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Article

Constructions of Helicoidal Surfaces in a 3-Dimensional Complete Manifold with Density

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Abstract: In this paper, we construct a helicoidal surface with a prescribed weighted mean curvature and weighted extrinsic curvature in a 3-dimensional complete manifold with a positive density function. We get a result for the minimal case. Additionally, we give examples of a helicoidal surface with a weighted mean curvature and weighted extrinsic curvature.

Keywords: complete manifold; manifold with density; weighted curvature; helicoidal surface

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1. Introduction

It is well known that a helicoidal surface is a generalization of a rotation surface. There are many studies about these surfaces under some given certain conditions [1–12]. Recently, the popular question has become whether a helicoidal surface can be constructed when its curvatures are prescribed. Several researchers have worked on this problem and obtained useful results. Firstly, Baikoussis et al. studied helicoidal surfaces with a prescribed mean and Gaussian curvature in $\mathbb{R}^3$ [13]. Then, Beneki et al. [14] and Ji et al. [15] studied similar work in $\mathbb{R}^3_1$. Furthermore, Dae Won Yoon et al. studied the helicoidal surfaces with a prescribed weighted mean and Gaussian curvature in $\mathbb{R}^3$ with density [16] and Yıldız et al. have studied the helicoidal surfaces with prescribed weighted curvatures in $\mathbb{R}^3_1$ with density [17]. For more details on manifolds with density and surfaces in manifold with density, see References [18–25].

This problem is extended to complete manifolds. Lee et al. studied the helicoidal surfaces with a prescribed extrinsic curvature or mean curvature in a conformally flat 3-space [10]. It is well known that a metric on a complete manifold is conformal to the Euclidean metric. For a given surface in a complete manifold with a conformal factor function F, the mean curvature and the extrinsic curvature are given by:

$$H_{g_F} = FH_{g_0} - \langle \mathbf{N}, \operatorname{grad} F \rangle\,, \tag{1}$$

$$G_{g_F} = F^2 G_{g_0} - 2H_{g_0} F \langle \mathbf{N}, \nabla F \rangle + \langle \mathbf{N}, \nabla F \rangle^2\,, \tag{2}$$

where $\mathbf{N}$ is the unit normal vector of a surface and ∇F is the gradient of F, H_{g_0} is the mean curvature of the surface in Euclidean 3-space, and G_{g_0} is the Gaussian curvature of a surface in Euclidean 3-space [26].

In this paper, we study helicoidal surfaces in a 3-dimensional complete manifold with density. We construct a helicoidal surface with a prescribed weighted mean and weighted extrinsic curvature. Then, we give examples to illustrate our result.

2. Preliminaries

Let M be a 3-dimensional complete manifold $\left(\mathbb{R}^3, \langle\,,\rangle_g\right)$ equipped with a metric $\langle\,,\rangle_g$ that is conformal to the Euclidean metric $\langle\,,\rangle$ such that:

$$\langle\,,\rangle_g = \frac{1}{F^2}\langle\,,\rangle,$$

where $F : \mathbb{R}^3 \to \mathbb{R}^+$ is a positive differentiable function.

A manifold with a positive density function φ is used to weight the volume and the hypersurface area. In terms of the underlying Riemannian volume dV_0 and area dA_0, the new, weighted volume and area are given by $dV = \varphi dV_0$ and $dA = \varphi dA_0$, respectively. One of the most important examples of manifolds with density, with applications to probability and statistics, is a Gauss space with density $\varphi = e^{a\left(-x^2-y^2-z^2\right)}$ for $a \in \mathbb{R}$ [22].

In Euclidean 3-space with density e^{φ}, the weighted mean curvature is given by:

$$H_{\varphi_{g_0}} = H_{g_0} - \frac{1}{2}\langle \mathbf{N}, \triangledown\varphi\rangle, \tag{3}$$

where H_{g_0} is the mean curvature of the surface, $\mathbf{N}$ is the unit normal vector of the surface, and $\triangledown\varphi$ is the gradient vector of φ [23]. If $H_{\varphi_{g_0}} = 0$, then the surface is called a weighted minimal surface. In Euclidean 3-space with density e^{φ}, the weighted Gaussian curvature with density is:

$$G_{\varphi_{g_0}} = G_{g_0} - \triangle\varphi, \tag{4}$$

where G_{g_0} is the Gaussian curvature of the surface and $\triangle$ is the Laplacian operator [27].

Throughout this paper, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we consider the positive density function and the conformal factor function as $e^{\varphi} = e^{-x_1^2-x_2^2}$ and $F = \sqrt{x_1^2 + x_2^2}$, respectively.

Let γ be a C^2-curve on x_1x_3-plane, of type $\gamma(u) = (u, 0, f(u))$, where $u \in I$ for an open interval $I \subset \mathbb{R}^+$. Using helicoidal motion on γ, we can obtain the helicoidal surface M as:

$$X(u,v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ 0 \\ f(u) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ hv \end{bmatrix},$$

with x_3-axis and a pitch $h \in \mathbb{R}$, so the parametric equation can be given in the form:

$$X(u,v) = (u\cos v, u\sin v, f(u) + hv).$$

It is straightforward to see that the mean curvature H_{g_0}, the Gaussian curvature G_{g_0}, and the unit normal vector of helicoidal surface are:

$$\begin{aligned}
H_{g_0} &= \frac{\left(u^2+h^2\right)uf''(u) + u^2 f'^3(u) + \left(u^2+2h^2\right)f'(u)}{2\left(u^2 f'^2(u) + u^2 + h^2\right)^{3/2}}, \\
G_{g_0} &= \frac{u^3 f'(u) f''(u) - h^2}{\left(u^2 f'^2(u) + u^2 + h^2\right)^2}, \\
\mathbf{N} &= \frac{\left(h\sin v - uf'(u)\cos v, -uf'(u)\sin v - h\cos v, u\right)}{\left(u^2 f'^2(u) + u^2 + h^2\right)^{1/2}}.
\end{aligned}$$

Using Equations (3) and (4), the weighted mean curvature $H_{\varphi_{g_0}}$ and the weighted Gaussian curvature $G_{\varphi_{g_0}}$ are obtained as:

$$
\begin{aligned}
H_{\varphi_{g_0}} &= \frac{\left(u^2+h^2\right) u f''(u) + \left(u^2-2u^4\right) f'^3(u) + \left(u^2+2h^2-2u^4-2h^2u^2\right) f'(u)}{2\left(u^2 f'^2(u)+u^2+h^2\right)^{3/2}}, \\
G_{\varphi_{g_0}} &= \frac{u^3 f'(u) f''(u) - h^2}{\left(u^2 f'^2(u)+u^2+h^2\right)^2} + 4.
\end{aligned}
$$

We assume that M is a surface in a 3-dimensional complete manifold with density. By considering Equations (1)–(4), we can define the weighted mean curvature $H_{\varphi_{g_F}}$ and the weighted extrinsic curvature $G_{\varphi_{g_F}}$ as:

$$
\begin{aligned}
H_{\varphi_{g_F}} &= F H_{g_0} - \frac{1}{2} F \langle \mathbf{N}, \bigtriangledown\varphi \rangle - \langle \mathbf{N}, \bigtriangledown F \rangle \\
G_{\varphi_{g_F}} &= F^2 G_{g_0} - 2F H_{g_0} \langle \mathbf{N}, \bigtriangledown F \rangle - F^2 \bigtriangleup \varphi + \langle \mathbf{N}, \bigtriangledown\varphi \rangle \langle \mathbf{N}, \bigtriangledown F \rangle F + \langle \mathbf{N}, \bigtriangledown F \rangle^2 .
\end{aligned}
$$

We obtain $H_{\varphi_{g_F}}$ and $G_{\varphi_{g_F}}$ for M as:

$$
H_{\varphi_{g_F}} = \frac{u\left[\left(3u^2-2u^4-2h^2\left(u^2-2\right)\right) f'(u) + \left(3u^2-2u^4\right) f'^3(u) + u\left(u^2+h^2\right) f''(u)\right]}{2\left(u^2 f'^2(u)+u^2+h^2\right)^{3/2}}, \tag{5}
$$

$$
G_{\varphi_{g_F}} = \frac{\begin{array}{c} u^2\left[4h^4+4u^4+h^2\left(8u^2-1\right)+\left(h^2\left(6u^2+3\right)+2\left(u^2+3u^4\right)\right) f'^2(u)\right. \\ \left. +2\left(u^2+u^4\right) f'^4(u) + u\left(2u^2+h^2\right) f'(u) f''(u)\right] \end{array}}{\left(u^2 f'^2(u)+u^2+h^2\right)^2}. \tag{6}
$$

3. Helicoidal Surfaces with Prescribed Weighted Mean or Weighted Extrinsic Curvature

In this section, we construct helicoidal surfaces with a prescribed weighted mean curvature and weighted extrinsic curvature in a 3-dimensional complete manifold with density $e^{\varphi} = e^{-x_1^2-x_2^2}$, where conformal factor $F = \sqrt{x_1^2+x_2^2}$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Theorem 1. *Let* $\gamma(u) = (u, 0, f(u))$ *be a profile curve of the helicoidal surface given by* $X(u,v) = (u\cos v, u \sin v, f(u) + hv)$ *in the 3-dimensional complete manifold with density and* $H_{\varphi_{g_F}}(u)$ *be the weighted mean curvature at the point* $(u, 0, f(u))$. *Then, there exists a two-parameter family of the helicoidal surface given by the curves:*

$$
\gamma\left(u, H_{\varphi_{g_F}}(u), h, c_1, c_2\right) = (u, 0, f(u)),
$$

where:

$$
f = \pm \int \frac{\sqrt{u^2+h^2}\left(\frac{1}{e^{-u^2}u^4}\left(\int\left(e^{-u^2}u^4\right) H_{\varphi_{g_F}} du + c_1\right)\right)}{\sqrt{1-u^2\left(\frac{1}{e^{-u^2}u^4}\left(\int\left(e^{-u^2}u^4\right) H_{\varphi_{g_F}} du + c_1\right)\right)^2}} du + c_2.
$$

Conversely, for a given smooth function $H_{\varphi_{g_F}}(u)$, *one can obtain the two-parameter family of curves* $\gamma\left(u, H_{\varphi_{g_F}}(u), h, c_1, c_2\right)$ *being the two-parameter family of helicoidal surfaces, accepting* $H_{\varphi_{g_F}}(u)$ *as the weighted mean curvature h as a pitch.*

Proof. Let us solve Equation (5), which is a second-order nonlinear ordinary differential equation. If we apply:

$$
\Psi = \frac{f'(u)}{\sqrt{u^2 f'^2(u)+u^2+h^2}}, \tag{7}
$$

into the equation, then we obtain the first-order linear ordinary differential equation:

$$H_{\varphi_{g_F}} = \left(2u - u^3\right)\Psi + \frac{u^2}{2}\Psi'. \tag{8}$$

Then, the general solution of Equation (8) is:

$$\Psi = \frac{1}{e^{-u^2}u^4}\left(\int \left(e^{-u^2}u^4\right) H_{\varphi_{g_F}} du + c_1\right), \tag{9}$$

where $c_1 \in \mathbb{R}$. Using Equations (7) and (9), we obtain:

$$\left(1 - u^2\left(\frac{1}{e^{-u^2}u^4}\left(\int \left(e^{-u^2}u^4\right) H_{\varphi_{g_F}} du + c_1\right)\right)^2\right) f'^2$$
$$= \left(u^2 + h^2\right)\left(\frac{1}{e^{-u^2}u^4}\left(\int \left(e^{-u^2}u^4\right) H_{\varphi_{g_F}} du + c_1\right)\right)^2. \tag{10}$$

From the above equation, we obtain:

$$f' = \pm\frac{\sqrt{u^2 + h^2}\Psi}{\sqrt{1 - u^2\Psi^2}}. \tag{11}$$

By integrating Equation (11), we obtain:

$$f = \pm\int \frac{\sqrt{u^2 + h^2}\Psi}{\sqrt{1 - u^2\Psi^2}} du + c_2, \tag{12}$$

where $c_2 \in \mathbb{R}$.

By contrast, for a given constant $h \in \mathbb{R} - \{0\}$, a real-valued smooth function $H_{\varphi_{g_F}}(u)$ defined on an open interval $I \subset \mathbb{R}^+$ and an arbitrary $u_0 \in I$, there exists an open subinterval $u_0 \in I' \subset I$ and an open interval $J \subset \mathbb{R}$ which contains:

$$\tilde{c}_1 = -\left(\int \left(e^{-u^2}u^4\right) H_{\varphi_{g_F}} du\right)(u_0),$$

such that:

$$S(u, c_1) = 1 - u^2\left(\frac{1}{e^{-u^2}u^4}\left(\int \left(e^{-u^2}u^4\right) H_{\varphi_{g_F}} du + c_1\right)\right)^2 > 0,$$

for arbitrary (u, c_1). Since $S(u_0, \tilde{c}_1) = 1 > 0$ and S is continuous, S is positive on $I' \times J \subset \mathbb{R}^2$. Thus, the two-parameter family of the curves can be given as:

$$\gamma\left(u, H_{\varphi_{g_F}}(u), h, c_1, c_2\right) = (u, 0, f(u)),$$

where:

$$f = \pm\int \frac{\sqrt{u^2 + h^2}\left(\frac{1}{e^{-u^2}u^4}\left(\int \left(e^{-u^2}u^4\right) H_{\varphi_{g_F}} du + c_1\right)\right)}{\sqrt{1 - u^2\left(\frac{1}{e^{-u^2}u^4}\left(\int \left(e^{-u^2}u^4\right) H_{\varphi_{g_F}} du + c_1\right)\right)^2}} du + c_2.$$

□

The following corollary is an immediate consequence of Theorem 1 and the definition of a minimal surface.

Corollary 1. *Let M be a minimal helicoidal surface in a complete manifold with density e^{φ}. Then, M is an open part of either a helicoid or a surface parametrized by:*

$$X(u, v) = \left(u\cos v, u\sin v, \pm\int \frac{c_1\sqrt{u^2 + h^2}}{\sqrt{e^{-2u^2}u^8 - u^2c_1^2}} du + c_2 + hv\right),$$

where $c_1, c_2 \in \mathbb{R}$.

Example 1. *Consider a helicoidal surface with the weighted mean curvature:*

$$H_{\varphi_{g_F}} = \frac{5e^{-u^2}u^4 - 2e^{-u^2}u^6}{e^{-u^2}u^4},$$

and the pitch $h = 1$ *in a complete manifold with density. Using Equation (12), we get* $\gamma(u)$. *Thus, we obtain the parametrization of the surface as follows:*

$$X(u,v) = \left(u\cos v, u\sin v, -\sqrt{1-u^2} + v\right),$$

and the figure of the domain:

$$\begin{cases} 0 < u < 1 \\ -4 < v < 4 \end{cases},$$

is given in Figure 1.

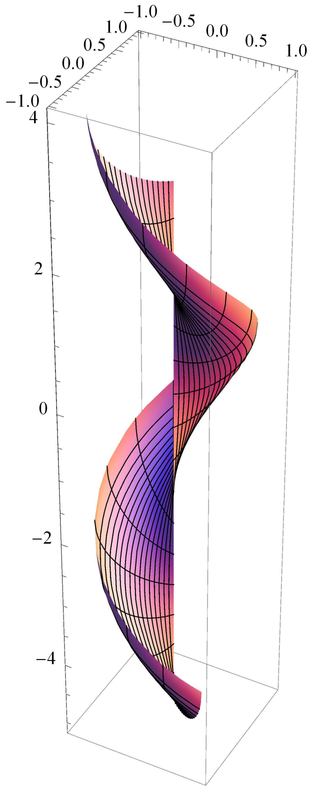

Figure 1. The helicoidal surface with the weighted mean curvature.

The difference between $H_{\varphi_{g_0}}$ *and* $H_{\varphi_{g_F}}$ *of the helicoidal surface with density can be seen in Figure 2.*

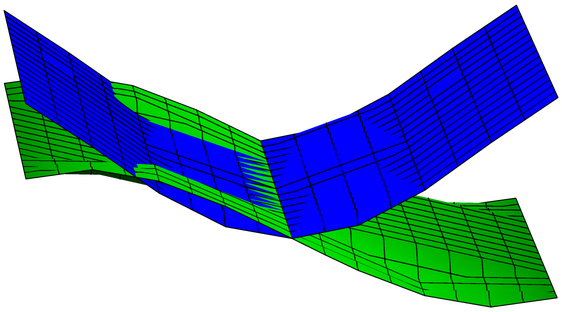

Figure 2. $H_{\varphi_{g_0}}$ (Green) and $H_{\varphi_{g_F}}$ (Blue).

Example 2. *Consider a helicoidal surface with the weighted mean curvature:*

$$H_{\varphi_{g_F}} = \frac{1-2u^2}{u^4},$$

and the pitch $h = 1$ in a complete manifold with density. Using Equation (12), we get $\gamma(u)$. Thus, we obtain the parametrization of the surface as follows:

$$X(u,v) = \left(u\cos v, u\sin v, -\frac{\sqrt{-1+u^4}\arctan\left(\frac{\sqrt{1+u^2}}{\sqrt{-1+u^4}}\right)}{\sqrt{1-\frac{1}{u^4}u^2}} + v \right),$$

and the figure of the domain:

$$\begin{cases} 2 < u < 10 \\ -5 < v < 5 \end{cases},$$

is given in Figure 3.

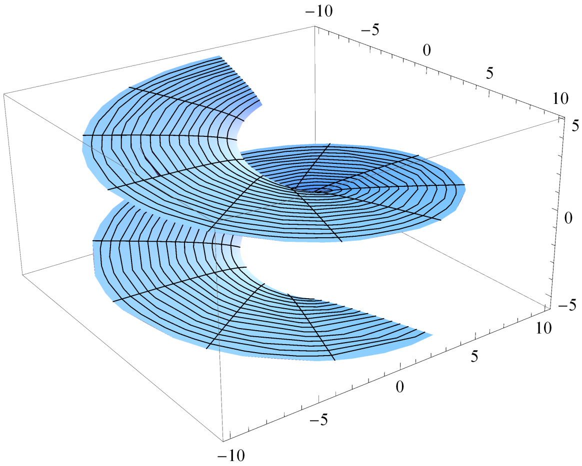

Figure 3. The helicoidal surface with the weighted mean curvature.

Theorem 2. *Let $\gamma(u) = (u, 0, f(u))$ be a profile curve of the helicoidal surface given by $X(u,v) = (u\cos v, u\sin v, f(u) + hv)$ in a 3-dimensional complete manifold with density and $G_{\varphi_{g_F}}(u)$ be the weighted extrinsic curvature at the point $(u, 0, f(u))$. Then, there exists a two-parameter family of the helicoidal surface, which is given by the curves:*

$$\gamma\left(u, G_{\varphi_{g_F}}, h, c_1, c_2\right) = \left(u, 0, \pm\int \left(\frac{\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)u + \left(u^2+h^2\right)B}{\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)u - u^2 B} \right)^{\frac{1}{2}} du + c_2 \right),$$

where:

$$B = \int \left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)\left(\left(\frac{-4-8h^2-12u^2}{2u^2+h^2}\right) + \frac{2}{u^2}G_{\varphi_{g_F}}\right) du + c_1,$$

and c_1 and c_2 are constants. Conversely, for a given smooth function $G_{\varphi_{g_F}}$, one can obtain the two-parameter family of curves $\gamma\left(u, G_{\varphi_{g_F}}(u), h, c_1, c_2\right)$, being the two-parameter family of the helicoidal surfaces, accepting $G_{\varphi_{g_F}}$ as the weighted extrinsic curvature h as a pitch.

Proof. Let's solve the second-order nonlinear ordinary differantial Equation (6). We can rewrite Equation (6) as follows:

$$\Phi' + \left(\frac{5h^2+6u^2-4h^2u^2-4u^4}{2u^3+uh^2}\right)\Phi = \left(\frac{-4-8h^2-12u^2}{2u^2+h^2}\right) + \frac{2}{u^2}G_{\varphi_{g_F}}, \tag{13}$$

where:

$$\Phi = \frac{uf'^2(u) - u}{u^2 f'^2(u) + u^2 + h^2}. \tag{14}$$

The general solution of Equation (13) is:

$$\Phi = \frac{\left(\int\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)\left(\left(\frac{-4-8h^2-12u^2}{2u^2+h^2}\right)+\frac{2}{u^2}G_{\varphi_{g_F}}\right)du+c_1\right)}{e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}}, \tag{15}$$

where $c_1 \in \mathbb{R}$. Combining Equations (14) and (15),we get:

$$\begin{aligned}
&\Bigg(\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)u\\
&-u^2\left(\int\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)\left(\left(\frac{-4-8h^2-12u^2}{2u^2+h^2}\right)+\frac{2}{u^2}G_{\varphi_{g_F}}\right)du+c_1\right)\Bigg)f'^2\left(u\right)\\
&=u\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)\\
&+\left(u^2+h^2\right)\left(\int\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)\left(\left(\frac{-4-8h^2-12u^2}{2u^2+h^2}\right)+\frac{2}{u^2}G_{\varphi_{g_F}}\right)du+c_1\right).
\end{aligned}$$

If we set:

$$B=\int\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)\left(\left(\frac{-4-8h^2-12u^2}{2u^2+h^2}\right)+\frac{2}{u^2}G_{\varphi_{g_F}}\right)du+c_1,$$

then:

$$f'^2=\frac{\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)u+\left(u^2+h^2\right)B}{\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)u-u^2B}. \tag{16}$$

It follows that:

$$f\left(u\right)=\pm\int\left(\frac{\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)u+\left(u^2+h^2\right)B}{\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)u-u^2B}\right)^{\frac{1}{2}}du+c_2 \tag{17}$$

where $c_2 \in \mathbb{R}$.

Conversely, for a given $h \in \mathbb{R}$ and a smooth function $G_{\varphi_{g_F}}(u)$, defined on an open interval $I \subset \mathbb{R}^+$ and an arbitrary $u_0 \in I$, there exists an open subinterval $I' \subset I$ containing u_0 and an open interval $J \subset \mathbb{R}$ containing:

$$\widetilde{c}_1=-\left(\int\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)\left(\left(\frac{-4-8h^2-12u^2}{2u^2+h^2}\right)+\frac{2}{u^2}G_{\varphi_{g_F}}\right)du\right)\left(u_0\right),$$

such that:

$$S\left(u,c_1\right)=\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)-uB>0,$$

which is defined on $I' \times J$. Thus, a two-parameter family of the curves can be given as:

$$\gamma\left(u,G_{\varphi_{g_F}},h,c_1,c_2\right)=\left(u,0,\pm\int\left(\frac{\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)u+\left(u^2+h^2\right)B}{\left(e^{-u^2}u^5\left(h^2+2u^2\right)^{-1-\frac{h^2}{2}}\right)u-u^2B}\right)^{\frac{1}{2}}du+c_2\right),$$

where $(u,c_1) \in I' \times J$; $c_2, h \in \mathbb{R}$ and $G_{\varphi_{g_F}}$ is a smooth function. □

Example 3. *Consider a helicoidal surface with the weighted extrinsic curvature:*

$$G_{\varphi_{g_F}}(u)=\frac{4+11u^2+14u^4+4u^6}{\left(2+u^2\right)^2},$$

in a complete manifold with density. Using Equation (17), we obtain $f(u) = \ln u$ *for* $h = 1, c_1 = 0, c_2 = 0$ *and the parametrization of the surface as follows:*

$$X(u,v) = (u\cos v, u\sin v, \ln u + v)\,.$$

The figure of the surface of the domain:

$$\begin{cases} 0 < u < 3 \\ -4 < v < 4 \end{cases},$$

is given in Figure 4.

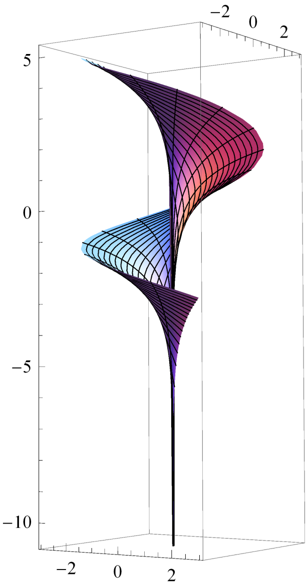

Figure 4. The helicoidal surface with the weighted Gaussian curvature.

The difference between $G_{\varphi_{g_0}}$ *and* $G_{\varphi_{g_F}}$ *of the helicoidal surface with density can be seen in Figure 5.*

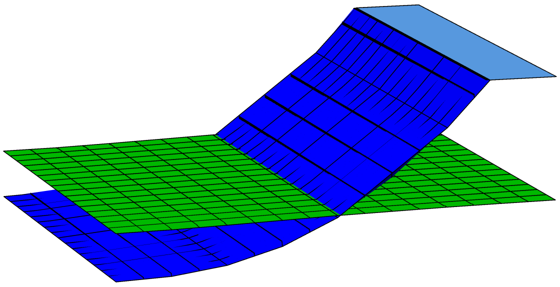

Figure 5. $G_{\varphi_{g_0}}$ (Green) and $G_{\varphi_{g_F}}$ (Blue).

4. Conclusions and Future Work

In this paper, using the conformal factor function $F = \sqrt{x_1^2 + x_2^2}$, we constructed a helicoidal surface with a prescribed weighted mean curvature and Gaussian curvature in a complete manifold with a positive density function. Different helicoidal surfaces can be obtained in a complete manifold with density using different conformal factor functions. In addition, if conformal factor function F is bounded, a manifold is called a conformally flat space. Thus, by considering a bounded function, one can study helicoidal surface in a conformally flat space with density.

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