Physica A 568 (2021) 125729

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Comparison of three types of superstatistics, superstatistic thermodynamic relations and paramagnet model

Won Sang Chung [a], Ali Ihsan Goker [b], Hassan Hassanabadi [c,*]

[a] Department of Physics and Research Institute of Natural Science, College of Natural Science, Gyeongsang National University, Jinju 660-701, Republic of Korea
[b] Bilecik University, Department of Physics, 11210, Gulumbe, Bilecik, Turkey
[c] Faculty of Physics, Shahrood University of Technology, Shahrood, Iran

A R T I C L E I N F O

Article history:
Received 1 July 2020
Received in revised form 14 December 2020
Available online 6 January 2021

Keywords:
Superstatistics
Boltzmann factor
Partition function
Superstatistical probability
Superstatistical internal energy
Superstatistical entropy

A B S T R A C T

In this paper we compare three types of superstatistics by computing superstatistical internal energies for continuous energy and quantum discrete energies and discuss spin $\frac{1}{2}$ paramagnet model based on superstatistics. We demonstrate for Spin $\frac{1}{2}$ paramagnet that the magnetic moment per spin depends on the number of spins for finite variance in contradiction with the case with zero variance. Moreover, we show that the superstatistical internal energy is not proportional to the magnetic field for finite variance on contrary to the zero variance case.

1. Introduction

Recently, superstatistics has garnered tremendous interest among theoretical physicists and applied scientists studying complex systems or non-equilibrium thermodynamics in order to tackle statistical systems composed of a superposition of different Boltzmann distributions [1–24]. The classic signature of a superstatistical system is its intrinsically contrasting behavior on microscopic and macroscopic levels. The system typically obeys a single Boltzmann statistic with a local $\beta_0 = \frac{1}{k_B T_0}$ on microscopic level. The local β_0 is no longer fixed on macroscopic level but it fluctuates dictated by the superstatistical kernel.

Mathematically, the superstatistics is a generalization of the probability distribution in Boltzmann–Gibbs statistical mechanics. For one-parameter probability density $p(x, \lambda_0)$, if we regard the parameter λ_0 as another stochastic variable λ and consider the probability density for λ, $f(\lambda, \lambda_0)$, we have a new one-parameter probability density $P(x, \lambda_0)$ in the form

$$P(x, \lambda_0) = \int d\lambda p(x, \lambda) f(\lambda, \lambda_0) \tag{1}$$

Based on [24], we know that x is a fast variable while λ is a slow variable. Therefore, $p(x, \lambda)$ is regarded as a conditional probability and $f(\lambda, \lambda_0)$ is marginal. For the well known probability density $p(x)$, we can obtain the infinite class of the probability densities based on Eq. (1) from different choice of the marginal probability.

* Corresponding author.
E-mail addresses: mimip44@naver.com (W.S. Chung), aihsan.goker@bilecik.edu.tr (A.I. Goker), h.hasanabadi@shahroodut.ac.ir (H. Hassanabadi).

https://doi.org/10.1016/j.physa.2021.125729
0378-4371/

In thermodynamics, one-parameter probability density is the probability density for the exponential distribution with the parameter $\beta_0 = 1/kT_0$ where T_0 denotes the equilibrium temperature. Thus, superstatistics in thermodynamics has another stochastic parameter β instead of fixed β_0. Here the distribution of β can be interpreted as the fluctuation of the equilibrium inverse temperature around β_0 which we call global equilibrium inverse temperature.

Up to now we have three different approaches to application of superstatistics to thermodynamics:

1. Type A superstatistics: This is an application of superstatistics to Boltzmann factor which gives superstatistical Boltzmann factor.

2. Type B superstatistics: This is an application of superstatistics to probability which gives superstatistical probability.

3. Type C superstatistics: This is an application of superstatistics to internal energy [11].

In this paper we compare three types of superstatistics by computing the superstatistical internal energies for continuous energy and quantum discrete energies and discuss spin $\frac{1}{2}$ paramagnet for Type C superstatistics. This paper is organized as follows: In Section 2 we discuss three types of superstatistics. In Section 3 we discuss comparison of superstatistical internal energy in three types of superstatistics with two level distribution. In Section 4 we discuss thermodynamical relations for Type C superstatistics and Spin $\frac{1}{2}$ paramagnet. Our results for Spin $\frac{1}{2}$ paramagnet clearly indicate that the magnetic moment per spin depends on the number of spins for finite variance in contradiction with the case with zero variance. Moreover, we show that the superstatistical internal energy is not proportional to the magnetic field for finite variance on contrary to the zero variance case.

2. Three types of superstatistics

Superstatistics deals with the complex non-equilibrium systems with spatio-temporal fluctuations of an intensive parameter on a large scale. In a canonical ensemble, the intensive parameter is inverse temperature, hence due to fluctuation in inverse temperature we should regard the inverse temperature as stochastic variable β. We take the distribution of stochastic inverse temperature, $f(\beta, \beta_0)$ so that it may obey the following:

1 *The probability density* $f(\beta, \beta_0)$ *should be non-negative and obey*

$$\int_0^\infty f(\beta, \beta_0) d\beta = 1 \tag{2}$$

2 *The average of* β *should be* $\beta_0 = 1/kT_0$:

$$\langle \beta \rangle = \int_0^\infty \beta f(\beta, \beta_0) d\beta = \beta_0 \tag{3}$$

3 *The distribution has non-zero variance* :

$$\langle \beta^2 \rangle = \int_0^\infty \beta^2 f(\beta, \beta_0) d\beta = (1+\eta^2)\beta_0^2 \tag{4}$$

The ordinary thermodynamics corresponds to $f(\beta, \beta_0) = \delta(\beta - \beta_0)$ which gives zero variance. For simplicity, let the entire system S be divided into M subsystems called local cells denoted by S_i, each of which has the local equilibrium inverse temperature β_i. Let q_i be the probability that we find S_i in S. This can be described in terms of the multi-level distribution with different weight,

$$f(\beta) = \sum_{i=1}^{M} q_i \delta(\beta - \beta_i) \tag{5}$$

From Eqs. (2)–(4) we get

$$\sum_{i=1}^{M} q_i = 1 \tag{6}$$

$$\sum_{i=1}^{M} q_i \beta_i = \beta_0 \tag{7}$$

$$\sum_{i=1}^{M} q_i \beta_i^2 = (1+\eta^2)\beta_0^2 \tag{8}$$

where η is real. Hence $\eta^2 \geq 0$. Consider the case that the energy levels are given by E_n, $n = 0, 1, 2, \ldots$ at the global equilibrium. Now let us discuss three types of superstatistics for this distribution.

2.1. Type A

Let us apply superstatistics to Boltzmann factor which gives superstatistical Boltzmann factor. This is referred to as Type A. In this case Type A superstatistical Boltzmann factor is

$$B_s(E_n) = \int d\beta e^{-\beta E_n} f(\beta) = \sum_{i=1}^{M} q_i e^{-\beta_i E_n} \tag{9}$$

and Type A superstatistical partition function

$$Z_s^A = \sum_{n=0}^{\infty} B_s(E) = \sum_{i=1}^{M} q_i Z(\beta_i) = \sum_{n=0}^{\infty}\sum_{i=1}^{M} q_i e^{-\beta_i E_n}, \tag{10}$$

where the partition function in the ith local cell is

$$Z(\beta_i) = \sum_{n=0}^{\infty} e^{-\beta_i E_n} \tag{11}$$

Thus, Type A superstatistical probability finding a system with energy E_n is

$$P_s^A(E_n) = \frac{B_s(E_n)}{Z_s^A} = \frac{1}{Z_s^A}\sum_{i=1}^{M} q_i e^{-\beta_i E_n} \tag{12}$$

Type A superstatistical internal energy is then given by

$$U_s^A(\beta_0) = -\frac{1}{Z_s^A}\sum_{i=1}^{M} q_i \frac{\partial}{\partial \beta_i} Z(\beta_i) = -\frac{1}{Z_s^A}\sum_{i=1}^{M} q_i Z(\beta_i)\frac{\partial}{\partial \beta_i}\ln Z(\beta_i) = \frac{1}{Z_s^A}\sum_{i=1}^{M} q_i Z(\beta_i) U(\beta_i), \tag{13}$$

where the internal energy in the ith local cell is

$$U(\beta_i) = \frac{1}{Z(\beta_i)}\sum_{n=0}^{\infty} E_n e^{-\beta_i E_n} = -\frac{\partial}{\partial \beta_i}\ln Z(\beta_i) \tag{14}$$

Type A superstatistical entropy is then

$$S_s^A(\beta_0) = -\sum_{n=0}^{\infty} P_s^A(E_n)\ln P_s^A(E_n) = -\sum_{n=0}^{\infty} P_s^A(E_n)\ln B_s(E_n) + \ln Z_s^A, \tag{15}$$

where and from now on we set $k = 1$.

2.2. Type B

Let us apply superstatistics to the probability which gives Type B superstatistical probability. This is referred to as Type B. In this case Type B superstatistical probability finding a system with energy E_n is

$$P_s^B(E_n) = \int d\beta p(E_n) f(\beta) = \int d\beta \frac{e^{-\beta E_n}}{Z(\beta)} f(\beta) = \sum_{i=1}^{M} q_i \frac{e^{-\beta_i E_n}}{Z(\beta_i)}, \tag{16}$$

where

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta E_n} \tag{17}$$

Type B superstatistical internal energy is then given by

$$U_s^B(\beta_0) = \sum_{i=1}^{M} q_i U(\beta_i) = -\sum_{i=1}^{M} q_i \frac{\partial}{\partial \beta_i}\ln Z(\beta_i) \tag{18}$$

Type B superstatistical entropy is then

$$S_s^B(\beta_0) = -\sum_{n=0}^{\infty} P_s^B(E_n)\ln P_s^B(E_n) = -\sum_{n=0}^{\infty} P_s^B(E_n)\ln\left(\sum_{i=1}^{M} q_i \frac{e^{-\beta_i E_n}}{Z(\beta_i)}\right) \tag{19}$$

2.3. Type C

Type C superstatistics differs from Type A and Type B in that the energy levels in local cell are different from the energy levels in the entire system S at the global equilibrium. We will refer to the energy levels as *effective energy levels* because it is not determined from Schrödinger equation. Following Ref. [11], we consider the effective energy levels $\tilde{E}_n^{(i)}$ in each local cell S_i with a local equilibrium inverse temperature β_i. Then we can define the internal energy E_n of entire system S corresponding to the quantum number n in terms of the average of the effective internal energies of the local systems S_i's with the distribution $f(\beta)$. Indeed, we have

$$E_n = \sum_{i=1}^{M} q_i \tilde{E}_n^{(i)} \tag{20}$$

or

$$\sum_{i=1}^{M} q_i \frac{\tilde{E}_n^{(i)}}{E_n} = 1 \tag{21}$$

From Eq. (7), we know

$$\sum_{i=1}^{M} q_i \frac{\beta_i}{\beta_0} = 1 \tag{22}$$

Comparing Eqs. (21) and (22), we can set

$$\tilde{E}_n^{(i)} = \frac{\beta_i}{\beta_0} E_n, \tag{23}$$

Here we know that the effective energy corresponding to a quantum number n in a local cell S_i is dependent upon both a local temperature and a global temperature, more precisely, the ratio of these two temperatures.

Now we can define Type C superstatistical probability at local cell S_i as the conditional probability in the form,

$$P_s^C(E_n|i) = \frac{1}{Z_s^C(\beta_0)} e^{-\beta_0 \tilde{E}_n^{(i)}} = \frac{1}{Z_s^C(\beta_0)} e^{-\beta_i E_n}, \tag{24}$$

where Type C superstatistical partition function is defined as

$$Z_s^C(\beta_0) = \sum_{n=0}^{\infty} \sum_{i=1}^{M} q_i e^{-\beta_0 \tilde{E}_n^{(i)}} = \sum_{i=1}^{M} q_i Z(\beta_i) \tag{25}$$

Thus, Type C superstatistical internal energy is

$$U_s^C(\beta_0) = \sum_{n=1}^{\infty} \sum_{i=1}^{M} q_i \tilde{E}_n^{(i)} P_s^C(E_n|i) = \sum_{n=1}^{\infty} \sum_{i=1}^{M} q_i \frac{1}{Z_s^C(\beta_0)} e^{-\beta_0 \tilde{E}_n^{(i)}} \tilde{E}_n^{(i)} \tag{26}$$

or

$$U_s^C(\beta_0) = -\frac{\partial}{\partial \beta_0} \ln Z_s^C(\beta_0) \tag{27}$$

Expressing Type C superstatistical internal energy in terms of E_n's, we get

$$U_s^C(\beta_0) = \frac{1}{\beta_0 Z_s^C} \sum_{n=1}^{\infty} \sum_{i=1}^{M} q_i \beta_i E_n e^{-\beta_i E_n} = \frac{1}{\beta_0 Z_s^C} \sum_{i=1}^{M} q_i \left(-\beta_i \frac{\partial}{\partial \beta_i} \right) (Z(\beta_i) U(\beta_i)) \tag{28}$$

Type C superstatistical entropy in entire system is defined as

$$S_s^C(\beta_0) = \sum_{i=1}^{M} q_i S_s^{(i)}, \tag{29}$$

where superstatistical entropy in the ith local cell is defined as

$$S_s^{(i)} = -\sum_{n=0}^{\infty} P_s^C(E_n|i) \ln P_s^C(E_n|i) \tag{30}$$

Inserting Eq. (30) into Eq. (29) we get

$$S_s^C(\beta_0) = \beta_0 U_s^C(\beta_0) + \ln Z_s^C(\beta_0) \tag{31}$$

Using Eq. (27), we have

$$S_s^C(\beta_0) = \frac{\partial}{\partial T_0}(T_0 \ln Z_s^C) \tag{32}$$

If we set

$$F_s^C = -T_0 \ln Z_s^C, \tag{33}$$

we have

$$F_s^C = U_s^C - T_0 S_s^C, \tag{34}$$

which is Type C superstatistical Helmholtz energy.

3. Comparison of superstatistical internal energy in three types of superstatistics with two level distribution

In this section we compare three types of superstatistics with two level distribution by computing their corresponding superstatistical internal energies.

3.1. Non-relativistic ideal gas

Now let us compute the density of states for the non-relativistic ideal gas consisting of N identical particles in a d-dimensional volume V. The mass of each particle is m. The energy of jth particle is

$$E_j = \frac{p_j^2}{2m}, \tag{35}$$

The partition function in local cell S_i is given by

$$Z(\beta_i) = \frac{1}{N!}\left(\frac{2m\pi}{h^2}\right)^{Nd/2} V^N \beta_i^{-Nd/2} \tag{36}$$

Now let us consider two level distribution obeying Eqs. (2)–(4),

$$f(\beta) = q_1\delta(\beta - \beta_0(1-\eta s)) + q_2\delta(\beta - \beta_0(1+\eta/s)), \tag{37}$$

where

$$0 < q_1, q_2 < 1, \quad q_1 + q_2 = 1 \quad \beta_1 < \beta_2 \tag{38}$$

and

$$s = \sqrt{\frac{q_2}{q_1}} \tag{39}$$

Here $s > 1$ for $q_2 > q_1$; $s = 1$ for $q_2 = q_1$; $0 < s < 1$ for $q_2 < q_1$. For three types of superstatistics, superstatistical internal energies are

$$U_s^A = \frac{1}{2}NdT_0\left[\frac{q_1(1-\eta s)^{-\frac{Nd}{2}-1} + q_2(1+\eta/s)^{-\frac{Nd}{2}-1}}{q_1(1-\eta s)^{-\frac{Nd}{2}} + q_2(1+\eta/s)^{-\frac{Nd}{2}}}\right] \tag{40}$$

$$U_s^B = \frac{1}{2}NdT_0\left[\frac{q_1}{1-\eta s} + \frac{q_2}{1+\eta/s}\right] \tag{41}$$

$$U_s^C = \frac{1}{2}NdT_0 \tag{42}$$

Fig. 1 shows the plot of U versus η with $\frac{Nd}{2} = 0.3$, $T_0 = 0.5$ for U_s^A (Line), U_s^B (Dashed) and U_s^C (Dot-Dashed) with $q_1 = 0.8$ and $q_2 = 0.2$. The superstatistical internal energy for Type C is the same as the ordinary case.

3.2. Quantum harmonic oscillator: photon

Now let us consider the photon case whose energy is $E_n = nh\nu$, where $n = 0, 1, 2, \ldots$. The partition function in local cell S_i is given by

$$Z(\beta_i) = \frac{1}{1 - e^{-\beta_i h\nu}} \tag{43}$$

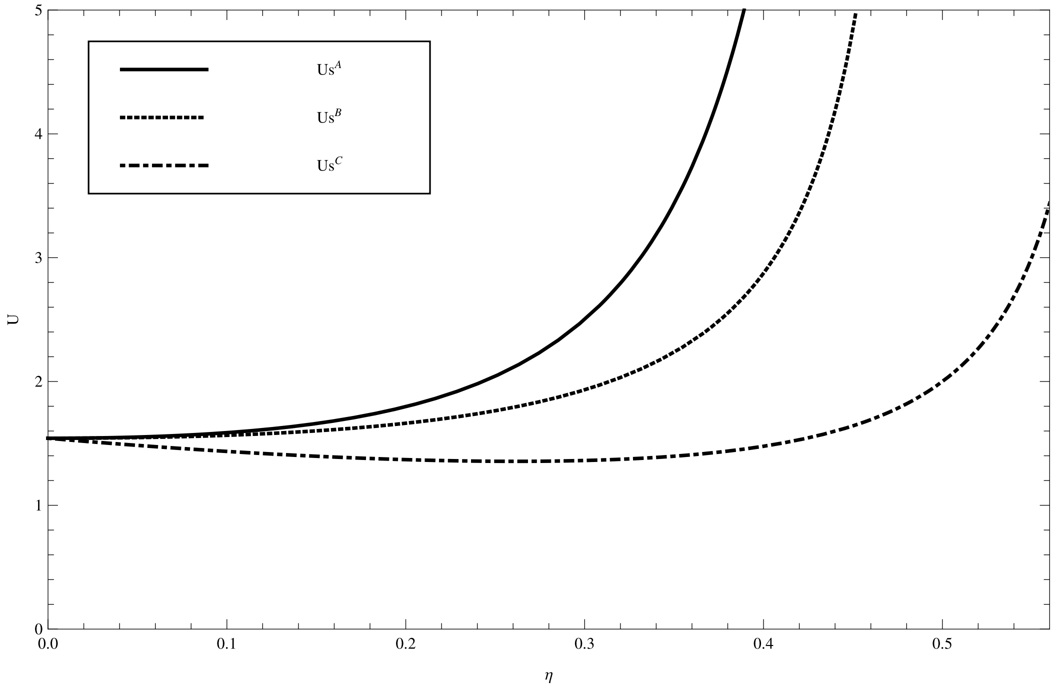

Fig. 1. Plot of U versus η with $\frac{Nd}{2}=0.3, T_0=0.5$ for U_s^A (Dashed), U_s^B (Dot-Dashed) and U_s^C (Line) with $q_1=0.8$ and $q_2=0.2$.

For three types of superstatistics, superstatistical internal energies are

$$U_s^A = h\nu\left[\frac{\frac{q_1 e^{-\beta_1 h\nu}}{(1-e^{-\beta_1 h\nu})^2}+\frac{q_2 e^{-\beta_2 h\nu}}{(1-e^{-\beta_2 h\nu})^2}}{\frac{q_1}{1-e^{-\beta_1 h\nu}}+\frac{q_2}{1-e^{-\beta_2 h\nu}}}\right] \tag{44}$$

$$U_s^B = h\nu\left[\frac{q_1}{e^{\beta_1 h\nu}-1}+\frac{q_2}{e^{\beta_2 h\nu}-1}\right] \tag{45}$$

$$U_s^C = h\nu\left[\frac{\frac{q_1(1-\eta s)e^{-\beta_1 h\nu}}{(1-e^{-\beta_1 h\nu})^2}+\frac{q_2(1-\eta/s)e^{-\beta_2 h\nu}}{(1-e^{-\beta_2 h\nu})^2}}{\frac{q_1}{1-e^{-\beta_1 h\nu}}+\frac{q_2}{1-e^{-\beta_2 h\nu}}}\right], \tag{46}$$

where

$$\beta_1=\beta_0(1-\eta s), \quad \beta_2=\beta_0(1+\eta/s), \quad \beta_1<\beta_2 \tag{47}$$

Figs. 2 and 3 show behavior of U versus η and ν with $q_1=0.8$, $q_2=0.2$ and $\beta_1,\beta_2=0.5$. For Type A superstatistical entropy is then

$$S_s^A = Ln\left(\frac{\prod_{i=0}^{n} f(n_i)}{\sum_{i=0}^{n} f(n_i)}\right), \tag{48}$$

where

$$f(n_i)=(q_1 e^{-\beta_1 nh\nu}+q_2 e^{-\beta_2 nh\nu}) \tag{49}$$

and Type B superstatistical entropy is then

$$S_s^B = -(q_1+q_2)\sum_{n=0}^{\infty} Ln\left(\frac{q_1 e^{-\beta_1 nh\nu}}{\sum_{n=0}^{\infty} e^{-\beta_1 nh\nu}}+\frac{q_2 e^{-\beta_2 nh\nu}}{\sum_{n=0}^{\infty} e^{-\beta_2 nh\nu}}\right) \tag{50}$$

and Type C superstatistical entropy is then

$$S_s^C = \frac{\partial}{\partial T_0}[T_0 Ln\left(\frac{q1}{1-(e^{-\frac{h\nu}{kT_0}})^{1-\eta s}}+\frac{q_2}{1-(e^{-\frac{h\nu}{kT_0}})^{1+\frac{\eta}{s}}}\right)] \tag{51}$$

Fig. 4 show that the S_s^C versus η with $h\nu=1$ and $\eta=0.2$ for $q_1=0.6$, $q_2=0.4$ and $T_0=0.5$.

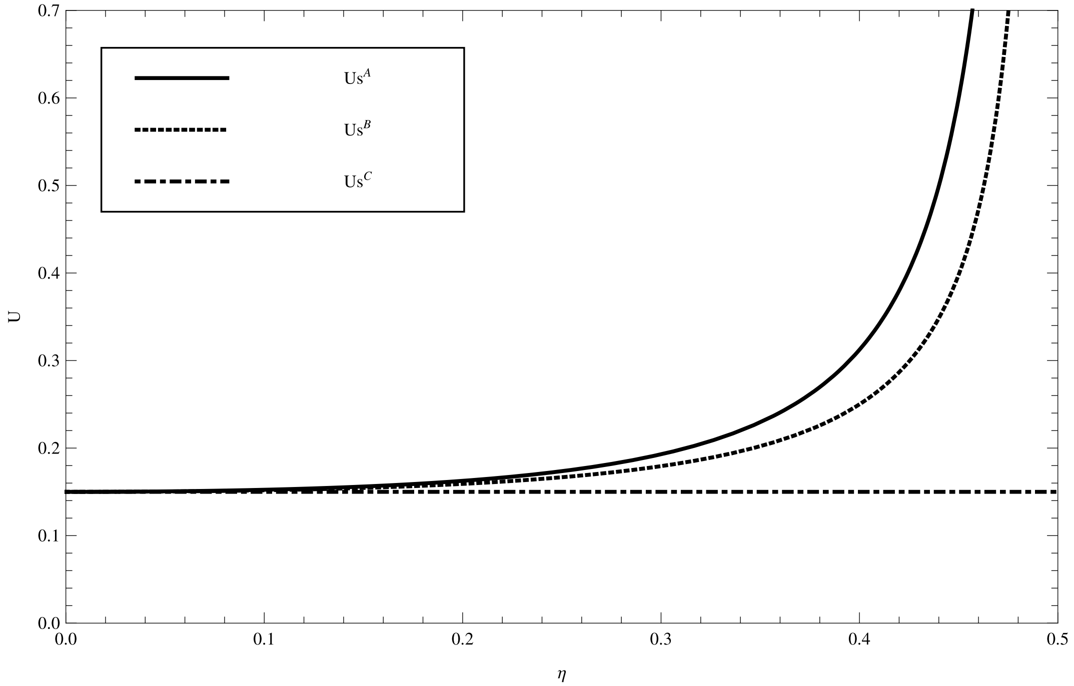

Fig. 2. Plot of U versus η with $\beta_1 = 0.1$, $\beta_2 = 0.5$ and $h\nu = 1$ for U_s^A (Line), U_s^B (Dashed) and U_s^C (Dot-Dashed) with $q_1 = 0.8$, $q_2 = 0.2$ and $\beta_0 = 0.5$.

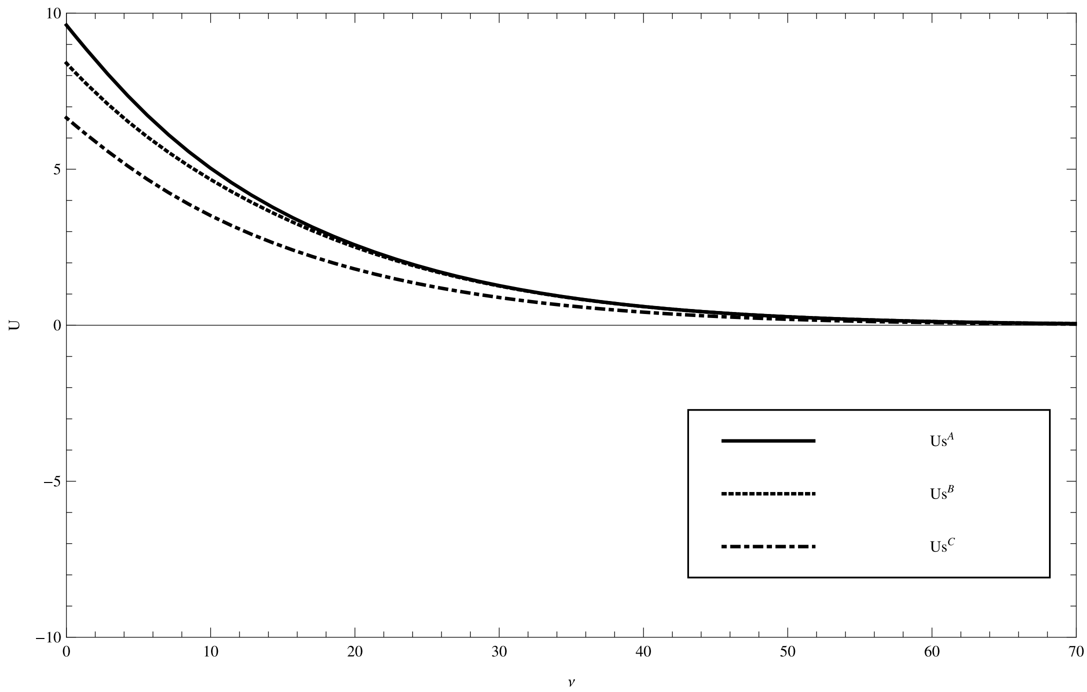

Fig. 3. Plot of U versus ν with $\beta_1 = 0.1$, $\beta_2 = 0.5$, $h = 1$ for U_s^A (Line), U_s^B (Dashed) and U_s^C (Dot-Dashed) with $q_1 = 0.8$, $q_2 = 0.2$ and $\eta = 0.8$.

4. Thermodynamical relations for Type C superstatistics and spin $\frac{1}{2}$ paramagnet

Although we can obtain superstatistical probabilities and superstatistical internal energies for three types of superstatistics, we cannot derive thermodynamic relations from Type A and Type B. However, we can construct the

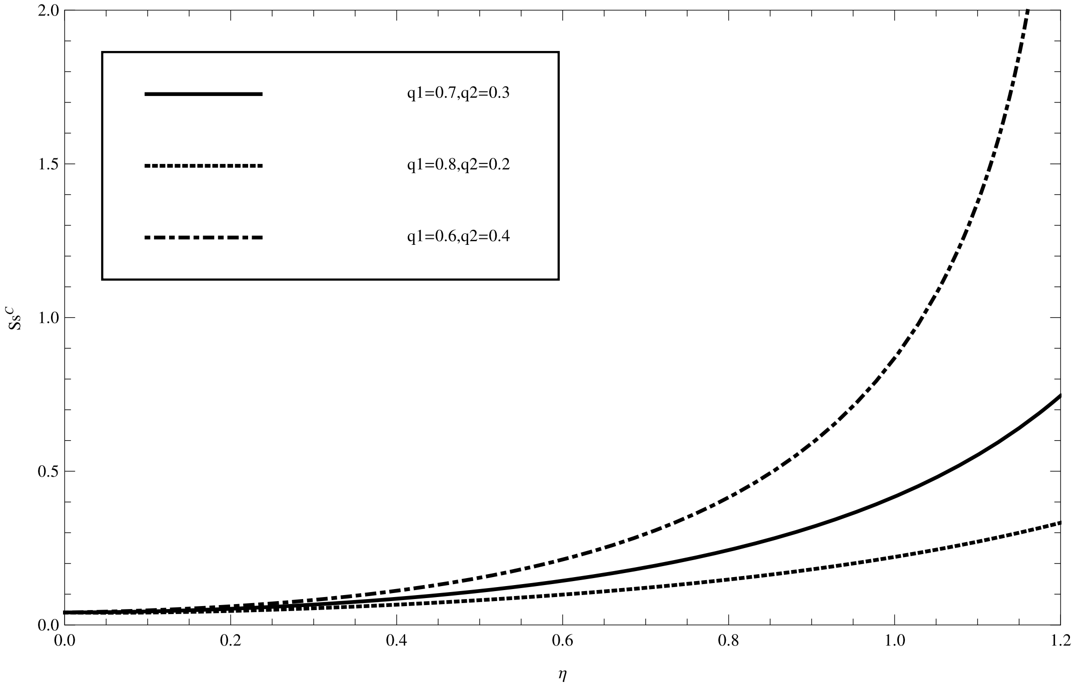

Fig. 4. Plot of S_s^C versus η with $T_0 = 0.5$.

thermodynamic relations from Type C superstatistics, hence we will discuss thermodynamics related to Type C superstatistics.

Let us denote the heat supplied to the system by Q. In ith local cell, we have

$$dQ = T_i dS_s^{(i)} \tag{52}$$

Using Eq. (26) we have

$$\begin{aligned} dS_s^C &= \sum_{i=1}^{M} q_i dS_s^{(i)} \\ &= \sum_{i=1}^{M} q_i \beta_i dQ \\ &= \beta_0 dQ \end{aligned} \tag{53}$$

Let us denote the volume of ith local cell by $V^{(i)}$. Then we have

$$\begin{aligned} dV &= \sum_{i=1}^{M} dV^{(i)} \\ &= -\sum_{i=1}^{M} \frac{dW^{(i)}}{P} \\ &= -\frac{dW}{P}, \end{aligned} \tag{54}$$

where P is pressure and $W^{(i)}$ is thermodynamic work in ith local cell. Thus we have the first law of thermodynamics in the form

$$dU_s^C = T_0 dS_s^C - PdV, \tag{55}$$

Thus we have the following thermodynamic relations

$$\left.\frac{\partial U_s^C}{\partial S_s^C}\right|_V = T_0, \quad \left.\frac{\partial U_s^C}{\partial V}\right|_{S_s^C} = -P, \tag{56}$$

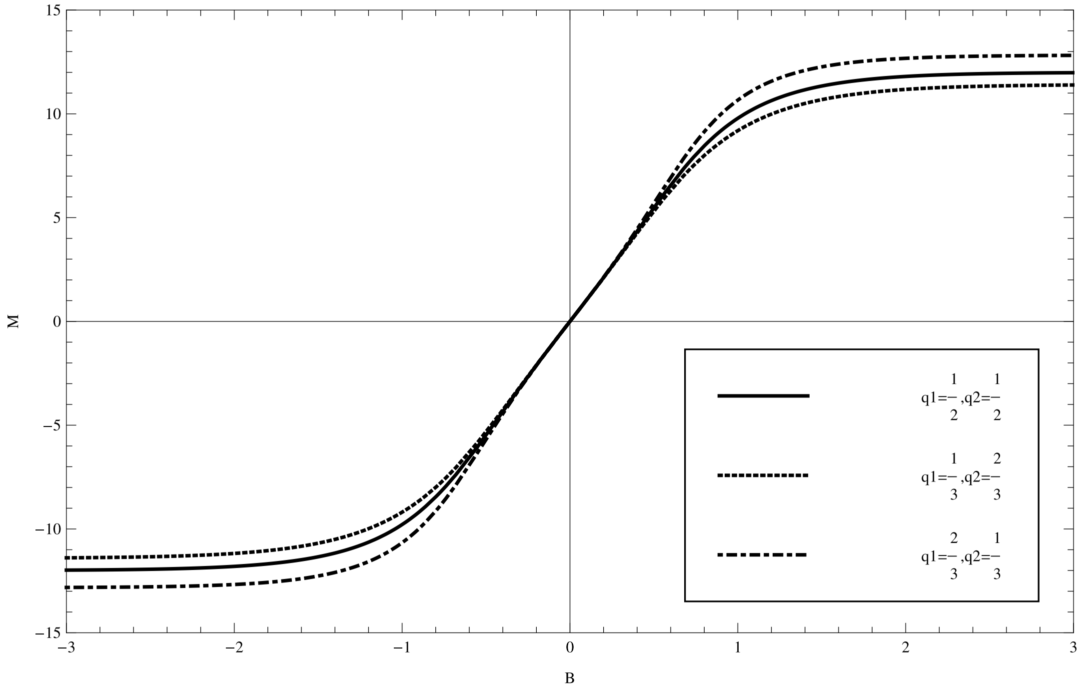

Fig. 5. Plot of M versus B with $\beta_0 = 1, \mu = 1, N = 10$ and $\eta = 0.2$.

From superstatistical Helmholtz energy we get

$$dF_s^C = -S_s^C dT_0 - PdV \tag{57}$$

Thus we have the following thermodynamic relations

$$\left.\frac{\partial F_s^C}{\partial T_0}\right|_V = -S_s^C, \quad \left.\frac{\partial F_s^C}{\partial V}\right|_{T_0} = -P, \tag{58}$$

4.1. Spin $\frac{1}{2}$ paramagnet

Now let us consider the spin $\frac{1}{2}$ paramagnet which composed of N spins. The Hamiltonian is given by

$$H = -\mu B \sum_{n=1}^{N} S_n, \tag{59}$$

where μ is the magnetic moment and B denotes magnetic field. The partition function in local cell S_i is

$$Z(\beta_i) = (2\cosh(\beta_i \mu B))^N \tag{60}$$

Now let us consider two level distribution. Thus, superstatistical Helmholtz energy is

$$F_s^C = -T_0 \ln\left[q_1\,(2\cosh(\beta_1\mu B))^N + q_2\,(2\cosh(\beta_2\mu B))^N\right] \tag{61}$$

We can work out the total magnetic moment M of the paramagnet by computing

$$\begin{aligned} M &= -\left(\frac{\partial F}{\partial B}\right)_T \\ &= N\mu\left[\frac{q_1(1-\eta s)\,(\cosh(\beta_1\mu B))^{N-1}\sinh(\beta_1\mu B) + q_2(1+\eta/s)\,(\cosh(\beta_2\mu B))^{N-1}\sinh(\beta_2\mu B)}{q_1\,(\cosh(\beta_1\mu B))^N + q_2\,(\cosh(\beta_2\mu B))^N}\right] \end{aligned} \tag{62}$$

where

$$\beta_1 = \beta_0(1-\eta s), \quad \beta_2 = \beta_0(1+\eta/s) \tag{63}$$

As is different from the ordinary case ($\eta = 0$), the magnetic moment per spin, M/N depends on the number of spins. Fig. 5 shows the plot of M versus B with $\beta_0 = 1, \mu = 1, N = 10$ for $q_1 = q_2 = 1/2$ (Line), $q_1 = 1/3, q_2 = 2/3$ (Dashed) and $q_1 = 2/3, q_2 = 1/3$ (Dot-Dashed) with $\eta = 0.2$. Beside, superstatistical internal energy is not proportional to the magnetic field while for $\eta = 0$ we have $U = -MB$.

5. Conclusion

In this paper we compared three types of superstatistics by constructing the corresponding superstatistical probabilities, superstatistical internal energies and superstatistical entropy. Here, superstatistical Helmholtz energy was defined only for Type C. We compared superstatistical internal energies in three types of superstatistics with two level distribution. As examples we computed superstatistical internal energies for continuous energy and quantum discrete energies. We found that thermodynamic relations and first law of thermodynamics could be derived only for Type C. As example we discussed the spin $\frac{1}{2}$ paramagnet. As is different from the ordinary case ($\eta = 0$), the magnetic moment per spin, M/N depends on the number of spins for $\eta \neq 0$. Besides we found that superstatistical internal energy was not proportional to the magnetic field while for $\eta = 0$ we have $U = -MB$. In this paper we discussed the canonical ensemble. We think that extension of this work into the grand canonical ensemble is more interesting.

CRediT authorship contribution statement

Won Sang Chung: Conceptualization, Methodology, Writing, Investigation, Writing - review & editing. **Ali Ihsan Goker:** Conceptualization, Methodology, Writing, Investigation, Writing - review & editing. **Hassan Hassanabadi:** Conceptualization, Methodology, Writing, Investigation, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

The authors thank the referee for a thorough reading of our manuscript and for constructive suggestion.

References

[1] C. Beck, E.G.D. Cohen, Physica 322A (2003) 267.
[2] A.G. Bashkirov, A.D. Sukhanov, J. Exp. Theor. Phys. 95 (2002) 440.
[3] C. Tsallis, A.M.C. Souza, Phys. Rev. E 67 (2003) 026106.
[4] C. Beck, E.G.D. Cohen, Physica A 344 (2004) 393.
[5] H. Touchette, C. Beck, Phys. Rev. E 71 (2005) 016131.
[6] C. Beck, E.G.D. Cohen, H.L. Swinney, Phys. Rev. E 72 (2005) 056133.
[7] C. Beck, E.G.D. Cohen, S. Rizzo, Europhys. News 36 (2005) 189.
[8] C. Vignat, A. Plastino, A.R. Plastino, Nuovo Cimento B 120 (2005) 951.
[9] P.-H. Chavanis, Physica A 359 (2006) 177.
[10] T. Yamano, Progr. Theoret. Phys. Suppl. 162 (2006) 87.
[11] C. Beck, Philos. Trans. R. Soc. A369 (2011) 453.
[12] A.M. Reynolds, Phys. Rev. Lett. 91 (2003) 084503.
[13] S. Jung, H.L. Swinney, Phys. Rev. E 72 (2005) 026304.
[14] S. Abe, S. Thurner, Phys. Rev. E 72 (2005) 036102.
[15] S. Abe, S. Thurner, Internat. J. Modern Phys. C 17 (2006) 1303.
[16] A. Porporato, G. Vico, P.A. Fay, Geophys. Res. Lett. 33 (2006) L15402.
[17] L.L. Chen, C. Beck, Physica A 387 (2008) 3162.
[18] S. Sargolzaeipor, H. Hassanabadi, W. Chung, Eur. Phys. J. Plus 133 (2018) 125.
[19] S. Sargolzaeipor, H. Hassanabadi, W. Chung, Eur. Phys. J. Plus 133 (2018) 157.
[20] W. Chung, H. Hassanabadi, Int. J. Geom. Methods Mod. Phys. 15 (2018) 1850123.
[21] H. Hassanabadi, S. Sargolzaeipor, W. Chung, Physica A 508 (2018) 740.
[22] W. Chung, H. Hassanabadi, Modern Phys. Lett. B33 (2019) 1950140.
[23] W. Chung, H. Hassanabadi, Physica A 516 (2019) 496.
[24] S. Abe, C. Beck, E. Cohen, Phys. Rev. E 76 (2007) 031102.