(t_j) - \widetilde{M}(t_{j-1})\right), \quad i = 1, \ldots .p \tag{4.7}$$

For the computational simplicity, the uniform partitions $\{0, t/p, 2t/p, \ldots, t\}$ and $\{0, 2t/p, 4t/p, \ldots, t\}$ of the interval $[0, t]$ can be used for M_d and M from Equations (4.5) and (4.7), and V_d from Equation (4.6), respectively. By writing $F(i)$, $M_d(i)$, $G(i)$, $M(i)$ and $V_d(i)$ instead of $F((i + 0.5)t/p)$, $\widetilde{M}_d(t_i)$, $G(t_i)$, $\widetilde{M}(t_i)$ and $\widetilde{V}_d(t_i)$, we obtain

$$M_d(i) = \frac{G(i) + \sum_{j=1}^{i} F(i-j)(M_d(j) - M_d(j-1)) - F(0)M_d(i-1)}{1 - F(0)}, \quad i = 1, \ldots, p$$

and

$$V_d(i) = M_d(2i)(1 - M_d(2i)) + 2\sum_{j=1}^{i} M_d(2(i-j) + 1)(M(2j) - M(2(j-1))), \quad i = 1, \ldots, p/2$$

These formulas are easy to program and quite understandable.

Table 1. Case 1 with $\beta_1 = 1.25,\ \beta_2 = 1.75.$

t	$M_d(t)$	$\hat{M}_d(t)$ $n,m=10$	$\hat{M}_d(t)$ $n,m=30$	$\hat{M}_d(t)$ $n,m=50$	$V_d(t)$	$\hat{V}_d(t)$ $n,m=10$	$\hat{V}_d(t)$ $n,m=30$	$\hat{V}_d(t)$ $n,m=50$
0.1	0.0578	0.0640	0.0603	0.0588	0.0591	0.0652	0.0615	0.0601
		(0.0005)	(0.0001)	(0.0001)		(0.0004)	(0.0001)	(0.0001)
0.5	0.3006	0.3335	0.3101	0.3084	0.3286	0.3608	0.3373	0.3363
		(0.0108)	(0.0028)	(0.0018)		(0.0101)	(0.0027)	(0.0017)
1	0.6259	0.6724	0.6460	0.6369	0.7217	0.7823	0.7471	0.7340
		(0.0368)	(0.0098)	(0.0060)		(0.0683)	(0.0154)	(0.0086)
3	2.0720	2.2746	2.1343	2.1006	2.5370	2.9593	2.6723	2.5851
		(0.4343)	(0.1027)	(0.0571)		(2.3556)	(0.4089)	(0.1980)
5	3.6230	3.9628	3.7119	3.6633	4.3527	5.1067	4.6140	4.4547
		(1.2986)	(0.3396)	(0.1879)		(6.2155)	(1.5883)	(0.7424)
8	6.0041	6.6217	6.1819	6.1463	6.9029	8.3344	7.2973	7.2097
		(4.7821)	(1.0808)	(0.6225)		(21.4431)	(3.5675)	(2.0412)
10	7.6013	8.2205	7.8580	7.7085	8.5376	9.9553	9.0536	8.7651
		(7.0742)	(1.7615)	(0.9882)		(23.8481)	(4.9812)	(2.5249)

5. Simulation study

In this section a simulation study is carried out to evaluate the performance of the plug-in estimators $\hat{M}_d$ and $\hat{V}_d$. For this purpose, the following distributions have been taken into account as possible candidates for the distribution functions F and G :

1. Exponential distribution $\mathrm{Exp}(\beta)$ with mean β having the probability density function (pdf) as

$$f(x) = \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right),\ \ x \geq 0;\ \ \beta > 0$$

2. Gamma distribution $\Gamma(\alpha, \beta)$ with shape parameter α and scale parameter β having the pdf as

$$f(x) = \frac{x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right)}{\Gamma(\alpha)\beta^{\alpha}},\ \ x \geq 0;\ \ \alpha,\ \ \beta > 0$$

3. Weibull distribution $\mathrm{W}(\alpha, \beta)$ with shape parameter α and scale parameter β having the pdf as

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right),\ \ x \geq 0;\ \ \alpha,\ \ \beta > 0$$

The cases considered in the simulation are given as below.

- Case 1: F is $\mathrm{Exp}(\beta_1)$, G is $\mathrm{Exp}(\beta_2)$.
- Case 2: F is $\mathrm{Exp}(\beta)$, G is $\mathrm{Exp}(2\beta)$.
- Case 3: F is $\mathrm{Exp}(\beta_1)$, G is $\Gamma(2, \beta_2)$.
- Case 4: F is $\mathrm{Exp}(2\beta)$, G is $\Gamma(2, \beta)$.
- Case 5: F is $\Gamma(\alpha, \beta_1)$, G is $\mathrm{Exp}(\beta_2)$.
- Case 6: F is $\Gamma(\alpha, \beta)$, G is $\mathrm{Exp}(\beta)$.
- Case 7: F is $\mathrm{W}(\alpha, \beta_1)$, G is $\mathrm{Exp}(\beta_2)$.
- Case 8: F is $\mathrm{W}(\alpha, \beta)$, G is $\mathrm{Exp}(\beta)$.

Table 2. Case 2 with $\beta = 0.5$.

t	$M_d(t)$	$\hat{M}_d(t)$ $n = 10$	$\hat{M}_d(t)$ $n = 30$	$\hat{M}_d(t)$ $n = 50$	$V_d(t)$	$\hat{V}_d(t)$ $n = 10$	$\hat{V}_d(t)$ $n = 30$	$\hat{V}_d(t)$ $n = 50$
0.1	0.1048	0.1178	0.1092	0.1075	0.1145	0.1310	0.1199	0.1178
		(0.0023)	(0.0005)	(0.0003)		(0.0036)	(0.0007)	(0.0004)
0.5	0.6065	0.6890	0.6334	0.6254	0.8125	0.9470	0.8566	0.8430
		(0.1060)	(0.0217)	(0.0110)		(0.2582)	(0.0521)	(0.0263)
1	1.3679	1.5953	1.4337	1.4064	2.0253	2.3747	2.1301	2.0868
		(0.5749)	(0.1106)	(0.0628)		(1.3594)	(0.2841)	(0.1624)
3	5.0498	5.7365	5.2649	5.1956	7.3503	8.0431	7.5663	7.5032
		(5.7522)	(1.4538)	(0.7779)		(7.6712)	(2.1495)	(1.1771)
5	9.0067	10.1627	9.4267	9.1709	11.8585	12.9786	12.2637	12.0090
		(14.5670)	(4.0923)	(2.2503)		(15.5874)	(4.5011)	(2.4938)
8	15.0003	16.7621	15.6084	15.4153	17.9889	19.7324	18.5909	18.4004
		(39.2835)	(10.0291)	(5.7988)		(39.5905)	(10.1231)	(5.8536)
10	19.0000	21.1845	19.6424	19.3291	21.9981	24.1746	22.6383	22.3258
		(66.1894)	(14.4882)	(9.1534)		(66.3298)	(14.5198)	(9.1712)

Table 3. Case 3 with $\beta_1 = 1.75, \;\; \beta_2 = 1.25$.

t	$M_d(t)$	$\hat{M}_d(t)$ $n,m = 10$	$\hat{M}_d(t)$ $n,m = 30$	$\hat{M}_d(t)$ $n,m = 50$	$V_d(t)$	$\hat{V}_d(t)$ $n,m = 10$	$\hat{V}_d(t)$ $n,m = 30$	$\hat{V}_d(t)$ $n,m = 50$
0.1	0.0031	0.0035	0.0033	0.0032	0.0032	0.0037	0.0034	0.0033
		(0.0000)	(0.0000)	(0.0000)		(0.0000)	(0.0000)	(0.0000)
0.5	0.0678	0.0766	0.0705	0.0693	0.0767	0.0873	0.0797	0.0783
		(0.0011)	(0.0003)	(0.0002)		(0.0013)	(0.0003)	(0.0002)
1	0.2327	0.2536	0.2434	0.2367	0.2744	0.3006	0.2869	0.2791
		(0.0076)	(0.0025)	(0.0013)		(0.0101)	(0.0029)	(0.0016)
3	1.2624	1.3370	1.2873	1.2789	1.4126	1.5420	1.4604	1.4394
		(0.1310)	(0.0361)	(0.0197)		(0.4187)	(0.0957)	(0.0505)
5	2.4155	2.6093	2.4470	2.4478	2.5254	2.9891	2.6126	2.6103
		(0.4958)	(0.1183)	(0.0772)		(2.1861)	(0.3978)	(0.2389)
8	4.1405	4.5266	4.2473	4.2337	4.1842	4.9505	4.3972	4.3819
		(1.6529)	(0.4766)	(0.2639)		(5.6240)	(1.2645)	(0.6991)
10	5.2851	5.8231	5.3990	5.3786	5.3131	6.3426	5.5272	5.4824
		(3.8529)	(0.6922)	(0.4425)		(11.0239)	(1.5891)	(1.0214)

Table 4. Case 4 with $\beta = 0.5$.

t	$M_d(t)$	$\hat{M}_d(t)$ $n = 10$	$\hat{M}_d(t)$ $n = 30$	$\hat{M}_d(t)$ $n = 50$	$V_d(t)$	$\hat{V}_d(t)$ $n = 10$	$\hat{V}_d(t)$ $n = 30$	$\hat{V}_d(t)$ $n = 50$
0.1	0.0181	0.0235	0.0195	0.0188	0.0190	0.0249	0.0205	0.0198
		(0.0003)	(0.0001)	(0.0000)		(0.0004)	(0.0001)	(0.0000)
0.5	0.3161	0.3706	0.3345	0.3283	0.3340	0.3802	0.3502	0.3451
		(0.0480)	(0.0100)	(0.0052)		(0.0394)	(0.0093)	(0.0050)
1	0.8647	0.9598	0.9024	0.8887	0.8200	0.8855	0.8454	0.8364
		(0.1613)	(0.0473)	(0.0263)		(0.0994)	(0.0290)	(0.0161)
3	2.9926	3.2272	3.0867	3.0445	2.5458	2.7969	2.6462	2.6023
		(1.1697)	(0.3545)	(0.2081)		(1.0706)	(0.3146)	(0.1826)
5	4.9998	5.5095	5.1766	5.1026	4.5023	5.0176	4.6810	4.6064
		(3.5834)	(0.9395)	(0.5199)		(3.5573)	(0.9318)	(0.5153)
8	8.0000	8.7709	8.2737	8.1403	7.5000	8.2713	7.7738	7.6404
		(9.4425)	(2.3890)	(1.3527)		(9.4404)	(2.3887)	(1.3525)
10	10.0000	11.1296	10.3662	10.1379	9.5000	10.6297	9.8662	9.6379
		(14.6183)	(3.9460)	(2.0100)		(14.6178)	(3.9460)	(2.0100)

For Case 1,3,5,7 in which F and G do not share common parameters, n unit samples from F (say X_1, X_2, ..., X_n) and m unit samples from G (say Y_1, Y_2, ..., Y_m) are generated. As for the other cases that F and G share common parameters, one sample is generated from G (say X_1) and $n-1$ unit samples are generated from F (say

Table 5. Case 5 with $\alpha = 1.5,\ \beta_1 = 2,\ \beta_2 = 3$.

t	$M_d(t)$	$\hat{M}_d(t)$ $n,m=10$	$\hat{M}_d(t)$ $n,m=30$	$\hat{M}_d(t)$ $n,m=50$	$V_d(t)$	$\hat{V}_d(t)$ $n,m=10$	$\hat{V}_d(t)$ $n,m=30$	$\hat{V}_d(t)$ $n,m=50$
0.1	0.0329	0.0358	0.0343	0.0335	0.0320	0.0347	0.0334	0.0326
		(0.0002)	(0.0000)	(0.0000)		(0.0001)	(0.0000)	(0.0000)
0.5	0.1590	0.1702	0.1650	0.1614	0.1448	0.1509	0.1487	0.1464
		(0.0030)	(0.0009)	(0.0004)		(0.0016)	(0.0006)	(0.0003)
1	0.3109	0.3269	0.3205	0.3145	0.2716	0.2719	0.2743	0.2722
		(0.0088)	(0.0030)	(0.0015)		(0.0050)	(0.0018)	(0.0010)
3	0.9162	0.9393	0.9341	0.9216	0.7725	0.7634	0.7720	0.7701
		(0.0489)	(0.0176)	(0.0089)		(0.0759)	(0.0253)	(0.0136)
5	1.5440	1.5859	1.5726	1.5520	1.2908	1.3069	1.2988	1.2918
		(0.1273)	(0.0430)	(0.0223)		(0.2849)	(0.0930)	(0.0503)
8	2.5164	2.6019	2.5655	2.5306	2.0506	2.1175	2.0769	2.0608
		(0.3474)	(0.1104)	(0.0584)		(0.8972)	(0.2828)	(0.1529)
10	3.1751	3.2957	3.2397	3.1943	2.5356	2.6322	2.5731	2.5518
		(0.5749)	(0.1789)	(0.0958)		(1.4765)	(0.4582)	(0.2471)

Table 6. Case 6 with $\alpha = 2.5,\ \beta = 1.5$.

t	$M_d(t)$	$\hat{M}_d(t)$ $n=10$	$\hat{M}_d(t)$ $n=30$	$\hat{M}_d(t)$ $n=50$	$V_d(t)$	$\hat{V}_d(t)$ $n=10$	$\hat{V}_d(t)$ $n=30$	$\hat{V}_d(t)$ $n=50$
0.1	0.0645	0.0852	0.0712	0.0691	0.0604	0.0763	0.0657	0.0641
		(0.0021)	(0.0005)	(0.0002)		(0.0013)	(0.0003)	(0.0002)
0.5	0.2849	0.3488	0.3070	0.3005	0.2066	0.2144	0.2105	0.2103
		(0.0207)	(0.0059)	(0.0031)		(0.0012)	(0.0006)	(0.0004)
1	0.4991	0.5705	0.5247	0.5183	0.2752	0.2492	0.2656	0.2699
		(0.0291)	(0.0100)	(0.0056)		(0.0034)	(0.0006)	(0.0003)
3	1.1020	1.1440	1.1135	1.1146	0.3975	0.3767	0.3880	0.3898
		(0.0401)	(0.0124)	(0.0071)		(0.0295)	(0.0109)	(0.0065)
5	1.6343	1.6962	1.6507	1.6516	0.6001	0.5876	0.5956	0.5957
		(0.1007)	(0.0276)	(0.0156)		(0.0673)	(0.0216)	(0.0127)
8	2.4334	2.5264	2.4586	2.4595	0.9230	0.8932	0.9097	0.9124
		(0.2483)	(0.0665)	(0.0379)		(0.1759)	(0.0551)	(0.0320)
10	2.9667	3.0792	2.9970	2.9982	1.1366	1.0985	1.1195	1.1230
		(0.3854)	(0.1025)	(0.0584)		(0.2767)	(0.0866)	(0.0504)

Table 7. Case 7 with $\alpha = 1.5,\ \beta_1 = 2.5,\ \beta_2 = 2$.

t	$M_d(t)$	$\hat{M}_d(t)$ $n,m=10$	$\hat{M}_d(t)$ $n,m=30$	$\hat{M}_d(t)$ $n,m=50$	$V_d(t)$	$\hat{V}_d(t)$ $n,m=10$	$\hat{V}_d(t)$ $n,m=30$	$\hat{V}_d(t)$ $n,m=50$
0.1	0.0489	0.0537	0.0503	0.0498	0.0469	0.0510	0.0481	0.0476
		(0.0003)	(0.0001)	(0.0000)		(0.0003)	(0.0001)	(0.0000)
0.5	0.2294	0.2466	0.2345	0.2326	0.1936	0.1993	0.1954	0.1948
		(0.0054)	(0.0015)	(0.0008)		(0.0019)	(0.0006)	(0.0004)
1	0.4362	0.4588	0.4428	0.4405	0.3354	0.3307	0.3340	0.3349
		(0.0139)	(0.0041)	(0.0023)		(0.0061)	(0.0020)	(0.0011)
3	1.2373	1.2726	1.2452	1.2440	0.8672	0.8670	0.8643	0.8668
		(0.0694)	(0.0225)	(0.0127)		(0.1120)	(0.0340)	(0.0188)
5	2.0827	2.1513	2.0975	2.0955	1.3930	1.4264	1.3987	1.3985
		(0.1861)	(0.0585)	(0.0332)		(0.4032)	(0.1127)	(0.0631)
8	3.3941	3.5225	3.4217	3.4177	2.0939	2.1806	2.1146	2.1090
		(0.5142)	(0.1544)	(0.0880)		(1.1457)	(0.3051)	(0.1719)
10	4.2771	4.4502	4.3146	4.3087	2.5258	2.6361	2.5533	2.5453
		(0.8474)	(0.2492)	(0.1424)		(1.7732)	(0.4706)	(0.2652)

$X_2, \ldots, X_{n-1})$. For all cases maximum likelihood estimators are used to estimate the model parameters of F and G in order to guarantee the strong consistency. Note that, maximum likelihood estimators for Case 1, 3 have been given in Section 3. For Case 2, the maximum likelihood estimator for β is found as $\left(2\sum_{i=1}^{n} X_i - X_1\right)/(2n)$. For Case 4,

Table 8. Case 8 with $\alpha = 2, \quad \beta = 1.5$.

t	$M_d(t)$	$\hat{M}_d(t)$ $n = 10$	$\hat{M}_d(t)$ $n = 30$	$\hat{M}_d(t)$ $n = 50$	$V_d(t)$	$\hat{V}_d(t)$ $n = 10$	$\hat{V}_d(t)$ $n = 30$	$\hat{V}_d(t)$ $n = 50$
0.1	0.0646	0.0667	0.0651	0.0649	0.0606	0.0625	0.0610	0.0609
		(0.0002)	(0.0000)	(0.0000)		(0.0001)	(0.0000)	(0.0000)
0.5	0.2946	0.3030	0.2965	0.2956	0.2303	0.2352	0.2312	0.2307
		(0.0032)	(0.0008)	(0.0005)		(0.0023)	(0.0006)	(0.0003)
1	0.5638	0.5771	0.5662	0.5651	0.4075	0.4107	0.4071	0.4068
		(0.0146)	(0.0037)	(0.0022)		(0.0136)	(0.0035)	(0.0020)
3	1.8432	1.8984	1.8548	1.8502	1.2476	1.2398	1.2416	1.2424
		(0.1819)	(0.0456)	(0.0282)		(0.1250)	(0.0338)	(0.0192)
5	3.2901	3.3832	3.3098	3.3022	1.9296	1.8994	1.9151	1.9178
		(0.5297)	(0.1336)	(0.0833)		(0.3092)	(0.0822)	(0.0455)
8	5.5291	5.6771	5.5604	5.5487	2.6890	2.6391	2.6655	2.6695
		(1.3563)	(0.3423)	(0.2148)		(0.7916)	(0.2076)	(0.1127)
10	7.0315	7.2163	7.0708	7.0563	3.1250	3.0660	3.0966	3.1009
		(2.1075)	(0.5316)	(0.3342)		(1.2772)	(0.3356)	(0.1815)

the maximum likelihood estimator for β is $(X_1 + \sum_{i=1}^{n} X_i)/(2n+2)$. Maximum likelihood estimators of α and β_1 in Case 5, 7 and of α and β in Case 6, 8 can not be found analytically. Then, the maximum likelihood estimators of the parameters can be numerically computed over joint log-likelihood function by using a non-linear optimization technique.

All the simulations are conducted in MATLAB. Throughout the simulation, replication number is taken as 1000. Sample sizes n and m are choosen as 10,30 and 50. The estimators $\hat{M}_d(t)$ and $\hat{V}_d(t)$ are computed at points $t = 0.1, 0.5, 1, 3, 5, 8, 10$. In order to compute the maximum likelihood estimators of the parameters when they have not analytical solutions, fmincon subroutine of MATLAB is performed. When $\hat{M}_d(t)$ and $\hat{V}_d(t)$ don't have analytical expressions, they are computed numerically as described in Section 4. For the computation of $\hat{M}_d(t)$ and $\hat{V}_d(t)$, the interval $[0, 10]$ is divided into 2000 subintervals with equal width $1/200$ for $\hat{M}_d(t)$ and into 1000 subintervals with equal width $1/100$ for $\hat{V}_d(t)$ for all cases 1 to 8. The same partition lengths are also used to compute the true values of $M_d(t)$ and $V_d(t)$ when analytical forms are not available. True values of functions and estimated values are given in Tables 1–8. Mean square error (MSE) values of the estimators are also given. Note that, values between parentheses denote MSE values of the related estimator.

It can be concluded from the Tables 1–8 that the plug-in estimators $\hat{M}_d(t)$ and $\hat{V}_d(t)$ work well for even small sample sizes: the simulated values $\hat{M}_d(t)$ and $\hat{V}_d(t)$ are getting closer to the true values and MSE values are decreasing rapidly as the sample size grows.

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Appendix

We provide the Matlab function for computation of $M_d(t)$ and $V_d(t)$, and give simulation codes for the Case7. Codes for other cases can be easily adapted from the present one.

```
% Function for computation of Md and Vd when
% G is Weibull with shape parameter alphaG and scale parameter betaG
% and F is Weibull with shape parameter alphaF and scale parameter betaF
% t: vector of time (computation points)
% pl: partition length for computation of Md
function [Md,Vd]=MdVdFwblGwbl(alphaF,betaF,alphaG,betaG,t,pl)
n = length(t);
k=(1/pl)*t(n);
l = k/2;
```

```
H(1)=0;
F1 = wblcdf(pl/2,betaF,alphaF);
for I = 2:k + 1
F(I)=wblcdf((I-.5)*pl,betaF,alphaF);
H(I)=wblcdf((I-1)*pl,betaF,alphaF);
end
for I = 2:k + 1
for J = 2:I
H(I)=H(I)+F(I-J + 1)*(H(J)-H(J-1));
end
H(I)=(H(I)-F1*H(I-1))/(1-F1);
end
for I = 1:k
H(I)=H(I + 1);
end
dH(1)=0;
F1 = wblcdf(pl/2,betaF,alphaF);
for I = 2:k + 1
F(I)=wblcdf((I-.5)*pl,betaF,alphaF);
dH(I)=wblcdf((I-1)*pl,betaG,alphaG);
end
for I = 2:k + 1
for J = 2:I
dH(I)=dH(I)+F(I-J + 1)*(dH(J)-dH(J-1));
end
dH(I)=(dH(I)-F1*dH(I-1))/(1-F1);
end
for I = 1:k
dH(I)=dH(I + 1);
end
for J = 1:l
ZZ = 0;
ZZ = ZZ+(dH((2*(J-1))+1)*(H(2)));
for I = 2:J
ZZ = ZZ+(dH((2*(J-I))+1)*(H(2*I)-H(2*(I-1))));
end
dV(J)=(2*ZZ)+(dH(2*J)*(1-dH(2*J)));
end
Md = dH(t*(1/pl));
Vd = dV(t*((1/2)/pl));
```

```
%Simulation for Case 7:
clear all
close all
clc
m = 30;
n = 30;
alphaF = 3/2;
betaF = 5/2;
alphaG = 1;
betaG = 2;
N = 1000;
t = [0.1 0.5 1 3 5 8 10];
pl = 1/200;
```

```
for i = 1:N
   y = exprnd(betaG,1,m);
   x = wblrnd(betaF,alphaF,1,n);
   betaGest(i)=mean(y);
   est = wblfit(x);
   betaFest(i)=est(1);
   alphaFest(i)=est(2);
   [Mdest(i,:),Vdest(i,:)]=MdVdFwblGwbl(alphaFest(i),betaFest(i),1,betaGest(i),t,pl);
   end
[Mdtrue,Vdtrue]=MdVdFwblGwbl(alphaF,betaF,1,betaG,t,pl);
Mdbias = Mdtrue-mean(Mdest);
Vdbias = Vdtrue-mean(Vdest);
MdMSE = Mdbias.^2 + var(Mdest);
VdMSE = Vdbias.^2 + var(Vdest);
```