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Approximation by Sampling Durrmeyer Operators in Weighted Space of Functions

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ABSTRACT
The present article deals with local and global approximation behaviors of sampling Durrmeyer operators for functions belonging to weighted spaces of continuous functions. After giving some fundamental notations of sampling type approximation methods and presenting well definiteness of the operators on weighted spaces of functions, we examine pointwise and uniform convergence of the family of operators and determine the rate of convergence via weighted modulus of continuity. A quantitative Voronovskaja theorem is also proved in order to obtain rate of pointwise convergence and upper estimate for this convergence. The last section is devoted to some numerical evaluations of sampling Durrmeyer operators with suitable kernels.

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1. Introduction

In approximation theory, generalized sampling series was constructed by P. L. Butzer and his school [1–3] in order to reconstruct a function f with its sample values at some discrete points. It is the approximate analogue of well-known Whittaker–Kotel'nikov-Shannon (WKS) type sampling theorem [4,5] and is defined by

$$(S_w^\tau f)(x) := \sum_{k\in\mathbb{Z}} \tau(wx-k) f\left(\frac{k}{w}\right), \quad w>0, \ x\in\mathbb{R} \tag{1.1}$$

for functions $f:\mathbb{R}\to\mathbb{R}$ for which the series is convergent for every $x\in\mathbb{R}$ and $\tau:\mathbb{R}\to\mathbb{R}$ (called the *kernel* of the operator) denotes a continuous, discrete approximate identity which satisfies suitable assumptions. The operators S_w^τ represent a method in order to study in continuous functions spaces [1, 2, 4, 5], simultaneous approximation and linear prediction [6], and for more see [24–26, 28, 29].

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The L^1-version of the generalized sampling operators, called sampling Kantorovich operators, was first introduced by Bardaro et al. [7] and has been intensively studied in, among the others, [8–13]. On the other hand, while the Kantorovich modification of the operators (1.1) furnishes a method to approximate functions belonging to L^1, the method to approximate functions belonging to L^p spaces (or more general Orlicz and modular spaces) is to construct Durrmeyer modification [27]. Durrmeyer modifications of the generalized sampling operators (1.1) were first introduced by Bardaro et al. [14], using a general convolution integral instead of the integral means, by

$$\left(S_w^{\tau,\tau}f\right)(x)=\sum_{k=-\infty}^{+\infty}\tau(wx-k)w\int_{\mathbb{R}}\tau(wu-k)f(u)du,\quad x\in\mathbb{R},$$

where τ is a kernel function which satisfies suitable conditions. In ref. [14], the authors established an asymptotic formula for functions f with a polynomial growth and as a consequence, a Voronovskaja type formula for the family of operators $(S_w^{\tau,\tau})$ was obtained.

In general, the kernels in the construction of Durrmeyer modifications do not need to be the same, that is, the kernels in convolution and in the series may be different. Such a consideration allows us to make weaker assumptions on the kernels than those made for the classical ones. In the recent article [15], Costarelli et al. have constructed a generalized sampling Durrmeyer operators in this frame as

$$\left(S_w^{\tau,\vartheta}f\right)(x)=\sum_{k=-\infty}^{+\infty}\tau(wx-k)w\int_{\mathbb{R}}\vartheta(wu-k)f(u)du,\quad x\in\mathbb{R}$$

for any functions f for which the series is absolutely convergent. Here the kernel functions τ,ϑ satisfy some conditions (see [15, p. 4]), for which the operators form an approximation process.

While studying generalized sampling series (1.1) in continuous function spaces, the space $C^0(\mathbb{R})$ of the uniformly continuous and bounded functions on $\mathbb{R}$ is generally considered. To enlarge the class of target functions, in the recent article [16] the authors considered polynomial weighted spaces of continuous functions and extended the studies of approximation behaviors of generalized sampling series to a more general class of continuous functions. Similar considerations for generalized sampling Kantorovich series and generalized exponential sampling series were studied in refs. [17] and [18], respectively.

Although the family of generalized sampling Durrmeyer operators $(S_w^{\tau,\vartheta})$ is defined for functions $f\in L^p$ (or Orlicz space, modular space), it is meaningful for continuous functions as well.

In the present article, we investigate convergence of $(S_w^{\tau,\vartheta})$ for functions belonging to polynomial weighted space of continuous functions. Our results consist of pointwise and uniform convergence of $(S_w^{\tau,\vartheta})$ and a rate of convergence by means of weighted modulus of continuity. Also, a quantitative Voronovskaja type theorem is obtained. The final section of the article is devoted to some numerical examples, describing the convergence results.

2. Preliminaries

We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{R}$ the sets of positive integers, non-negative integers, integers and real numbers, respectively.

Let $\tau \in C^0(\mathbb{R})$ and $\vartheta \in L^1(\mathbb{R})$ be functions such that

$$\sum_{k\in\mathbb{Z}} \tau(u-k) = 1, \text{for every } u \in \mathbb{R} \text{ and } \int_{\mathbb{R}} \vartheta(u)du = 1. \tag{2.1}$$

For any $j \in \mathbb{N}_0$ let us define the algebraic moments of τ and ϑ, respectively, by

$$m_j(\tau, u) = \sum_{k\in\mathbb{Z}} \tau(u-k)(k-u)^j,$$

$$\mathfrak{m}_j(\vartheta) = \int_{\mathbb{R}} y^j \vartheta(y)dy$$

and the absolute moments for $l > 0$ by

$$M_l(\tau) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\tau(u-k)||u-k|^l$$

and

$$\mathcal{M}_l(\vartheta) := \int_{\mathbb{R}} |y|^l |\vartheta(y)|dy.$$

From now on, τ and ϑ will be called kernels if they satisfy the condition (2.1) such that there exists $\alpha, \beta > 0$ with $M_\alpha(\tau) < +\infty$ and $\mathcal{M}_\beta(\vartheta) < +\infty$.

Remark 1.

1. Let τ and ϑ be kernels, then $(S_w^{\tau,\vartheta}\mathbf{1})(x) = 1$, where $\mathbf{1}(x) = 1$ for every $x \in \mathbb{R}$.
2. Let $\gamma, \eta > 0$ with $\eta \le \gamma$. Then $M_\gamma(\tau) < +\infty$ implies $M_\eta(\tau) < +\infty$, see, [11]. In the case of τ has compact support, it is immediately obtained that $M_\gamma(\tau) < +\infty$, for every $\gamma \ge 0$.
3. Similarly, let $\gamma, \eta > 0$ with $\eta \le \gamma$. Then $\mathcal{M}_\gamma(\vartheta) < +\infty$ implies $\mathcal{M}_\eta(\vartheta) < +\infty$, see [15].

Lemma 1 ([16, Lemma 1]). *Let χ be a function satisfying the conditions:*

1. *χ is continuous on $\mathbb{R}$,*
2. *there exists $\beta > 0$, such that*

$$M_\beta(\chi) = \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-k)||u-k|^\beta,$$

is finite.

For every $\delta > 0$ there holds:

$$\lim_{w\to\infty} \sum_{|k-wx|>w\delta} |\chi(wx-k)| = 0,$$

uniformly with respect to $x \in \mathbb{R}$.

Let $C(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$. A function $\rho \in C(\mathbb{R})$ is called a weight function if it is positive on the entire real axis $\mathbb{R}$. In this study, we consider the weight function

$$\rho(x) = \frac{1}{1+x^2}, \; x \in \mathbb{R}.$$

$B_\rho(\mathbb{R})$ will denote the space of real functions whose product with the weight function ρ on $\mathbb{R}$ is bounded, that is,

$$B_\rho(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \sup_{x\in\mathbb{R}} \rho(x)|f(x)| \in \mathbb{R} \right\}.$$

Also, we shall consider the following natural subspaces of $B_\rho(\mathbb{R})$:

$$\begin{aligned} C_\rho(\mathbb{R}) &:= C(\mathbb{R}) \cap B_\rho(\mathbb{R}), \\ C^*_\rho(\mathbb{R}) &:= \left\{ f \in C_\rho(\mathbb{R}) : \exists \lim_{x\to\mp\infty} \rho(x) f(x) \in \mathbb{R} \right\}, \\ U_\rho(\mathbb{R}) &:= \left\{ f \in C_\rho(\mathbb{R}) : \rho f \text{ is uniformly continuous} \right\}. \end{aligned}$$

The linear space of functions $B_\rho(\mathbb{R})$, and its above subspaces are normed spaces with the norm

$$\|f\|_\rho := \sup_{x\in\mathbb{R}} \rho(x)|f(x)|,$$

see [19–22].

The norm of operator acting from $B_\rho(\mathbb{R})$ to $B_\rho(\mathbb{R})$ will be denoted by $\|\cdot\|_{B_\rho(\mathbb{R})\to B_\rho(\mathbb{R})}$.

The weighted modulus of continuity for functions $f \in C_\rho(\mathbb{R})$ is denoted by $\Omega(f;\cdot)$ and given for $\delta > 0$ by

$$\Omega(f;\delta) := \sup_{|h|<\delta,\ x\in\mathbb{R}} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}. \tag{2.2}$$

In the following lemma we recall some properties of the weighted modulus of continuity. For more details about (2.2) and the proof of given properties we refer the readers to [23].

Lemma 2. ([23]) *Let $\delta > 0, x \in \mathbb{R}$. Then,*

1. $\Omega(f;\delta)$ is a monotonically increasing function of δ,
2. $\Omega(f;\delta) \to 0$ as $\delta \to 0$ for functions $f \in C_\rho^*(\mathbb{R})$,
3. for each $\lambda > 0$ and $f \in C_\rho(\mathbb{R})$, the inequality

$$\Omega(f;\lambda\delta) \leq 2(1+\lambda)(1+\delta^2)\Omega(f;\delta) \tag{2.3}$$

holds.

Remark 2 ([16, p. 156]). Let $x, y \in \mathbb{R}, \delta > 0$. By considering the definition of the weighted modulus of continuity we have

$$|f(y)-f(x)| \leq 16(1+x^2)\Omega(f;\delta)\left(1+\frac{|y-x|^3}{\delta^3}\right) \tag{2.4}$$

with the choice of $\delta \leq 1$.

3. Main results

Let us first prove the well definiteness of the operators $S_w^{\tau,\vartheta}$ on weighted spaces of functions. To do this, we first need to the following proposition.

Proposition 1. *Let τ, ϑ be kernels with $\alpha = \beta = 2$. Further we denote by $\nu(x) := 1/\rho(x) = 1+x^2, x \in \mathbb{R}$. Then,*

$$\left|\left(S_w^{\tau,\vartheta}\nu\right)(x)\right| \leq M_0(\tau)\left(\mathcal{M}_0(\vartheta) + \frac{2}{w^2}\mathcal{M}_2(\vartheta)\right) + 4\mathcal{M}_0(\vartheta)\left(\frac{1}{w^2}M_2(\tau) + x^2 M_0(\tau)\right).$$

Proof. By using definition of sampling Durrmeyer operators, Remark 1 (1) and ν we have

$$\begin{aligned}\left|\left(S_w^{\tau,\vartheta}\nu\right)(x)\right| &\le \sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|w\int_{\mathbb{R}}\left|\vartheta(wu-k)\right|(1+u^2)du\\ &= \sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|w\int_{\mathbb{R}}\left|\vartheta(wu-k)\right|du\\ &+\sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|w\int_{\mathbb{R}}\left|\vartheta(wu-k)\right|u^2du.\end{aligned}$$

By the hypothesis of Proposition, since τ, ϑ are the kernels with $\alpha = \beta = 2,$ and using Remark 1 (2), we immediately have

$$\begin{aligned}\left|\left(S_w^{\tau,\vartheta}\nu\right)(x)\right| &\le M_0(\tau)\mathcal{M}_0(\vartheta) + \sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|\frac{1}{w}\int_{\mathbb{R}}\left|\vartheta(wu-k)\right|(wu-k+k)^2du\\ &\le M_0(\tau)\mathcal{M}_0(\vartheta) + \frac{2}{w}\sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|\left|\int_{\mathbb{R}}\left|\vartheta(wu-k)\right|\left[(wu-k)^2+k^2\right]du\\ &= M_0(\tau)\left(\mathcal{M}_0(\vartheta)+\frac{2}{w^2}\mathcal{M}_2(\vartheta)\right)\\ &+\frac{2}{w^2}\mathcal{M}_0(\vartheta)\sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|(k-wx+wx)^2\\ &\le M_0(\tau)\left(\mathcal{M}_0(\vartheta)+\frac{2}{w^2}\mathcal{M}_2(\vartheta)\right)+4\mathcal{M}_0(\vartheta)\left(\frac{1}{w^2}M_2(\tau)+x^2M_0(\tau)\right)\end{aligned}$$

which completes the proof. □

Theorem 1. *Let τ, ϑ be kernels with $\alpha = \beta = 2$. For any fixed $w > 0$, the operator $S_w^{\tau,\vartheta}$ is a linear operator from $B_\rho(\mathbb{R})$ to $B_\rho(\mathbb{R})$ and the inequality*

$$\begin{aligned}\|S_w^{\tau,\vartheta}\|_{B_\rho(\mathbb{R})\to B_\rho(\mathbb{R})} &\le M_0(\tau)\left(\mathcal{M}_0(\vartheta)+\frac{2}{w^2}\mathcal{M}_2(\vartheta)\right)\\ &+4\mathcal{M}_0(\vartheta)\left(\frac{1}{w^2}M_2(\tau)+M_0(\tau)\right)\end{aligned}$$

holds.

Proof. By using definition of $S_w^{\tau,\vartheta}$ and Proposition 1 we have

$$\begin{aligned}\left|\left(S_w^{\tau,\vartheta}f\right)(x)\right| &\le \sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|w\int_{\mathbb{R}}\left|\vartheta(wu-k)\right|\rho(u)|f(u)|\frac{1}{\rho(u)}du\\ &\le \|f\|_\rho\sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|w\int_{\mathbb{R}}\left|\vartheta(wu-k)\right|\frac{1}{\rho(u)}du\\ &\le \|f\|_\rho\left[M_0(\tau)\left(\mathcal{M}_0(\vartheta)+\frac{2}{w^2}\mathcal{M}_2(\vartheta)\right)+4\mathcal{M}_0(\vartheta)\left(\frac{1}{w^2}M_2(\tau)+x^2M_0(\tau)\right)\right].\end{aligned}$$

Now, if we multiply both sides with $\rho(x)$ we get inequality

$$\rho(x)\left|\left(S_w^{\tau,\vartheta}f\right)(x)\right| \leq \|f\|_\rho \left[M_0(\tau)\left(\mathcal{M}_0(\vartheta)+\frac{2}{w^2}\mathcal{M}_2(\vartheta)\right)\right.$$
$$\left.+4\mathcal{M}_0(\vartheta)\left(\frac{1}{w^2}M_2(\tau)+M_0(\tau)\right)\right]$$

for every $x \in \mathbb{R}$. By assumptions we conclude $||S_w^{\tau,\vartheta}f||_\rho < +\infty$, that is $S_w^{\tau,\vartheta}f \in B_\rho(\mathbb{R})$. Now, taking supremum over $x \in \mathbb{R}$ and the supremum with respect to $f \in B_\rho(\mathbb{R})$ with $\|f\| \leq 1$ it turns out

$$\|S_w^{\tau,\vartheta}\|_{B_\rho(\mathbb{R})\to B_\rho(\mathbb{R})} \leq M_0(\tau)\left(\mathcal{M}_0(\vartheta)+\frac{2}{w^2}\mathcal{M}_2(\vartheta)\right)$$
$$+4\mathcal{M}_0(\vartheta)\left(\frac{1}{w^2}M_2(\tau)+M_0(\tau)\right)$$

which completes the proof. □

Theorem 2. *Let $f \in C_\rho(\mathbb{R})$ be fixed and τ, ϑ be kernels with $\alpha = \beta = 2$. Then,*

$$\lim_{w\to\infty}\left(S_w^{\tau,\vartheta}f\right)(x) = f(x), \quad x \in \mathbb{R}. \tag{3.1}$$

Moreover, if $f \in U_\rho(\mathbb{R})$, then

$$\lim_{w\to\infty}\|S_w^{\tau,\vartheta}f - f\|_\rho = 0. \tag{3.2}$$

Proof. For all $x, u \in \mathbb{R}$, we can write

$$|f(u)-f(x)| \leq \rho(u)|f(u)|\left|\frac{1}{\rho(u)}-\frac{1}{\rho(x)}\right| + \frac{1}{\rho(x)}|\rho(u)f(u)-\rho(x)f(x)|.$$

By using above inequality, the linearity of the operators and Remark 1 (1) we have

$$
\begin{aligned}
\left|\left(S_w^{\tau,\vartheta}f\right)(x)-f(x)\right| &\le \sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|w\int_{\mathbb{R}}|\vartheta(wu-k)||f(u)-f(x)|du \\
&\le \sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|w\int_{\mathbb{R}}|\vartheta(wu-k)|\left[\rho(u)|f(u)|\left|\frac{1}{\rho(u)}-\frac{1}{\rho(x)}\right|\right. \\
&\left.+\frac{1}{\rho(x)}|\rho(u)f(u)-\rho(x)f(x)|\right]du \\
&= \sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|w\int_{\mathbb{R}}|\vartheta(wu-k)|\rho(u)|f(u)|\left|u^2-x^2\right|du \\
&+ \sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|w\int_{\mathbb{R}}|\vartheta(wu-k)|\frac{1}{\rho(x)}|\rho(u)f(u)-\rho(x)f(x)|du \\
&:= I_1+I_2.
\end{aligned}
$$

Let us first estimate I_1. Since

$$
\begin{aligned}
\left|u^2-x^2\right| &= |u-x||u+x| \\
&= \frac{1}{w^2}|wu-wx||wu+wx| \\
&\le \frac{1}{w^2}(|wu-k|+|k-wx|)(|wu-k|+|k+wx|) \\
&= \frac{1}{w^2}\left(|wu-k|^2+|wu-k||k+wx|+|k-wx||wu-k|+|k-wx||k+wx|\right),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
I_1 &\le \frac{\|f\|_\rho}{w^2}\left[\sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right|w\int_{\mathbb{R}}\left|\vartheta(wu-k)\right|\left|wu-k\right|^2du\right. \\
&+\sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right||k+wx|w\int_{\mathbb{R}}\left|\vartheta(wu-k)\right||wu-k|du \\
&+\sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right||k-wx|w\int_{\mathbb{R}}\left|\vartheta(wu-k)\right||wu-k|du \\
&\left.+\sum_{k\in\mathbb{Z}}\left|\tau(wx-k)\right||k-wx||k+wx|w\int_{\mathbb{R}}\left|\vartheta(wu-k)\right|du\right] \\
&\le \frac{\|f\|_\rho}{w^2}\{M_0(\tau)\mathcal{M}_2(\vartheta)+M_1(\tau)\mathcal{M}_1(\vartheta)+2|wx|M_0(\tau)\mathcal{M}_1(\vartheta) \\
&+M_1(\tau)\mathcal{M}_1(\vartheta)+M_2(\tau)\mathcal{M}_0(\vartheta)+2|wx|M_1(\tau)\mathcal{M}_0(\vartheta)\}.
\end{aligned}
$$

Let $\varepsilon>0$ be fixed. Then there exists $\delta>0$ such that $|\rho(u)f(u)-\rho(x)f(x)|<\varepsilon$ when $|u-x|<\delta$. Hence, we can write

$$
\begin{aligned}
I_2 = \frac{1}{\rho(x)} \Bigg[& \sum_{|wx-k| \le \frac{w\delta}{2}} \left|\tau(wx-k)\right| w \int_{|wu-k| \le \frac{w\delta}{2}} \left|\vartheta(wu-k)\right| |\rho(u)f(u) - \rho(x)f(x)| du \\
& + \sum_{|wx-k| \le \frac{w\delta}{2}} \left|\tau(wx-k)\right| w \int_{|wu-k| > \frac{w\delta}{2}} \left|\vartheta(wu-k)\right| |\rho(u)f(u) - \rho(x)f(x)| du \\
& + \sum_{|wx-k| > \frac{w\delta}{2}} \left|\tau(wx-k)\right| w \int_{\mathbb{R}} \left|\vartheta(wu-k)\right| |\rho(u)f(u) - \rho(x)f(x)| du \Bigg] \\
& := \frac{1}{\rho(x)} [I_{2,1} + I_{2,2} + I_{2,3}].
\end{aligned}
$$

For $u \in \mathbb{R}$ with the property $|wu-k| \le \frac{w\delta}{2}$ if we also have $|wx-k| \le \frac{w\delta}{2}$ then we have

$$
|u-x| \le \left|u - \frac{k}{w}\right| + \left|\frac{k}{w} - x\right| < \delta.
$$

Since $\rho f \in C_\rho(\mathbb{R})$ we get

$$
I_{2,1} \le \varepsilon \sum_{|wx-k| \le \frac{w\delta}{2}} \left|\tau(wx-k)\right| w \int_{|wu-k| \le \frac{w\delta}{2}} \left|\vartheta(wu-k)\right| du \le \varepsilon M_0(\tau) \mathcal{M}_0(\vartheta).
$$

Taking supremum for $u \in \mathbb{R}$ we can write

$$
I_{2,2} \le 2\|f\|_\rho \sum_{|wx-k| \le \frac{w\delta}{2}} \left|\tau(wx-k)\right| w \int_{|wu-k| > \frac{w\delta}{2}} \left|\vartheta(wu-k)\right| du
$$

and

$$
w \int_{|wu-k| > \frac{w\delta}{2}} \left|\vartheta(wu-k)\right| du = \int_{|t| > \frac{w\delta}{2}} |\vartheta(t)| dt \to 0 \text{ as } w \to +\infty
$$

for sufficiently large w. Hence, we get

$$
I_{2,2} \le 2\|f\|_\rho M_0(\tau)\varepsilon.
$$

Finally, by Lemma 1, since

$$
\lim_{w \to +\infty} \sum_{|wx-k| > \gamma} \left|\tau(wx-k)\right| = 0
$$

uniformly w.r.t. $x \in \mathbb{R}$, we have

$$
I_{2,3} \le 2\|f\|_\rho \mathcal{M}_0(\vartheta)\varepsilon
$$

for sufficiently large w. Combining the above estimates, we have

$$\begin{aligned}
& \left|\left(S_w^{\tau,\vartheta}f\right)(x)-f(x)\right| \\
\leq\ & I_1+I_{2,1}+I_{2,2}+I_{2,3} \\
=\ & \frac{\|f\|_\rho}{w^2}\{M_0(\tau)\mathcal{M}_2(\vartheta)+M_1(\tau)\mathcal{M}_1(\vartheta)+2|wx|M_0(\tau)\mathcal{M}_1(\vartheta) \\
+\ & M_1(\tau)\mathcal{M}_1(\vartheta)+M_2(\tau)\mathcal{M}_0(\vartheta)+2|wx|M_1(\tau)\mathcal{M}_0(\vartheta)\} \\
+\ & \frac{\varepsilon}{\rho(x)}\left[M_0(\tau)\mathcal{M}_0(\vartheta)+2\|f\|_\rho M_0(\tau)+2\|f\|_\rho\mathcal{M}_0(\vartheta)\right].
\end{aligned} \tag{3.3}$$

By taking limit as $w\to+\infty$ we get (3.1).

Let $f\in U_\rho(\mathbb{R})$. Following the same steps of first part of the proof and multiplying both sides of (3.3) with $\rho(x)$ we have

$$\begin{aligned}
& \rho(x)\left|\left(S_w^{\tau,\vartheta}f\right)(x)-f(x)\right| \\
& \leq\frac{\|f\|_\rho}{w^2}\{M_0(\tau)\mathcal{M}_2(\vartheta)+M_1(\tau)\mathcal{M}_1(\vartheta)+2wM_0(\tau)\mathcal{M}_1(\vartheta) \\
& +M_1(\tau)\mathcal{M}_1(\vartheta)+M_2(\tau)\mathcal{M}_0(\vartheta)+2wM_1(\tau)\mathcal{M}_0(\vartheta)\} \\
& +\varepsilon\left[M_0(\tau)\mathcal{M}_0(\vartheta)+2\|f\|_\rho M_0(\tau)+2\|f\|_\rho\mathcal{M}_0(\vartheta)\right]
\end{aligned}$$

and taking supremum over $x\in\mathbb{R}$ we obtain (3.2) for $w\to+\infty$. □

Theorem 3. *Let $f\in C_\rho(\mathbb{R})$ and τ,ϑ be kernels with $\alpha=\beta=3$. Then,*

$$\begin{aligned}
\left|\left(S_w^{\tau,\vartheta}f\right)(x)-f(x)\right| \leq\ & 16(1+x^2)\Omega\left(f;w^{-1}\right)\{M_0(\tau)\mathcal{M}_0(\vartheta) \\
& +4[M_0(\tau)\mathcal{M}_3(\vartheta)+M_3(\tau)\mathcal{M}_0(\vartheta)]\}.
\end{aligned}$$

Proof. By using definition of $S_w^{\tau,\vartheta}$, Remark 1 (1) and the inequality (2.4) we have

$$\begin{aligned}
\left|\left(S_w^{\tau,\vartheta}f\right)(x)-f(x)\right| & \leq\sum_{k\in\mathbb{Z}}|\tau(wx-k)|w\int_{\mathbb{R}}|\vartheta(wu-k)||f(u)-f(x)|du \\
& \leq 16(1+x^2)\Omega(f;\delta)\sum_{k\in\mathbb{Z}}|\tau(wx-k)| \\
& \times w\int_{\mathbb{R}}|\vartheta(wu-k)|\left(1+\frac{|u-x|^3}{\delta^3}\right)du \\
& =16(1+x^2)\Omega(f;\delta)\left[\sum_{k\in\mathbb{Z}}|\tau(wx-k)|w\int_{\mathbb{R}}|\vartheta(wu-k)|du\right. \\
& \left.+\frac{1}{\delta^3}\sum_{k\in\mathbb{Z}}|\tau(wx-k)|w\int_{\mathbb{R}}|\vartheta(wu-k)||u-x|^3\right] \\
& := 16(1+x^2)\Omega(f;\delta)(I_1+I_2).
\end{aligned}$$

It can be easily seen that

$$I_1 \le M_0(\tau)\mathcal{M}_0(\vartheta).$$

Now, we need to estimate I_2. By straightforward computation, we can write

$$\begin{aligned}
I_2 &\le \frac{1}{\delta^3 w^3}\sum_{k\in\mathbb{Z}}|\tau(wx-k)|w\int_{\mathbb{R}}|\vartheta(wu-k)||wu-k+k-wx|^3 du \\
&\le \frac{4}{\delta^3 w^3}\left[\sum_{k\in\mathbb{Z}}|\tau(wx-k)|w\int_{\mathbb{R}}|\vartheta(wu-k)||wu-k|^3 du\right. \\
&\left.+\sum_{k\in\mathbb{Z}}|\tau(wx-k)||k-wx|^3 w\int_{\mathbb{R}}|\vartheta(wu-k)|du\right] \\
&= \frac{4}{\delta^3 w^3}[M_0(\tau)\mathcal{M}_3(\vartheta)+M_3(\tau)\mathcal{M}_0(\vartheta)].
\end{aligned}$$

Substituting I_1 and I_2 and choosing $\delta = w^{-1}$ we immediately obtain the thesis. □

Remark 3 ([16]). Let us consider the Taylor expansion of a function $f \in C^{(r)}(\mathbb{R})$ at the point $x \in \mathbb{R}$, that is,

$$f(t) = \sum_{k=0}^{r}\frac{f^{(k)}(x)}{k!}(t-x)^k + R_r(f;t,x), \tag{3.4}$$

where

$$R_r(f;t,x) = \frac{(t-x)^r}{r!}\left(f^{(r)}(\xi) - f^{(r)}(x)\right)$$

and ξ lying between t and x.

Hence using the inequality $|\xi - x| \le |t-x|$, we can write

$$\begin{aligned}
\left|R_r^*(f;t,x)\right| &:= \frac{\left|f^{(r)}(\xi) - f^{(r)}(x)\right|}{r!} \\
&\le \frac{16}{r!}(1+x^2)\Omega\left(f^{(r)};\delta\right)\left(1+\frac{|t-x|^3}{\delta^3}\right).
\end{aligned}$$

Hence we have the estimate

$$R_r(f;t,x) \le \frac{16}{r!}(1+x^2)\Omega\left(f^{(r)};\delta\right)\left(|t-x|^r + \frac{|t-x|^{r+3}}{\delta^3}\right). \tag{3.5}$$

In order to present Voronovskaja theorem in quantitative form, we need further assumption on kernel function τ, that is, there exists $r \in \mathbb{N}$ such that for every $j \in \mathbb{N}_0, j \le r$, there holds:

$$m_j(\tau, u) =: m_j(\tau) \in \mathbb{R}, \text{ is independent of } u. \tag{3.6}$$

Now we can state the Voronovskaja theorem.

Theorem 4. *Let τ, ϑ be kernels with $\alpha = \beta = r+3, r \geq 1,$ where r is the parameter of assumption (3.6). Then for $f \in C^{(r)}(\mathbb{R})$ such that $f^{(r)} \in C_w^*(\mathbb{R})$, we have*

$$\left| w\left[\left(S_w^{\tau,\vartheta} f \right)(x) - f(x) \right] - \sum_{m=1}^{r} \frac{f^{(m)}(x)}{m!} \frac{1}{w^{m-1}} \sum_{j=0}^{m} \binom{m}{j} m_{m-j}(\tau) \mathfrak{m}_j(\vartheta) \right|$$
$$\leq \frac{2^{r+3}}{r! w^{r-1}} (1+x^2) \Omega\left(f^{(r)}; \frac{1}{w} \right) \{ M_0(\tau) \mathcal{M}_r(\vartheta) + M_r(\tau) \mathcal{M}_0(\vartheta)$$
$$+ 8(M_0(\tau) \mathcal{M}_{r+3}(\vartheta) + M_{r+3}(\tau) \mathcal{M}_0(\vartheta)) \}.$$

Proof. Considering the Taylor expansion of f we have

$$f(t) = \sum_{k=0}^{r} \frac{f^k(x)}{k!} (t-x)^r + R_r(f; t, x),$$

where $R_r(f; t, x)$ is the Lagrange remainder (see (3.5)). Using definition of the $S_w^{\tau,\vartheta}$ and (3.4) we get

$$\begin{aligned} \left(S_w^{\tau,\vartheta} f \right)(x) &= \sum_{k \in \mathbb{Z}} \tau(wx-k) w \int_{\mathbb{R}} \vartheta(wu-k) f(u) du \\ &= \sum_{k \in \mathbb{Z}} \tau(wx-k) w \int_{\mathbb{R}} \vartheta(wu-k) \sum_{m=0}^{r} \frac{f^{(m)}(x)}{m!} (u-x)^m du \\ &+ \sum_{k \in \mathbb{Z}} \tau(wx-k) w \int_{\mathbb{R}} \vartheta(wu-k) R_r(f; u, x) du \\ &:= I_1 + I_2. \end{aligned} \tag{3.7}$$

By straightforward computation, we get

$$\begin{aligned}
I_1 &= \sum_{k\in\mathbb{Z}} \tau(wx-k) w \sum_{m=0}^{r} \frac{f^{(m)}(x)}{m!} \int_{\mathbb{R}} \vartheta(wu-k)(u-x)^m du \\
&= \sum_{k\in\mathbb{Z}} \tau(wx-k) \sum_{m=0}^{r} \frac{f^{(m)}(x)}{m!} \frac{1}{w^{m-1}} \int_{\mathbb{R}} \vartheta(wu-k)(wu-k+k-wx)^m du \\
&= \sum_{m=0}^{r} \frac{f^{(m)}(x)}{m!} \frac{1}{w^{m-1}} \sum_{k\in\mathbb{Z}} \tau(wx-k) \int_{\mathbb{R}} \vartheta(wu-k) \sum_{j=0}^{m} \binom{m}{j} (wu-k)^j (k-wx)^{m-j} du \\
&= \sum_{m=0}^{r} \frac{f^{(m)}(x)}{m!} \frac{1}{w^{m-1}} \sum_{j=0}^{m} \binom{m}{j} \sum_{k\in\mathbb{Z}} \tau(wx-k)(k-wx)^{m-j} \int_{\mathbb{R}} \vartheta(wu-k)(wu-k)^j du \\
&= \sum_{m=0}^{r} \frac{f^{(m)}(x)}{m!} \frac{1}{w^{m}} \sum_{j=0}^{m} \binom{m}{j} m_{m-j}(\tau)\mathfrak{m}_j(\vartheta).
\end{aligned}$$

Now we estimate the term I_2. Using the inequality (3.5) we get

$$\begin{aligned}
|I_2| &\le \sum_{k\in\mathbb{Z}} |\tau(wx-k)| w \int_{\mathbb{R}} |\vartheta(wu-k)| |R_r(f;u,x)| du \\
&\le \sum_{k\in\mathbb{Z}} |\tau(wx-k)| w \int_{\mathbb{R}} |\vartheta(wu-k)| \left[\frac{16}{r!}(1+x^2)\Omega\left(f^{(r)};\delta\right)\left(|u-x|^r + \frac{|u-x|^{r+3}}{\delta^3}\right)\right] du \\
&= \frac{16}{r!}(1+x^2)\Omega\left(f^{(r)};\delta\right)\left[\sum_{k\in\mathbb{Z}} |\tau(wx-k)| w \int_{\mathbb{R}} |\vartheta(wu-k)| |u-x|^r du \right.\\
&\quad \left. + \sum_{k\in\mathbb{Z}} |\tau(wx-k)| w \int_{\mathbb{R}} |\vartheta(wu-k)| \frac{|u-x|^{r+3}}{\delta^3} du \right] \\
&= \frac{2^{r-1}}{r! w^r}(1+x^2)\Omega\left(f^{(r)};\delta\right)\left[M_0(\tau)\mathcal{M}_r(\vartheta) + M_r(\tau)\mathcal{M}_0(\vartheta) \right.\\
&\quad \left. + \frac{8}{w^3}\frac{1}{\delta^3}\left(M_0(\tau)\mathcal{M}_{r+3}(\vartheta) + M_{r+3}(\tau)\mathcal{M}_0(\vartheta)\right)\right].
\end{aligned}$$

By choosing $\delta = w^{-1}$ we have

$$\begin{aligned}
|I_2| \le \frac{2^{r-1}}{r! w^r}(1+x^2)\Omega\left(f^{(r)};\frac{1}{w}\right)&\{M_0(\tau)\mathcal{M}_r(\vartheta) + M_r(\tau)\mathcal{M}_0(\vartheta) \\
&+ 8(M_0(\tau)\mathcal{M}_{r+3}(\vartheta) + M_{r+3}(\tau)\mathcal{M}_0(\vartheta))\}.
\end{aligned}$$

Hence by combining I_1 and I_2 in (3.7) we conclude that

$$\left| w\left[\left(S_w^{\tau,\vartheta}f\right)(x) - f(x)\right] - \sum_{m=1}^{r}\frac{f^{(m)}(x)}{m!}\frac{1}{w^{m-1}}\sum_{j=0}^{m}\binom{m}{j} m_{m-j}(\tau)\mathfrak{m}_j(\vartheta)\right|$$
$$\leq \frac{2^{r-1}}{r!w^{r-1}}(1+x^2)\Omega\left(f^{(r)};\frac{1}{w}\right)\{M_0(\tau)\mathcal{M}_r(\vartheta) + M_r(\tau)\mathcal{M}_0(\vartheta)$$
$$+ 8(M_0(\tau)\mathcal{M}_{r+3}(\vartheta) + M_{r+3}(\tau)\mathcal{M}_0(\vartheta))\}$$

which is desired result. □

Corollary 1. Under the assumptions of Theorem 4, it follows:

i. *In view of Lemma 2 (ii) we have qualitative form of Voronovskaja theorem for $S_w^{\tau,\vartheta}$ as*

$$\lim_{w\to\infty} w\left[\left(S_w^{\tau,\vartheta}f\right)(x) - f(x)\right] = f'(x)(m_1(\tau) + \mathfrak{m}_1(\vartheta)).$$

ii. *Moreover, if we also assume $m_j(\tau,u) = 0$ for $j = 1,2,\ldots,r-1,\quad r\in\mathbb{N}$, then we have*

$$\left| w^r\left[\left(S_w^{\tau,\vartheta}f\right)(x) - f(x)\right] - \frac{f^{(r)}(x)}{r!}m_r(\tau)\right|$$
$$\leq \frac{2^{r-1}}{r!}(1+x^2)w\left(f^{(r)};\frac{1}{w}\right)[M_0(\tau)\mathcal{M}_r(\vartheta) + M_r(\tau)\mathcal{M}_0(\vartheta)$$
$$+ 8(M_0(\tau)\mathcal{M}_{r+3}(\vartheta) + M_{r+3}(\tau)\mathcal{M}_0(\vartheta))]$$

and

$$\lim_{w\to\infty} w^r\left[\left(S_w^{\tau,\vartheta}f\right)(x) - f(x)\right] = \frac{f^{(r)}(x)}{r!}m_r(\tau).$$

4. Numerical evaluations

In this section, we give specific examples of kernels τ and ϑ which satisfies the assumptions employed in the previous sections. We consider a B-spline as kernel τ and a characteristic function as kernel ϑ. They satisfy the kernel assumptions given in Section 2 (see [15]).

The central B-spline of order $n\in\mathbb{N}$ is defined by:

$$\sigma_n(t) := \frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^i\binom{n}{j}\left(\frac{n}{2}+t-j\right)_+^{n-1}, \qquad t\in\mathbb{R},$$

where $(t)_+ := max\{t,0\}, t\in\mathbb{R}$. The support of σ_n is contained in the compact interval $[-n/2, n/2]$, and the Fourier transform is given by (see [15])

Table 1. $|(S_w^{\tau,\chi}f)(x) - f(x)|$ at some random values.

w	$\|(S_w^{\tau,\chi}f)(-1.2) - f(-1.2)\|$	$\|(S_w^{\tau,\chi}f)(1.4) - f(1.4)\|$	$\|(S_w^{\tau,\chi}f)(2.6) - f(2.6)\|$
5	0,120209	1,081473	0,680930
20	0,048204	0,183083	0,247636
50	0,021161	0,067269	0,105667
100	0,010916	0,032692	0,053943

Table 2. $|(S_w^{\tau,\chi}g)(x) - g(x)|$ at some random values.

w	$\|(S_w^{\tau,\chi}g)(-1.2) - g(-1.2)\|$	$\|(S_w^{\tau,\chi}g)(1.4) - g(1.4)\|$	$\|(S_w^{\tau,\chi}g)(2.6) - g(2.6)\|$
5	0,633541	0,815972	2,406237
20	0,106743	0,318299	1,004848
50	0,033731	0,124794	0,355468
100	0,015174	0,061529	0,165480

$$\hat{M}_n(v) = \mathrm{sinc}^n(v/2\pi), \qquad v \in \mathbb{R}.$$

Let us take, to greater detail, the kernel τ as the central B-spline of order 3, that is,

$$\sigma_3(t) := \frac{1}{2}\sum_{j=0}^{3}\binom{3}{j}\left(\frac{3}{2}+t-j\right)_+^2, \quad t \in \mathbb{R}. \tag{4.1}$$

Rewriting explicitly the expression in (4.1), we have

$$\sigma_3(t) = \begin{cases} \frac{3}{4}-t^2, & |t| \leq \frac{1}{2} \\ \frac{1}{2}\left(\frac{3}{2}-|t|\right)^2, & \frac{1}{2} < |t| < \frac{3}{2}, \\ 0, & |t| \geq \frac{3}{2} \end{cases}$$

where $t \in \mathbb{R}$.

Also, by choosing $\vartheta(t) = \chi_{[0,1]}(t)$, $t \in \mathbb{R}$, the operators $S_w^{\tau,\vartheta}$ reduces to

$$\left(S_w^{\sigma_3,\chi_{[0,1]}}f\right) = \sum_{k\in\mathbb{Z}} \sigma_3(wx-k)w\int_{\mathbb{R}} \chi_{[0,1]}(wu-k)f(u)du.$$

Now, we show that the convergence of the sampling Durrmeyer operators with the specific kernels, that is, $S_w^{\sigma_3,\chi_{[0,1]}}$, to the target function. We choose two functions defined by $f(t) = t^2e^{\cos(\pi t)}$ and $g(t) = t\sin(\pi t^2)$, $t \in \mathbb{R}$.

We present the numerical evaluations of difference of sampling Durrmeyer operators and functions f and g in the Tables 1 and 2, respectively.

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