Generalization of Zorn's Lemma: Maximal Crossed Ideals

E. Ilgaz Çağlayan and U. Ege Arslan

March 15, 2021

Abstract: In this paper we generalize some special ideals to crossed modules and give some relations. Also we prove one of the existence results in algebra for crossed modules using Zorn's lemma that a crossed module with nontrivial base ring with unity contains a maximal crossed ideal.

Keywords: The category of crossed module, Maximal crossed ideals, Prime crossed ideals, Coprime crossed ideals.

AMS Mathematics Subject Classification: 18A05, 18A99, 18D35.

1 Introduction

Crossed modules of groups, introduced by Whitehead [11], originate in algebraic topology and more particularly in homotopy theory. Mac Lane and Whitehead showed in [7] that crossed modules of groups modelled homotopy 2-types (3-types in their notation). The commutative algebra version of this construction has been given by T. Porter, [3]. Crossed modules are algebraic objects with rich structure. They provide a simultaneous generalization of the concepts of ideals and modules. Furthermore, we may consider any algebra as a crossed module. For any algebra C, we have the crossed module $id : C \to C$. Therefore it is of interest to investigate generalizations of algebra (ring) theoretic concepts and structures to crossed modules. We interest some possible such generalizations.

In this paper we generalize certain well known some special ideals such as maximal, prime and coprime ideals to crossed modules and study the connections between these concepts. Also, we give the main result that (C, R, ∂) has a maximal ideal where (C, R, ∂) is a crossed module such that R has an identity $1_R \neq 0$, as an application of Zorn's Lemma.

2 Preliminaries

We recall the definition and elementary theory of crossed modules of commutative algebras given in [10].

Let k be a fixed commutative ring, R be a k-algebra with identity. All k-algebras will be commutative and will not necessarily have identities unless this is specified.

Definition 1 *A crossed module, (C, R, ∂), consists of an R-algebra, C, and a k-algebra morphism $\partial : C \to R$ with an action R on C, written $(r, c) \longmapsto r \cdot c$ such that, for $r \in R$, $c_1, c_2 \in C$,*

$$CM1)\ \partial(r \cdot c) = r\partial(c) \ \ and \ \ CM2)\ \partial(c_1) \cdot c_2 = c_1c_2.$$

A morphism of crossed modules, $(\mu, \eta) : (C, R, \partial) \to (C', R', \partial')$ consists of a pair of k-algebra morphisms $\mu : C \to C'$, $\eta : R \to R'$ such that $\partial'\mu = \eta\partial$ and $\mu(r \cdot c) = \eta(r) \cdot \mu(c)$ for all $r \in R$, and $c \in C$.

Examples:

(i) Any ideal, I, in R gives an inclusion map $I \xhookrightarrow{inc.} R$ which is a crossed module. Conversely, given any crossed module $\partial : C \longrightarrow R$, the image $I = \partial(C)$ of C is an ideal in R.

(ii) Any R-module M can be considered as an R-algebra with zero multiplication and hence $0 : M \longrightarrow R$ is a crossed module. Conversely, if $\partial : C \longrightarrow R$ is a crossed R-module, $\ker\partial$ is an $R/\partial(C)$-module.

(iii) Let (C, R, ∂) be a crossed module and I be an ideal of R such that $I \subseteq Ann(C) = \{r \in R \mid r \cdot c = 0\}$. Then $(C, R/I, \overline{\partial})$ is a crossed module.

(iv) Let (C,R,∂) be a crossed module and C' be an ideal of C and $RC' \subseteq C'$. Then $(C/C',R,\overline{\partial})$ is a crossed module.

(v) Let (C,R,∂) be a crossed module and I be an ideal of R. Then $I \subseteq Ann(C/IC)$ and $(C/IC,R/I,\overline{\partial})$ is a crossed module.

(vi) Let $\mathrm{M}(R)$ be the multiplication algebra defined by Mac Lane [8] (see also [6]) as the set of all multipliers $\delta : R \to R$ such that for all $r,r' \in R$, $\delta(rr') = r\delta(r')$ where R is a commutative k-algebra and $Ann(R) = 0$ or $R^2 = R$. Then $\mu : R \to M(R)$ is a crossed module given by $\mu(r) = \delta_r$ with $\delta_r(r') = rr'$ for all $r,r' \in R$. (See [2],[5] for details).

A crossed module (C',R',∂') is a *subcrossed module* of a crossed module (C,R,∂) if C',R' are subalgebras of C,R, respectively, $\partial' = \partial|_{C'}$ and the action of R' on C' is induced by the action of R on C. Also, if C',R' are ideals of C,R, respectively and $R'C \subseteq C'$, $RC' \subseteq C'$, then a subcrossed module (C',R',∂') of a crossed module (C,R,∂) is called *crossed ideal* and the set of all ideals of (C,R,∂) is denoted by $\mathrm{Id}(C,R,\partial)$. Consequently, we have the *quotient crossed module* of (C,R,∂) by (C',R',∂'), $(C/C',R/R',\widetilde{\partial})$ with induced boundary map and action [9].

3 Some special crossed ideals

As an analogues of algebras, we can define some special ideals such as maximal, prime and coprime ideals. First we will explain some notations and then give some definitions and properties about these special crossed ideals and relations with factor crossed modules.

Let (C,R,∂) be a crossed module and let C_1,C_2 be ideals of C, R_1,R_2 be ideals of R. $\langle D_{R_1}(C_1)\rangle$ will denote the ideal of C generated by the set $\{r_1 \cdot c_1 : r_1 \in R_1, c_1 \in C_1\}$ and $\langle D_{R_2}(C_1), D_{R_1}(C_2)\rangle$ will denote the ideal of C generated by the set $\{r_2 \cdot c_1, r_1 \cdot c_2 : r_1 \in R_1, r_2 \in R_2, c_1 \in C_1, c_2 \in C_2\}$.

Proposition 2 *Let (C,R,∂) be a crossed module and $Id(C,R,\partial)$ be the set of all ideals of (C,R,∂). If we define addition and multiplication on $Id(C,R,\partial)$ by*

$$(C_1,R_1,\partial) + (C_2,R_2,\partial) = (C_1 + C_2, R_1 + R_2, \partial)$$

and

$$(C_1, R_1, \partial) \cdot (C_2, R_2, \partial) = (< D_{R_2}(C_1), D_{R_1}(C_2) >, R_1R_2, \partial).$$

We obtain an algebraic structure $(Id(C, R, \partial), +, \cdot)$ *with the following properties:*

(a) The addition in $Id(C, R, \partial)$ *is commutative and associative and posseses a neutral element,namely* $(0, 0, \partial)$.

(b) The multiplication in $Id(C, R, \partial)$ *is associative and commutative. If* R *has an identity* $(1_R \neq 0)$, *then* $(C_1, R_1, \partial) \cdot (C, R, \partial) = (C_1, R_1, \partial)$.

(c) The distributive laws $(C_1, R_1, \partial) \cdot ((C_2, R_2, \partial) + (C_3, R_3, \partial)) = (C_1, R_1, \partial) \cdot (C_2, R_2, \partial) + (C_1, R_1, \partial) \cdot (C_3, R_3, \partial)$ *and* $((C_1, R_1, \partial) + (C_2, R_2, \partial)) \cdot (C_3, R_3, \partial) = (C_1, R_1, \partial) \cdot (C_3, R_3, \partial) + (C_2, R_2, \partial) \cdot (C_3, R_3, \partial)$ *hold.*

(d) $(C_1, R_1, \partial) \cdot (C_2, R_2, \partial) \subseteq (C_1 \cap C_2, R_1 \cap R_2, \partial) \subseteq (C_1, R_1, \partial) + (C_2, R_2, \partial)$ *for all* (C_1, R_1, ∂), $(C_2, R_2, \partial) \in Id(C, R, \partial)$.

Remark: From the above properties of addition and multiplication, $Id(C, R, \partial)$ is "almost a ring"; the only axiom which is not satisfied is the existence of additive inverses.

Proposition 3 *If* (C_i, R_i, ∂) *is a family of ideals of a crossed module* (C, R, ∂), *then the intersection* $\cap(C_i, R_i, \partial) = (\cap C_i, \cap R_i, \partial)$ *is also an ideal of* (C, R, ∂).

Proof. It is clear from the definition of crossed ideal. ■

We can state isomorphism theorems for crossed modules that describe the relationship between quotients, homomorphisms, and subobjects.

Theorem 4 *(First Isomorphism Theorem) Let* $(\mu, \eta) : (C, R, \partial) \to (C', R', \partial')$ *be a morphism of crossed modules, then*

$$\frac{(C, R, \partial)}{Ker(\mu, \eta)} \cong \mathrm{Im}(\mu, \eta).$$

where $Ker(\mu, \eta)$ *is the crossed module* $(Ker\mu, Ker\eta, \partial)$ *and* $\mathrm{Im}(\mu, \eta)$ *is the crossed module* $(\mathrm{Im}\, \mu, \mathrm{Im}\, \eta, \partial')$.

Theorem 5 *(Second Isomorphism Theorem)Let* (C_1, R_1, ∂_1) *and* (C_2, R_2, ∂_2) *be subcrossed modules of* (C, R, ∂) *and* (C_1, R_1, ∂_1) *be an ideal of* (C, R, ∂)*. Then,*

$$\frac{(C_1, R_1, \partial_1) + (C_2, R_2, \partial_2)}{(C_1, R_1, \partial_1)} \cong \frac{(C_2, R_2, \partial_2)}{(C_1, R_1, \partial_1) \cap (C_2, R_2, \partial_2)}.$$

Theorem 6 *(Third Isomorphism Theorem) Let* (C_1, R_1, ∂_1) *and* (C_2, R_2, ∂_2) *be ideals of* (C, R, ∂) *such that* $(C_2, R_2, \partial_2) \subseteq (C_1, R_1, \partial_1)$*, then*

$$\frac{(C, R, \partial)/(C_2, R_2, \partial_2)}{(C_1, R_1, \partial_1)/(C_2, R_2, \partial_2)} \cong (C, R, \partial)/(C_1, R_1, \partial_1).$$

Theorem 7 *Suppose* R *has an identity* $1_R \neq 0$ *and let* (C_1, R_1, ∂) *be an ideal of crossed module* (C, R, ∂)*. Then for* $n \geq 2$*,*

$$(C_1, R_1, \partial)^n = (\langle D_{R_1^{n-1}}(C_1)\rangle, R_1^n, \partial)$$

where $D_{R_1^{n-1}}(C_1) = \{r_1 \cdot (r_2 \cdot (\cdots (r_{n-2} \cdot c_1) \cdots)) \mid r_1, \ldots, r_{n-2} \in R_1, c \in C_1\}$*. If* $C_1 = C$ *and* C *has an identity* $1_C \neq 0$*, then* $(C, R_1, \partial)^n = (C, R_1^n, \partial)$*.*

Proof. It is clear from the induction. ■

Proposition 8 *If* $(\alpha, \phi) : (C, R, \mu) \longrightarrow (C', R', \mu')$ *is a surjective crossed module morphism, then*

$$(\alpha, \phi)((C, R, \mu)^n) = (C', R', \mu')^n$$

for some $n \in N$*.*

Proof.

$$\begin{aligned} (\alpha, \phi)((C, R, \mu)^n) &= (\alpha, \phi)((D^{n-1}, R^n, \mu)) \\ &= (\alpha(D^{n-1}), \phi(R^n), \mu') \\ &\overset{(*)}{=} (D'^{n-1}, R'^n, \mu') \\ &= (C', R', \mu')^n \end{aligned}$$

where $D'^{n-1} = \{r'_1 \cdot (r'_2 \cdot (\ldots (r'_{n-1} \cdot c')\ldots)) \mid r'_1, \ldots, r'_{n-1} \in R',\ c' \in C\}$. We obtain the equation $(*)$ by induction on n. See [1] for details. ∎

The annihilator $Ann(C, R, \mu)$ of a crossed module (C, R, μ) is defined in [2] to be the kernel of a morphism of crossed modules $(C, R, \mu) \longrightarrow \mathcal{A}(C, R, \mu)$ where $\mathcal{A}(C, R, \mu)$ is an actor of crossed module (C, R, μ) and clearly $Ann(C, R, \mu)$ is an ideal of (C, R, μ).

We will give following results related with $Ann(C, R, \mu)$

Proposition 9 *Let (C', R', μ) be an ideal of (C, R, μ).*

a) If $(C', R', \mu) \cdot (C, R, \mu) = 0$ then $(C', R', \mu) \subseteq Ann(C, R, \mu)$

b) $(C, R, \mu) \cdot Ann(C, R, \mu) = 0$.

Proof. See [1] for details. ∎

4 Generalization of Zorn's Lemma: Maximal crossed ideals

Zorn's lemma is one of the most famous and useful result in mathematics. It is a result in set theory that appears in proofs of some non-constructive existence theorems throughout mathematics. It shows up in many areas including the functional analysis, algebra and topology. We will give a generalization of Zorn's lemma to crossed modules as follows.

We note that $Id(C, R, \partial)$ is partially ordered set as following order relation:

$$(C_1, R_1, \partial) \subseteq (C_2, R_2, \partial) \text{ if and only if } C_1 \subseteq C_2 \text{ and } R_1 \subseteq R_2.$$

for $(C_1, R_1, \partial), (C_2, R_2, \partial) \in Id(C, R, \partial)$.

Definition 10 *An ideal (C_1, R_1, ∂) of a crossed module (C, R, ∂) is said to be maximal if $(C_1, R_1, \partial) \neq (C, R, \partial)$ and for every ideal (C_2, R_2, ∂) such that $(C_1, R_1, \partial) \subset (C_2, R_2, \partial) \subset (C, R, \partial)$, either $(C_2, R_2, \partial) = (C_1, R_1, \partial)$ or $(C_2, R_2, \partial) = (C, R, \partial)$.*

We state the following results without proof.

Proposition 11 *Suppose R has an identity $1_R \neq 0$. (C, R, ∂) has no ideal (C_1, R, ∂) such that $C_1 \neq C$.*

Lemma 12 *Let (C, R, ∂) be a crossed module such that R has an identity $1_R \neq 0$ and (C_1, R_1, ∂) be a proper ideal. Then $R_1 \neq R$.*

We now introduce a generalization of Zorn's Lemma.

Theorem 13 *Let (C, R, ∂) be a crossed module such that R has an identity $1_R \neq 0$. Then (C, R, ∂) has a maximal ideal.*

Proof. The proof is a straightforward application of Zorn's Lemma to the partially ordered set of all proper ideals of $(C, R, \partial), \tau$. In order to apply Zorn's Lemma we must show that every chain $\Delta = \{(C_i, R_i, \partial) : i \in \Gamma\}$ in τ,indexed by some set Γ, has an upper bound in τ. Let

$$\bar{C} = \bigcup_{i\in\Gamma} C_i \text{ and } \bar{R} = \bigcup_{i\in\Gamma} R_i.$$

be the unions of all the C_i and R_i, respectively. Clearly, $C_1 \neq \emptyset, C_1 \subseteq C$ and $R_1 \neq \emptyset, R_1 \subseteq R$.

If x and y are in $\bar{C}$ then there are $i, j \in \Gamma$ with $x \in C_i$ and $y \in C_j$. Since Δ is a chain, either $C_i \subseteq C_j$ or $C_j \subseteq C_i$; without loss of generality we assume that $C_i \subseteq C_j$. Then $x, y \in C_j$. Since C_j is an ideal of C we have $x - y \in C_j \subseteq C$ and $cx, xc \in C_j \subseteq \bar{C}$ for $x \in C_j, c \in C$. This proves that $\bar{C}$ is an ideal of C. Similarly, $\bar{R}$ is an ideal of R. The reader can check $\bar{C}$ and $\bar{R}$ are algebras of C.

We show that $(\bar{C}, \bar{R}, \partial)$ is a subcrossed module of (C, R, ∂). Let $c \in \bar{C}$. There is a $(C_{i_0}, R_{i_0}, \partial) \in \Delta$ such that $c \in C_{i_0}$. Then $\partial(c) \in R_{i_0}$ and so $\partial(c) \in \bigcup_{i\in\Gamma} R_i = \bar{R}$. That is $\partial(\bar{C}) \subseteq \bar{R}$. Let $r \in \bar{R}$ and $c \in \bar{C}$. Then there are (C_i, R_i, ∂), $(C_j, R_j, \partial) \in \Delta$ such that $r \in R_i$ and $c \in C_j$. Since Δ is a chain, either $(C_i, R_i, \partial) \subseteq (C_j, R_j, \partial)$ or $(C_j, R_j, \partial) \subseteq (C_i, R_i, \partial)$. Thus, either $c \in C_i, r \in R_i$ or $c \in C_j, r \in R_j$. Hence, either $r_1 \cdot c_1 \in C_j$ or $r_1 \cdot c_1 \in C_i$. Therefore, $r_1 \cdot c_1 \in \bar{C}$ and so $(\bar{C}, \bar{R}, \partial)$ is a subcrossed module of (C, R, ∂).

So, to show that $(\bar{C}, \bar{R}, \partial)$ is an ideal of (C, R, ∂), we must show that $R\bar{C} \subseteq \bar{C}$ and $\bar{R}C \subseteq \bar{C}$.

Let $r \in R$ and $c_1 \in \bar{C}$. Then there is a $(C_{i_0}, R_{i_0}, \partial) \in \Delta$ such that $c_1 \in C_{i_0}$. Since $(C_{i_0}, R_{i_0}, \partial)$ is an ideal, $RC_{i_0} \subseteq C_{i_0}$. Thus, $rc_1 \in C_{i_0}$ and so $rc_1 \in C_1$. Hence $R\bar{C} \subseteq \bar{C}$.

Similarly we get $\bar{R}C \subseteq \bar{C}$.

To see that $(\bar{C}, \bar{R}, \partial)$ is a proper ideal, note that $1_R \notin R_i$ for each i since each $(C_i, R_i, \partial) \in \tau$ is a proper ideal of (C, R, ∂). Therefore $1_R \notin \cup R_i = \bar{R}$, so $(\bar{C}, \bar{R}, \partial) \neq (C, R, \partial)$ and is $(\bar{C}, \bar{R}, \partial)$ indeed a proper ideal of (C, R, ∂). It is an upper bound for the chain since $(C_i, R_i, \partial) \subseteq (\bar{C}, \bar{R}, \partial)$ for each $i \in \Gamma$. So the hypotheses of Zorn's lemma applies. There is then a maximal element (M, N, ∂) of τ. We claim that (M, N, ∂) is a maximal ideal of (C, R, ∂). To prove this, suppose that $(M, N, \partial) \neq (C, R, \partial)$ and $(M, N, \partial) \subseteq (I, J, \partial) \subseteq (C, R, \partial)$ with (I, J, ∂) an ideal of (C, R, ∂). Maximality of (M, N, ∂) then forces $(M, N, \partial) = (I, J, \partial)$. Thus either $(I, J, \partial) = (M, N, \partial)$ or $(I, J, \partial) = (C, R, \partial)$. ■

Corollary 14 *Let (C, R, ∂) be a crossed module such that R has an identity $1_R \neq 0$. Every proper ideal of (C, R, ∂) is contained in a maximal ideal.*

Proof. Apply Zorn's Lemma to the set of all proper ideals of (C, R, ∂) which contain any proper ideal (I, J, ∂) of (C, R, ∂),

$$\{(S, T, \partial) \in Id(C, R, \partial) | (S, T, \partial) \neq (C, R, \partial) \text{ and } (I, J, \partial) \subseteq (S, T, \partial)\}.$$

■

Theorem 15 *Let (C_1, R_1, ∂) be an ideal of (C, R, ∂) such that R has an identity $1_R \neq 0$. (C_1, R_1, ∂) is a maximal ideal if and only if R_1 is a maximal ideal of R and C_1 is a maximal ideal of C with respect to $\partial(C_1) \subseteq R_1$.*

Proof. Suppose that (C_1, R_1, ∂) is a maximal ideal but R_1 is not a maximal ideal of R. Hence, there is a proper ideal R_2 of R such that $R_1 \subsetneq R_2$. Then $(\langle D_R(C_1), D_{R_2}(C) \rangle, R_2, \partial)$ is an ideal of (C,R,∂). Moreover, $(\langle D_R(C_1),D_{R_2}(C)\rangle,R_2,\partial)\neq(C,R,\partial)$, $(\langle D_R(C_1),D_{R_2}(C)\rangle,R_2,\partial)\neq(C_1,R_1,\partial)$ and $(C_1,R_1,\partial)\subseteq(\langle D_R(C_1),D_{R_2}(C)\rangle,R_2,\partial)$. This contradicts to our assumption that (C_1,R_1,∂) is a maximal ideal of (C,R,∂). Hence R_1 is a maximal ideal of R. Now, we will show that there is no ideal C_2 of C such that $C_1\subsetneqq C_2$ and $\partial(C_2)\subseteq R_1$. Suppose that there is an ideal C_2 of C such that $C_1\subsetneqq C_2$ and $\partial(C_2)\subseteq R_1$. Then $(C_1,R_1,\partial)\subseteq(\langle D_R(C_2),D_{R_1}(C)\rangle,R_1,\partial)$ is an ideal of (C,R,∂). Since $R_1\neq R$ and (C_1,R_1,∂) is maximal, we have $C_1=\langle D_R(C_2),D_{R_1}(C)\rangle$. Thus, $D_R(C_2)\subseteq C_1$. Therefore, if $c_2\in C_2$, then $c_2=1_R.c_2\in D_R(C_2)\subseteq C_1$ and so $C_2\subseteq C_1$. This contradicts to $C_1\neq C_2, C_1\subseteq C_2$. Thus C_1 is a maximal ideal of C with respect to $\partial(C_1)\subseteq R_1$.

Let R_1 be a maximal ideal of R and C_1 be an ideal of C maximal with respect to $\partial(C_1)\subseteq R_1$. We will show that (C_1,R_1,∂) is a maximal ideal of (C,R,∂). Suppose (C_2,R_2,∂) is an ideal of (C,R,∂) such that $(C_1,R_1,\partial)\subseteq(C_2,R_2,\partial)$. Then $C_1\subseteq C_2\subseteq C$ and $R_1\subseteq R_2\subseteq R$. Since R_1 is a maximal ideal of R, we can write $R_1=R_2$ or $R_2=R$. If $R_2=R$, as (C,R,∂) has no ideal (C_2,R,∂) such that $C\neq C_2$, then $(C_2,R_2,\partial)=(C,R,\partial)$. If $R_1=R_2$, then $(C_1,R_1,\partial)\subseteq(C_2,R_1,\partial)$. Since (C_2,R_1,∂) is an ideal of (C,R,∂), we have $\partial(C_2)\subseteq R_1$. As C_1 is a maximal ideal of C with respect to $\partial(C_1)\subseteq R_1$, we get $C_2\subseteq C_1$. Consequently, we obtain $C_2=C_1$. That is $(C_1,R_1,\partial)=(C_2,R_2,\partial)$. Therefore, (C_1,R_1,∂) is a maximal ideal of (C,R,∂). ■

Now, we will explain some relations between maximal ideals and factor crossed modules.

Proposition 16 *Let R be a field and (C,R,∂) be a crossed module. Then, $\partial=0$ or $\partial(C)=R$.*

Proof. The proof is immediate. ■

Theorem 17 *Let (C_1,R_1,∂) be an ideal of (C,R,∂) and R be a field. Then*

1) $\widetilde{\partial}=0$ or

2) $\widetilde{\partial}(C/C_1)=R/R_1$. In this case, $(C/C_1,R/R_1,\widetilde{\partial})\cong(R/R_1,R/R_1,id)$.

where $\widetilde{\partial}:C/C_1\to R/R_1$ is the quotient map $\widetilde{\partial}(c+C_1)=\partial(c)+R_1$.

Proof. It is obvious from the Proposition 16. ■

Theorem 18 *Let* (C_1, R_1, ∂) *be an ideal of* (C, R, ∂) *such that* R *and* C *have identities* $1_R \neq 0, 1_C \neq 0$. (C_1, R_1, ∂) *is a maximal ideal if and only if*

a) R/R_1 *is a field and* $C = C_1$ *or*

b) R/R_1 *and* C/C_1 *are fields.*

Proof. Suppose that (C_1, R_1, ∂) is a maximal ideal of (C, R, ∂). Then R_1 is a maximal ideal of R, by Theorem 15. Hence R/R_1 is a field. On the other hand, since R/R_1 is a field, we can write by Theorem 17,

i) $\widetilde{\partial} = 0$ or

ii) $(C/C_1, R/R_1, \widetilde{\partial}) \cong (R/R_1, R/R_1, id)$.

If $\widetilde{\partial} = 0$, then $\widetilde{\partial}(c + C_1) = \partial(c) + R_1 = R_1$ for every $c \in C$ i.e. $\partial(c) \in R_1$ and so $\partial(C) \subseteq R_1$. Since (C_1, R_1, ∂) is a maximal ideal, we have, by the Theorem 15, $C = C_1$. Thus (a) is true.

If $(C/C_1, R/R_1, \widetilde{\partial}) \cong (R/R_1, R/R_1, id)$, then $C/C_1 \cong R/R_1$. Since R/R_1 is a field, C/C_1 is a field. Hence, (b) is true.

Now suppose that (a) or (b) is true. Let (a) be true. Then, R/R_1 is a field and $C = C_1$. Hence, R_1 is maximal and so $(C_1, R_1, \partial) = (C, R_1, \partial)$. This implies that (C_1, R_1, ∂) is a maximal ideal. Let (b) be true. Then, R/R_1 and C/C_1 both are fields and so R_1 and C_1 are both maximal ideals. So, we can write, by the Theorem 15, (C_1, R_1, ∂) is a maximal ideal of (C, R, ∂). ■

Theorem 19 *Let* (C_1, R_1, ∂) *be an ideal of* (C, R, ∂) *such that* R *and* C *have identities* $1_R \neq 0$ *and* $1_C \neq 0$. (C_1, R_1, ∂) *is a maximal ideal if and only if* R/R_1 *is a field and*

a) If $\partial(C) \subseteq R_1$, *then* $C_1 = C$;

b) If $\partial(C) \nsubseteq R_1$, *then* C/C_1 *is a field.*

Proof. Let (C_1, R_1, ∂) be a maximal ideal. Then, R_1 is maximal and so R/R_1 is a field. On the other hand, if $\partial(C) \subseteq R_1$ then (C, R_1, ∂) is an ideal of (C, R, ∂) and $(C, R_1, \partial) \neq (C, R, \partial)$. Moreover, $(C_1, R_1, \partial) \subseteq (C, R_1, \partial)$. Since (C_1, R_1, ∂) is a maximal ideal, we have $C_1 = C$. Thus (a) is true. Let $\partial(C) \nsubseteq R_1$. Then, in the factor crossed module $(C/C_1, R/R_1, \widetilde{\partial})$, $\widetilde{\partial} \neq 0$. Since R/R_1 is a field, we have $\widetilde{\partial}(C/C_1) = R/R_1$ and $(C/C_1, R/R_1, \widetilde{\partial}) \cong (R/R_1, R/R_1, id)$ by Theorem 17. Since R/R_1 is a field, C/C_1 is a field. Hence (b) is true.

Suppose that (a) is true. Then, R/R_1 is a field and $C = C_1$. Thus R_1 is a maximal ideal. This implies that $(C_1, R_1, \partial) = (C, R_1, \partial)$ is a maximal ideal. Now suppose that (b) is true. Then R/R_1 and C/C_1 are both fields and so R_1 and C_1 are both maximal ideals. (C, R_1, ∂) is not an ideal because of $\partial(C) \nsubseteq R_1$. Since (C, R, ∂) has not any ideal of the form (C_1, R, ∂), we can say that (C_1, R_1, ∂) is a maximal ideal. ■

Corollary 20 *Let (C_1, R_1, ∂) be an ideal of (C, R, ∂) such that R and C have identities $1_R \neq 0, 1_C \neq 0$ and $\partial(C) \nsubseteq R_1$. (C_1, R_1, ∂) is a maximal ideal if and only if C/C_1 and R/R_1 are fields i.e C_1 and R_1 are both maximal ideals.*

Proof. It is clear from Theorem 19, (C_1, R_1, ∂) is maximal if and only if C/C_1 and R/R_1 are both are fields. On the other hand, C/C_1 and R/R_1 are both fields if and only if C_1 and R_1 are both maximal ideals. ■

Theorem 21 *Let (C, R, ∂) be a crossed module such that R has an identity $1_R \neq 0$ and R_1 be a maximal ideal of R. Then there is a unique maximal ideal (C_1, R_1, ∂) of (C, R, ∂).*

Proof. Suppose that (C_2, R_1, ∂) and (C_3, R_1, ∂) are maximal ideals. Then $(C, R, \partial) \neq (\langle D_R(C_2 \cup C_3)\rangle, R_1, \partial)$ is an ideal and $(C_2, R_1, \partial), (C_3, R_1, \partial) \subseteq (\langle D_R(C_2 \cup C_3)\rangle, R_1, \partial)$. On the other hand, since (C_2, R_1, ∂) and (C_3, R_1, ∂) are maximal ideals, we have $(C_2, R_1, \partial) = (\langle D_R(C_2 \cup C_3)\rangle, R_1, \partial) = (C_3, R_1, \partial)$ and so $C_2 = C_3$. ■

Corollary 22 *Let (C, R, ∂) be a crossed module such that R has an identity $1_R \neq 0$. There is an one to one map from the set of maximal ideals of (C, R, ∂) onto the set of maximal ideals of R.*

Theorem 23 *Let (C, R, ∂) be a crossed ideal such that R and C have identities $1_R \neq 0$, $1_C \neq 0$ and (C_1, R_1, ∂) be an ideal of (C, R, ∂). If $\bigcap_{n=1}^{\infty} R_1^n = 0$, then $C_1 \neq C$.*

Proof. Suppose that $C_1 = C$. Since R has an identity, we have $\langle D_{R_1^{n-1}}(C)\rangle = C$ and $(C_1, R_1, \partial)^n = (\langle D_{R_1^{n-1}}(C)\rangle, R_1^n, \partial) = (C, R_1^n, \partial)$. Hence

$$\bigcap_{n=1}^{\infty}(C, R_1, \partial)^n = (\bigcap_{n=1}^{\infty}\langle D_{R_1^{n-1}}(C)\rangle, \bigcap_{n=1}^{\infty} R_1^n, \partial) = (C, 0, \partial).$$

That is, we have $\partial = 0$. This contradicts to C has an identity and $(C, R, 0)$ is a crossed module. Thus $C_1 \neq C$. ■

Corollary 24 *Let (C, R, ∂) be a crossed module such that R and C have identities $1_R \neq 0, 1_C \neq 0$ and R be a local ring with maximal ideal R_1. If $\bigcap_{n=1}^{\infty} R_1^n = 0$, then ∂ is onto.*

Proof. Suppose that ∂ is not onto. Then $\partial(C)$ is a proper ideal of R and so $\partial(C) \subseteq R_1$. On the other hand, (C, R_1, ∂) is a crossed ideal and

$$\bigcap_{n=1}^{\infty}(C, R_1, \partial)^n = (\bigcap_{n=1}^{\infty}\langle D_{R_1^{n-1}}(C)\rangle, \bigcap_{n=1}^{\infty} R_1^n, \partial) = (C, \bigcap_{n=1}^{\infty} R_1^n, \partial) = (C, 0, \partial).$$

That is, we have $\partial = 0$. This is a contradiction. So, ∂ is onto. ■

Corollary 25 *Let (C, R, ∂) be a crossed module such that R and C have identities $1_R \neq 0, 1_C \neq 0, R$ be a local ring with maximal ideal R_1 and ∂ be not onto. If $\bigcap_{n=1}^{\infty} R_1^n = 0$, then (C_1, R_1, ∂) is maximal if and only if R_1 and C_1 are maximal ideals.*

Theorem 26 *Let (C, R, ∂) be a crossed module such that R and C have identities $1_R \neq 0$, $1_C \neq 0$ and (C_1, R_1, ∂) be an ideal of (C, R, ∂). If $\bigcap_{n=1}^{\infty}(C_1, R_1, \partial)^n = (0, 0, \partial)$ then $C_1 \neq C$.*

Proof. Suppose that $C_1 = C$. Then $(C_1, R_1, \partial)^n = (\langle D_{R_1^{n-1}}(C)\rangle, R_1^n, \partial)$. As $C^n = C$ for every $n \in \mathbb{N}$, if $c_1c_2\cdots c_{n-1}c_n \in C^n$, then

$$c_1c_2\cdots c_{n-1}c_n = \partial(c_1c_2\cdots c_{n-1})c_n = \partial(c_1)\partial(c_2)\cdots\partial(c_{n-1})c_n \in D_{R_1^{n-1}}(C)$$

and so $C = \langle D_{R_1^{n-1}}(C)\rangle$. Since

$$\bigcap_{n=1}^{\infty}(C,R_1,\partial)^n = (\bigcap_{n=1}^{\infty}\langle D_{R_1^{n-1}}(C)\rangle, \bigcap_{n=1}^{\infty} R_1^n, \partial) = (0,0,\partial), \text{ then we have}$$

$$C = \bigcap_{n=1}^{\infty}\langle D_{R_1^{n-1}}(C)\rangle = 0 \text{ and } \bigcap_{n=1}^{\infty} R_1^n = 0.$$

Since $1_C \in C$ and $1_C \neq 0$, this is a contradiction . Thus $C_1 \neq C$. ■

Theorem 27 *Let (C,R,∂) be a crossed module such that R and C have identities $1_R \neq 0$, $1_C \neq 0$ and let (C_1,R_1,∂) be an ideal of (C,R,∂) such that for some $m \in \mathbb{N}$, $R_1^m = 0$. Then $C \neq C_1$.*

Proof. Suppose that $C_1 = C$. Then for every $n \in \mathbb{N}$, $(C,R_1,\partial)^n = (\langle D_{R_1^{n-1}}(C)\rangle, R_1^n, \partial) = (C, R_1^n, \partial)$. Since $R_1^m = 0$ for $m \in \mathbb{N}$ and $R_1^{m+1} = 0$, we have $\langle D_{R_1^m}(C)\rangle = 0$ and $(C,R_1,\partial)^{m+1} = (\langle D_{R_1^m}(C)\rangle, R_1^{m+1}, \partial) = (C, R_1^{m+1}, \partial)$. Thus $C = 0$ and so $C_1 \neq C$. ■

Definition 28 *An ideal (C_3,R_3,∂) of (C,R,∂) is said to be prime if $(C_3,R_3,\partial) \neq (C,R,\partial)$ and $(C_1,R_1,\partial)(C_2,R_2,\partial) \subseteq (C_3,R_3,\partial)$, for any ideals (C_1,R_1,∂), (C_2,R_2,∂) of (C,R,∂) then $(C_1,R_1,\partial) \subseteq (C,R,\partial)$ or $(C_2,R_2,\partial) \subseteq (C,R,\partial)$.*

We also note the following results, whose proofs take place in [4].

Proposition 29 *Let (C,R,∂) be a crossed module. If R' is a maximal ideal of R, then $(\partial^{-1}(R'), R', \partial)$ is prime.*

Corollary 30 *Let (C,R,∂) be a crossed module. If R' is a prime ideal of R, then $(\partial^{-1}(R'), R', \partial)$ is prime.*

Theorem 31 *Let (C,R,∂) be a crossed module. If $(C,R,\partial)^2 = (C,R,\partial)$, then every maximal ideal is also prime.*

Corollary 32 *Let C,R have identities and (C,R,∂) be a crossed module. If the crossed ideals (C_1,R_1,∂) and (C_2,R_2,∂), as well as the crossed ideals (C_1,R_1,∂) and (C_3,R_3,∂), are coprime, then (C_1,R_1,∂) and $(C_2,R_2,\partial)(C_3,R_3,\partial)$ are also coprime.*

References

[1] Arslan, U.E. (2010). Nilpotent crossed modules of commutative algebras. *Algebras Groups and Geometries*, **27**, 63-72.

[2] Arvasi, Z., Arslan, U.E. (2003). Annihilator, multipliers and crossed modules, *Applied Categorical Structures,* **11**, 487-506.

[3] Arvasi, Z., Porter, T. (1996). Simplicial and Crossed Resolutions of Commutative Algebras, Journal of Algebra, **181**, 426-448.

[4] Ilgaz, E. (2013). Değişmeli cebirler için çaprazlanmş modüller üzerinde tamlamalar, *Master's Thesis*, ESOGU.

[5] Lavendhomme, R., Lucas, Th. (1996). On modules and crossed modules, *Journal of Algebra,* **179,** 936-963.

[6] Lue, A.S.T. (1968). A non abelian cohomology of associative algebras, *Quart. J.Math. Oxford Ser.*, **2**, 159-180.

[7] Mac Lane , S., Whitehead, J.H.C. (1950). On the 3-type of a complex, *Proc. Nat. Acad. Sci.*, 41-48.

[8] Mac Lane, S. (1958). Extensions and obstructions for rings, *Illinois J. Math.,* **121**, 316-345.

[9] Shammu, N.M. (1992). Algebraic and categorical structure of category of crossed modules of algebras, *Ph.D. Thesis*, U.C.N.W.

[10] Porter, T. (1987). Some categorical results in the theory of crossed modules in commutative algebras, *Journal of Algebra*, **109**, 415-429.

[11] Whitehead J.H.C. (1949). Combinatorial Homotopy II, *Bull. American Math. Society,* **55** , 453-456.

U. Ege Arslan
Department of Mathematics-Computer,
Eskişehir Osmangazi University,
Eskişehir, Turkey
E-Mail: uege@ogu.edu.tr

E. Ilgaz Çağlayan
Department of Mathematics,
Bilecik Şeyh Edebali University,
Bilecik, Turkey
E-Mail: elif.caglayan@bilecik.edu.tr