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FOUNDATIONS

Soft partial metric spaces

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Abstract

This paper is an introduction to soft partial metric spaces. The aim is to create a soft topological model for a programming language described as a soft logic system, like in classical partial metric studies. Since the soft metric spaces have Hausdorff properties, they are not useful in examining non-Hausdorff soft topologies. This paper proposes a generalized soft metric for non-Hausdorff soft topologies and a new approach that guides how to expand soft metric implements like the Banach theorem to such topologies.

Keywords Soft set · Soft element · Soft partial metric · Soft quasi-metric · Banach theorem

1 Introduction

We present a soft partial metric concept via soft elements which is a generalization of soft metric in this paper, where for a soft element $\tilde{r}$, $p(\tilde{r}, \tilde{r})$ does not have to be equal to soft zero real number. We give the relations of a soft partial metric with both the classical metric, the soft metric and the soft quasi-metric that we introduce briefly here. We prove some completeness properties of the soft partial metric space and generalize Banach fixed point theorem in this space.

Matthews (1994) presented a partial metric space concept, considering that $d(r, r)$ does not have to be equal to zero, which is a generalization of metric spaces. This new area has wide application potentials in studying the semantics and computer domains (O'Neill 1995; Kopperman et al. 2005; Bukatin et al. 2009). There are distinct approaches to the partial metric space when it comes to application of the advancing mathematical notions to computer science (Romaguera and Schellekens 2001; Schellekens 2003; Waszkiewicz 2003; Künzi et al. 2006). Many researchers worked on topological properties and fixed point theorems in partial metric spaces and integration of this with fuzzy set theory (See Oltra and Valero (2004); Rabarison (2007); Altun et al. (2010); Karapınar and Erhan (2011); Aldemir et al. (2020); Aydoğdu et al. (2021) and others).

On the other hand, Molodtsov (1999) introduced the soft set theory, in 1999. He applied this theory to solve some practical problems in medical science, economics, social science, etc. After these, Maji et al. (2002, 2003) examined the soft set theory and also applied soft sets to decision-making problems. Some researchers (Shabir and Naz 2011; Hussain and Ahmad 2011; Hazra et al. 2012; Varol et al. 2012; Zorlutuna et al. 2012; Aygünoğlu and Aygün 2012; Georgiou and Megaritis 2014; Babitha and John 2015; Matejdes 2021a, b) established different topologies on the soft sets and investigated theirs topological properties. Das and Samanta (2012, 2013a, b) gave the concepts such as soft elements, soft real numbers, soft complex numbers and they also studied the soft metric concept via soft elements. Chiney and Samanta (2019) and Taşköprü and Altıntaş (2021) have established a new topological structure on soft sets using elementary operations. Some researchers (Wardowski 2013; Yazar et al. 2016; Abbas et al. 2016; Guler et al. 2016; Hosseinzadeh 2017; Altıntaş and Şimşek 1880; Öztunç and Aslan 2019; Çetkin et al. 2020; Altıntaş and Taşköprü 2020) studied fixed point theorems in soft metric space and its generalizations. Also, many researchers have applied the theory of soft sets to data science, and new mathematical structures have been

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continued to be established on soft sets (Chen et al. 2005; Pei and Miao 2005; Aktaş and Çağman 2007; Zou and Xiao 2008; Jun and Park 2008; Feng et al. 2010; Çetkin and Aygün 2016; Tahat et al. 2018; Kandemir 2018; Terepeta 2019; Alcantud 2020).

The paper is planned as below: Section 2 contains some basic information about soft set theory to be used in other sections. In Sect. 3, we first define a soft partial metric on an absolute soft set and some decisive example. We show that a soft partial metric space generates a T_0-soft topology. We discuss the relation between soft partial metrics and classical partial metrics. We then define the concept of soft quasi-metric and examine the relations soft partial metric with both weightable soft quasi-metric and soft metric. In the last part of the section, we briefly investigate the concept of completeness in a soft partial metric space. In Sect. 4, we consider Banach fixed point theorem. We generalize the extended fixed point theorems in the complete soft metric space to the complete soft partial metric space and prove their theoretical validity (Theorems 16 and 17). These theorems are also extensions of the Banach fixed point theorem in complete soft metric to complete soft partial metric space.

2 Preliminary information

Definition 1 (Matthews 1994; Bukatin et al. 2009) Assume that X is a non-empty set. We call a function $p : X \times X \to \mathbb{R}^+$ that meets the following axioms a partial metric on X and call the pair (X, p) a partial metric space. For each $r, s, t \in X$:

1. $r = s \Leftrightarrow p(r,r) = p(r,s) = p(s,s)$,
2. $p(r,r) \leq p(r,s)$,
3. $p(r,s) = p(s,r)$,
4. $p(r,s) \leq p(r,t) + p(t,s) - p(t,t)$.

Definition 2 (Bukatin et al. 2009) A sequence $\{r_n\}$ of pints in a partial metric space (X, p) is Cauchy sequence if the $p(r_n, r_m)$ converges to a point $r_0 \in X$ as n and m approach infinity, that is, if $\lim_{n,m\to\infty} p(r_n, r_m) = r_0$.

A partial metric space (X, p) is complete if every Cauchy sequence converges.

Banach fixed point theorem, which is an important theorem in metric spaces, is extended to partial metric spaces with the help of a contraction mapping and a monotone non-decreasing mapping as follows:

Theorem 1 *(*Bukatin et al. 2009; Rabarison 2007*) Suppose that (X, p) is a complete partial metric space.*

(a) *If $f : X \to X$ is a contraction (That is, there exists a constant $c \in [0, 1)$ such that for each $r, s \in X$, $p(f(r), f(s)) \leq cp(r,s)$). Then, there is a unique r_0 in X such that $f(r_0) = r_0$ and $p(r_0, r_0) = 0$.*

(b) *If for every fixed real number $r > 0$, $\psi : \mathbb{R} \to \mathbb{R}$ is monotone non-decreasing with $\lim_{n\to\infty} \psi(r) = 0$, then the mapping $f : X \to X$ defined by $p(f(r), f(s)) \leq p(r,s)$ for each $r, s \in X$ has a unique fixed point $r_0 \in X$ such that $f(r_0) = r_0$ and $p(r_0, r_0) = 0$.*

Definition 3 (Molodtsov 1999) Assume that P is a parameters set, X is a non-empty set and $P(X)$ is the power set of X. We call a pair (G, P) a soft set on X if $G : P \to P(X)$ is a mapping. That is, a soft set on X is a parameterized class of subsets in X. We can regard a soft set as the $\alpha-$approximate elements set of (G, P) for every $\alpha \in P$.

We call a function $\varepsilon : P \to X$ a soft element of X. Also, if for each $\alpha \in P$, $\varepsilon(\alpha) \in G(\alpha)$, then ε is called as belonging to (G, P). If $G(\alpha)$ is a singleton set for each $\alpha \in P$, then the soft set (G, P) itself can be taken as a soft element. The soft elements are denoted $\tilde{r}, \tilde{s}, \tilde{t}$, etc. and the class of soft elements of (G, P) by $SE(G, P)$. For details, see (Das and Samanta 2012).

Definition 4 (Das and Samanta 2012) Assume that P is a parameters set and $B(\mathbb{R})$ is the class of non-empty all bounded subsets of real numbers set $\mathbb{R}$. We call a mapping $G : P \to B(\mathbb{R})$ a soft real set. If $G(\alpha)$ is a singleton set for each $\alpha \in P$, then (G, P) is a soft real number. Soft real numbers are denoted $\tilde{s}, \tilde{t}$, etc. Specially, for each $\alpha \in P$, a soft real number satisfying $\tilde{s}(\alpha) = s$, is denoted by $\bar{s}$. We denote all the soft real number class and nonnegative soft real number class with $\mathbb{R}(P)$ and $\mathbb{R}(P)^*$, respectively.

Definition 5 Assume that (G, P) and (H, P) are soft sets on X. We call (G, P) a null soft set if $G(\alpha) = \emptyset$ and call (G, P) an absolute soft set if $G(\alpha) = X$ for each $\alpha \in P$, denote them by Φ and $\tilde{X}$, respectively.

Each class $\mathfrak{B}$ of soft elements of a soft set can generate a soft subset of this soft set. The soft set generated with $\mathfrak{B}$ is denoted by $SS(\mathfrak{B})$. For any soft set $(G, P) \in S(\tilde{X})$, $SS(SE(G, P)) = (G, P)$; whereas for a class $\mathfrak{B}$ of soft elements, $SE(SS(\mathfrak{B})) \supset \mathfrak{B}$.

We call (G, P) a soft subset of (H, P) if $G(\alpha) \subset H(\alpha)$ for each $\alpha \in P$ and denote by $(G, P)\tilde{\subset}(H, P)$. Then, we call (H, P) a soft upper set of (G, P) and denote by $(H, P)\tilde{\supset}(G, P)$. $(G, P) = (H, P)$ if and only if $(G, P)\tilde{\subset}(H, P)$ and $(H, P)\tilde{\supset}(G, P)$.

The union (F, P) of (H, P) and (G, P), denoted by $(F, P) = (H, P)\tilde{\cup}(G, P)$, is defined as $F(\alpha) = H(\alpha) \cup G(\alpha)$. The intersection (F, P) of (H, P) and (G, P), denoted by $(F, P) = (H, P)\tilde{\cap}(G, P)$, is defined as $F(\alpha) = H(\alpha) \cap G(\alpha)$. The difference (F, P) of (H, P) and (G, P), denoted by $(F, P) = (H, P)\tilde{\setminus}(G, P)$, is defined as $F(\alpha) = H(\alpha) \setminus G(\alpha)$. The complement $(G, P)^c = (G^c, P)$ of

(G, P) is defined as a mapping $G^c : P \to P(X)$ given by $G^c(\alpha) = X \setminus F(\alpha), \forall \alpha \in P$.

Throughout the work, the null soft set Φ and the soft sets (G, P) on X such that $G(\alpha) \neq \emptyset$ for every $\alpha \in P$ will be considered. The class of these soft sets denoted by $S(\tilde{X})$.

As it can be seen in Example 1, any soft element $\tilde{r}\tilde{\in}G\tilde{\cup}H$ does not necessarily imply that either $\tilde{r}\tilde{\in}G$ or $\tilde{r}\tilde{\in}H$. Also, the intersection of two soft sets or complement of a soft set of $S(\tilde{X})$ is not necessarily a member of $S(\tilde{X})$.

The elementary union $(F, P) = (H, P) \Cup (G, P)$, elementary intersection $(F, P) = (H, P) \Cap (G, P)$ of $(H, P), (G, P) \in S(\tilde{X})$ and the elementary complement $(G, P)^{\mathbb{C}}$ of (G, P) are defined by

$$
\begin{aligned}
(F, P) &= SS\{\tilde{r}\tilde{\in}\tilde{X} : \tilde{r}\tilde{\in}G \text{ or } \tilde{r}\tilde{\in}H\} \\
&= SS(SE(H, P) \cup SE(G, P)),
\end{aligned}
$$

$$
\begin{aligned}
(F, P) &= SS\{\tilde{r}\tilde{\in}\tilde{X} : \tilde{r}\tilde{\in}G \text{ and } \tilde{r}\tilde{\in}H\} \\
&= SS(SE(H, P) \cap SE(G, P)),
\end{aligned}
$$

and

$$
(G, P)^{\mathbb{C}} = SS\{\tilde{r}\tilde{\in}\tilde{X} : \tilde{r}\tilde{\in}(G, P)^c\},
$$

respectively. (For details, see (Taşköprü and Altıntaş 2021)).

Example 1 Let $P = \{\alpha, \beta\}$ and $X = \{r, s, t\}$. The class of all soft elements of $\tilde{X}$ is $SE(\tilde{X}) = \{\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_9\}$, where

$$
\begin{aligned}
\tilde{r}_1 &= \{(\alpha, r), (\beta, r)\}, & \tilde{r}_6 &= \{(\alpha, s), (\beta, t)\}, \\
\tilde{r}_2 &= \{(\alpha, r), (\beta, s)\}, & \tilde{r}_7 &= \{(\alpha, t), (\beta, r)\}, \\
\tilde{r}_3 &= \{(\alpha, r), (\beta, t)\}, & \tilde{r}_8 &= \{(\alpha, t), (\beta, s)\}, \\
\tilde{r}_4 &= \{(\alpha, s), (\beta, r)\}, & \tilde{r}_9 &= \{(\alpha, t), (\beta, t)\}. \\
\tilde{r}_5 &= \{(\alpha, s), (\beta, s)\},
\end{aligned}
$$

If we consider the classes of soft elements $\mathfrak{B}_1 = \{\tilde{r}_1, \tilde{r}_5\}$, $\mathfrak{B}_2 = \{\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_5\}$ and $\mathfrak{B}_3 = \{\tilde{r}_1, \tilde{r}_2, \tilde{r}_4, \tilde{r}_5\}$, then we obtain the soft sets

$$
\begin{aligned}
(F, P) &= SS(\mathfrak{B}_1) = SS(\mathfrak{B}_3) = \{(\alpha, \{r, s\}), (\beta, \{r, s\})\}, \\
(G, P) &= SS(\mathfrak{B}_2) = \{(\alpha, \{r, s\}), (\beta, \{r, s, t\})\}.
\end{aligned}
$$

Then, we have the classes of soft elements

$$
\begin{aligned}
&SE(F, P) = \{\tilde{r}_1, \tilde{r}_2, \tilde{r}_4, \tilde{r}_5\} \text{ and} \\
&SE(G, P) = \{\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4, \tilde{r}_5, \tilde{r}_6\}
\end{aligned}
$$

such that $\mathfrak{B}_1 \subset \mathfrak{B}_3 = SE(F, P)$ and $\mathfrak{B}_2 \subset SE(G, P)$. Also, if we consider the soft sets

$$
\begin{aligned}
&(H, P) = \{(\alpha, \{r, t\}), (\beta, \{t\})\} \text{ and} \\
&(K, P) = \{(\alpha, \{r, s, t\}), (\beta, \{s, t\})\},
\end{aligned}
$$

then we obtain

$$
(F, P)\tilde{\cup}(H, P) = \{(\alpha, \{r, s, t\}), (\beta, \{r, s, t\})\} \in S(\tilde{X}).
$$

We can see easily that $\tilde{r}_7\tilde{\in}(F, P)\tilde{\cup}(H, P)$, but $\tilde{r}_7\tilde{\notin}(F, P)$ and $\tilde{r}_7\tilde{\notin}(H, P)$. In addition, we encounter

$$
\begin{aligned}
(K, P)^c &= \{(\alpha, \emptyset), (\beta, \{r\})\} \notin S(\tilde{X}) \text{ and} \\
(F, P)\tilde{\cap}(H, P) &= \{(\alpha, \{r\}), (\beta, \emptyset)\} \notin S(\tilde{X}).
\end{aligned}
$$

Notice that

$$
\begin{aligned}
(F, P) \Cup (H, P) &= \{(\alpha, \{r, s, t\}), (\beta, \{r, s, t\})\} \in S(\tilde{X}), \\
(K, P)^{\mathbb{C}} &= \Phi \in S(\tilde{X}), \\
(F, P) \Cap (H, P) &= \Phi \in S(\tilde{X}).
\end{aligned}
$$

Definition 6 (Taşköprü and Altıntaş 2021) Assume that X, Y are two non-empty sets and P is a parameters set. We call a mapping $f : SE(\tilde{X}) \to SE(\tilde{Y})$ a soft mapping and call it a soft function if $\{f(\tilde{r})(\alpha) : \tilde{r}(\alpha) = a, \tilde{r} \in SE(\tilde{X}), \alpha \in P, a \in X\}$ is a singleton set.

Definition 7 (Das and Samanta 2013b) Assume that P is a parameters set and X is a non-empty set. We call a mapping $d : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$ that meets the following condition a soft metric on $\tilde{X}$ and call the triple $(\tilde{X}, d, P)$ a soft metric space. Let $\tilde{r}, \tilde{s}, \tilde{t}\tilde{\in}SE(\tilde{X})$:

M1. $d(\tilde{r}, \tilde{s})\tilde{\geq}\bar{0}$,
M2. $\tilde{r} = \tilde{s}$ if and only if $d(\tilde{r}, \tilde{s}) = \bar{0}$,
M3. $d(\tilde{r}, \tilde{s}) = d(\tilde{s}, \tilde{r})$,
M4. $d(\tilde{r}, \tilde{t})\tilde{\leq}d(\tilde{r}, \tilde{s}) + d(\tilde{s}, \tilde{t})$.

Definition 8 (Das and Samanta 2013b) Suppose that $(\tilde{X}, d, P)$ is a soft metric space. A sequence $\{\tilde{r}_n\}$ of soft elements in $\tilde{X}$ is called a Cauchy sequence in $\tilde{X}$ if corresponding to every $\tilde{\epsilon}\tilde{>}\bar{0}$, $\exists m \in \mathbb{N}$ such that $d(\tilde{r}_i, \tilde{r}_j)\tilde{<}\tilde{\epsilon}, \forall i, j \geq m$, i.e., $d(\tilde{r}_i, \tilde{r}_j) \to \bar{0}$ as $i, j \to \infty$.

The soft metric space $(\tilde{X}, d, P)$ is called complete if every Cauchy sequence in $\tilde{X}$ converges to a soft element of $\tilde{X}$.

A mapping $f : SE(\tilde{X}) \to SE(\tilde{X})$ is called a contraction mapping if there is a positive soft real number $\tilde{t}$ with $\bar{0} \leq \tilde{t}\tilde{<}\bar{1}$ such that $d(f(\tilde{r}), f(\tilde{s}))\tilde{\leq}\tilde{t}d(\tilde{s}, \tilde{s}), \forall \tilde{r}, \tilde{s}\tilde{\in}\tilde{X}$.

If there is a soft element $\tilde{r}_0\tilde{\in}\tilde{X}$ such that $f(\tilde{r}_0) = \tilde{r}_0$, then $\tilde{r}_0$ is called a fixed soft element of the mapping $f : SE(\tilde{X}) \to SE(\tilde{X})$.

Theorem 2 (Das and Samanta 2013b) *Suppose that $(\tilde{X}, d, P)$ is a complete soft metric space and $f : SE(\tilde{X}) \to SE(\tilde{X})$ is a contraction mapping. Then, f has a unique fixed soft element.*

Definition 9 (Taşköprü and Altıntaş 2021) We call a soft sets class $\mathcal{T} \subset S(\tilde{X})$ an elementary soft topology if it meets the following axioms:

1. $\Phi, \tilde{X} \in \mathcal{T}$,
2. The finite elementary intersection of soft sets of $\mathcal{T}$ belongs to $\mathcal{T}$,
3. The arbitrary elementary union of soft sets of $\mathcal{T}$ belongs to $\mathcal{T}$.

Then, we call the triple $(\tilde{X}, \mathcal{T}, P)$ an elementary soft topological space.

We call a soft sets class $\mathcal{B} \subset S(\tilde{X})$ a soft base for any elementary soft topology over $\tilde{X}$ if (1) there is a soft set $(B, P) \in \mathcal{B}$ with $\tilde{r}\tilde{\in}(B, P)$ for each soft element $\tilde{r}\tilde{\in}\tilde{X}$, (2) if for $(B_1, P), (B_2, P) \in \mathcal{B}$, $\tilde{r}\tilde{\in}(B_1, P) \Cap (B_2, P)$, there is a soft set (B_3, P) with $\tilde{r}\tilde{\in}(B_3, P)\tilde{\subset}(B_1, P) \Cap (B_2, P)$.

3 Soft partial metric spaces

3.1 Defining soft partial metric

Definition 10 Assume that P is a parameters set and X is a non-empty set. We call a mapping $p : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$ that meets the following conditions a soft partial metric on the soft set $\tilde{X}$ and call $(\tilde{X}, p, P)$ a soft partial space. For each $\tilde{r}, \tilde{s}, \tilde{t}\tilde{\in}\tilde{X}$,

P1. $\tilde{r} = \tilde{s} \Leftrightarrow p(\tilde{r}, \tilde{r}) = p(\tilde{r}, \tilde{s}) = p(\tilde{s}, \tilde{s})$,
P2. $p(\tilde{r}, \tilde{r})\tilde{\leq}p(\tilde{r}, \tilde{s})$,
P3. $p(\tilde{r}, \tilde{s}) = p(\tilde{s}, \tilde{r})$,
P4. $p(\tilde{r}, \tilde{s})\tilde{\leq}p(\tilde{r}, \tilde{t}) + p(\tilde{t}, \tilde{s}) - p(\tilde{t}, \tilde{t})$.

A soft partial metric space is a generalization of a soft metric space. In fact, if the self-distance of a soft element is soft zero real number, the above conditions are reduced to their soft metric equivalents. So, a soft metric can be defined to be a soft partial metric if every self-distance is soft zero. If $p(\tilde{r}, \tilde{s}) = \bar{0}$, $\tilde{r} = \tilde{s}$ by (P1) and (P2). But generally, the opposite is not true.

Definition 11 Assume that $(\tilde{X}, p, P)$ is a soft partial metric space and $\tilde{r} \in SE(\tilde{X})$ and $\tilde{\epsilon}\tilde{>}\bar{0}$ is a soft real number. We call the class

$$B_{\tilde{\epsilon}}(\tilde{r}) = \{\tilde{s} \in SE(\tilde{X}) : p(\tilde{s}, \tilde{r})\tilde{<}\tilde{\epsilon}\} \subset SE(\tilde{X})$$

and $SS(B_{\tilde{\epsilon}}(\tilde{r}))$ an open ball and a soft open ball, respectively.

Remark 1 In a soft partial metric space, some soft balls can be null soft set. For example, for every $\tilde{r} \in SE(\tilde{X})$, the open ball $B_{p(\tilde{r},\tilde{r})}(\tilde{r})$ is empty and the soft open ball $SS(B_{p(\tilde{r},\tilde{r})}(\tilde{r}))$ is null soft set.

Example 2 Suppose that P is finite parameters set and $\mathbb{R}(P)^*$ is the nonnegative soft real number class. Then, for each $\tilde{r}, \tilde{s} \in \mathbb{R}(P)^*$, the mapping

$$p : \mathbb{R}(P)^* \times \mathbb{R}(P)^* \to \mathbb{R}(P)^*$$

defined by $p(\tilde{r}, \tilde{s}) = max\{\tilde{r}, \tilde{s}\}$ is a soft partial metric. For each nonnegative soft real number, the self-distance is its value itself.

Example 3 Suppose that P is finite parameters set and $\mathbb{R}(P)^-$ is the non-positive soft reel number class. Then, for each $\tilde{r}, \tilde{s} \in \mathbb{R}(P)^-$, the mapping

$$p : \mathbb{R}(P)^- \times \mathbb{R}(P)^- \to \mathbb{R}(P)^*$$

defined by $p(\tilde{r}, \tilde{s}) = -min\{\tilde{r}, \tilde{s}\}$ is a soft partial metric. For each nonnegative soft real number the self-distance is its absolute value. We call this mapping the soft usual partial metric on $\mathbb{R}(P)^-$. In this space, if $\tilde{r}\tilde{\geq} - \tilde{\epsilon}$, $B_{\tilde{\epsilon}}(\tilde{r}) = \{\tilde{s} \in \mathbb{R}(P)^- : -min\{\tilde{r}, \tilde{s}\}\tilde{<}\tilde{\epsilon}\} = (-\tilde{\epsilon}, \bar{0})$. If $\tilde{r}\tilde{<} - \tilde{\epsilon}$, $p(\tilde{r}, \tilde{s}) = -\tilde{r}\tilde{>} - \tilde{\epsilon}$ and so $B_{\tilde{\epsilon}}(\tilde{r}) = \emptyset$. Let $\tilde{s} \in B_{\tilde{\epsilon}}(\tilde{r})$. Then, $-min\{\tilde{r}, \tilde{s}\}\tilde{<}\tilde{\epsilon}$ implies $min\{\tilde{r}, \tilde{s}\}\tilde{>} - \tilde{\epsilon}$ hence, $\tilde{s}\tilde{>} - \tilde{\epsilon}$.

Theorem 3 *Suppose that $(\tilde{X}, p, P)$ is a soft partial metric space. Then, the class of every soft open balls in $(\tilde{X}, p, P)$ forms a basis for T_0-soft topology on $\tilde{X}$. We call this topology the soft partial metric topology and denote by $\tau[p]$.*

Proof Obviously,

$$\tilde{X} = \tilde{\bigcup_{\tilde{r}\tilde{\in}\tilde{X}}} SS(B_{p(\tilde{r},\tilde{r})+\bar{1}}(\tilde{r})).$$

Because each soft open ball $SS(B_{p(\tilde{r},\tilde{r})+\bar{1}}(\tilde{r}))$ contains $\tilde{r}$. Let $SS(B_{\tilde{\epsilon}}(\tilde{r}))$ and $SS(B_{\tilde{\delta}}(\tilde{s}))$ are soft open balls in $(\tilde{X}, p, P)$ and $\tilde{t}\tilde{\in}SS(B_{\tilde{\epsilon}}(\tilde{r}))\tilde{\cap}SS(B_{\tilde{\delta}}(\tilde{s}))$. Take

$$\tilde{\alpha} = p(\tilde{t}, \tilde{t}) + min\{\tilde{\epsilon} - p(\tilde{r}, \tilde{t}), \tilde{\delta} - p(\tilde{s}, \tilde{t})\}.$$

Since $\tilde{\epsilon} - p(\tilde{r}, \tilde{t})\tilde{>}\bar{0}$, $\tilde{t}\tilde{\in}SS(B_{\tilde{\alpha}}(\tilde{r}))$, $\tilde{\delta} - p(\tilde{s}, \tilde{t})\tilde{>}\bar{0}$ and so $p(\tilde{t}, \tilde{t})\tilde{<}\tilde{\alpha}$. We show that $\tilde{t}_0\tilde{\in}SS(B_{\tilde{\epsilon}}(\tilde{r}))\tilde{\cap}SS(B_{\tilde{\delta}}(\tilde{s}))$ for $\tilde{t}_0\tilde{\in}SS(B_{\tilde{\alpha}}(\tilde{r}))$. From the axiom (P4), we have

$$\begin{aligned} p(\tilde{r}, \tilde{t}_0) &\tilde{\leq} p(\tilde{r}, \tilde{t}) + p(\tilde{t}, \tilde{t}_0) - p(\tilde{t}, \tilde{t}) \\ &\tilde{<} \tilde{\epsilon} - p(\tilde{t}, \tilde{t}) + p(\tilde{r}, \tilde{t}) \\ &\tilde{<} \tilde{\epsilon} - p(\tilde{r}, \tilde{t}) + p(\tilde{r}, \tilde{t}) = \tilde{\epsilon}. \end{aligned}$$

Thus, $\tilde{t}_0 \in B_{\tilde{\epsilon}}(\tilde{r})$ and $\tilde{t}_0 \tilde{\in} SS(B_{\tilde{\epsilon}}(\tilde{r}))$. Likewise, we have $\tilde{t}_0 \tilde{\in} SS(B_{\tilde{\delta}}(\tilde{s}))$ since

$$\begin{aligned} p(\tilde{s}, \tilde{t}_0) &\tilde{\leq} p(\tilde{s}, \tilde{t}) + p(\tilde{t}, \tilde{t}_0) - p(\tilde{t}, \tilde{t}) \\ &\tilde{<} \tilde{\delta} - p(\tilde{t}, \tilde{t}) + p(\tilde{s}, \tilde{t}) \\ &\tilde{<} \tilde{\delta} - p(\tilde{s}, \tilde{t}) + p(\tilde{s}, \tilde{t}) = \tilde{\delta}. \end{aligned}$$

As a result, $\tilde{t}_0 \tilde{\in} SS(B_{\tilde{\epsilon}}(\tilde{r})) \tilde{\cap} SS(B_{\tilde{\delta}}(\tilde{s}))$ and so

$$SS(B_{\tilde{\alpha}}(\tilde{r})) \tilde{\subset} SS(B_{\tilde{\epsilon}}(\tilde{r})) \tilde{\cap} SS(B_{\tilde{\delta}}(\tilde{s})).$$

Hence, the class of all soft open balls is a basis of an elementary soft topology $\tau[p]$ on $\tilde{X}$. We show that this topology is T_0. From the axiom (P4), $p(\tilde{r}, \tilde{r}) \tilde{<} p(\tilde{r}, \tilde{s})$ for two distinct soft elements $\tilde{r}, \tilde{s} \tilde{\in} \tilde{X}$. Taking $\tilde{\epsilon} = (p(\tilde{r}, \tilde{r}) + p(\tilde{r}, \tilde{s}))/2 \tilde{>} \bar{0}$, we have $\tilde{r} \tilde{\in} SS(B_{\tilde{\epsilon}}(\tilde{r}))$ but $\tilde{s} \tilde{\notin} SS(B_{\tilde{\epsilon}}(\tilde{r}))$ since

$$SS(B_{\tilde{\epsilon}}(\tilde{r})) = SS\left(\{\tilde{t} \in SE(\tilde{X}) : p(\tilde{r}, \tilde{t}) \tilde{<} \tilde{\epsilon}\}\right),$$

$2p(\tilde{r}, \tilde{r}) \tilde{<} p(\tilde{r}, \tilde{r}) + p(\tilde{r}, \tilde{s})$ and $2p(\tilde{r}, \tilde{s}) \tilde{\not<} p(\tilde{r}, \tilde{r}) + p(\tilde{r}, \tilde{s})$. □

The examinations of soft partial metrics constitute the partial ordering concept required for the study of soft domains in computer sciences.

Definition 12 We call a soft relation $\tilde{\preceq}$ on a soft set $\tilde{X}$ a soft partial order on $\tilde{X}$ if the following conditions are met. For $\tilde{r}, \tilde{s}, \tilde{t} \tilde{\in} \tilde{X}$,

1. $\tilde{r} \tilde{\preceq} \tilde{r}$,
2. $\tilde{r} \tilde{\preceq} \tilde{s}$ and $\tilde{s} \tilde{\preceq} \tilde{r} \Rightarrow \tilde{r} = \tilde{s}$,
3. $\tilde{r} \tilde{\preceq} \tilde{s}$ and $\tilde{s} \tilde{\preceq} \tilde{t} \Rightarrow \tilde{r} \tilde{\preceq} \tilde{t}$.

Once we have a T_0-soft topology on $\tilde{X}$, there is a natural soft partial ordering $\tilde{\sqsubseteq}$ defined by $\tilde{r} \tilde{\sqsubseteq} \tilde{s} \Leftrightarrow \tilde{r} \in (H, P)$ refer $\tilde{s} \notin (H, P)$ for $\tilde{r}, \tilde{s} \in SE(\tilde{X})$ on $\tilde{X}$, where $(H, P) \in \tau[p]$ is a soft open set.

Proposition 1 *In $(\tilde{X}, p, P)$, $\tilde{r} \tilde{\sqsubseteq} \tilde{s} \Leftrightarrow p(\tilde{r}, \tilde{r}) = p(\tilde{r}, \tilde{s})$.*

Proof Let $\tilde{r} \tilde{\sqsubseteq} \tilde{s}$. Then, we have

$$p(\tilde{r}, \tilde{r}) \tilde{\leq} p(\tilde{r}, \tilde{s}) \tilde{<} \tilde{\epsilon} \tilde{<} p(\tilde{r}, \tilde{r}) + \tilde{\epsilon}$$

for $\tilde{\epsilon} \tilde{>} \bar{0}$ and $\tilde{s} \in B_{\tilde{\epsilon}}(\tilde{r})$. So $p(\tilde{r}, \tilde{r}) = p(\tilde{r}, \tilde{s})$.

On the contrary, let $(H, P) \in \tau[p]$, $\tilde{r} \tilde{\in} (H, P)$ and $p(\tilde{r}, \tilde{r}) = p(\tilde{r}, \tilde{s})$. Then, there is a $\tilde{\epsilon} \tilde{>} \bar{0}$ satisfying $B_{\tilde{\epsilon}}(\tilde{r}) \subset SE(H, P)$. But, $p(\tilde{r}, \tilde{r}) = p(\tilde{r}, \tilde{r}) \tilde{<} \tilde{\epsilon} \Rightarrow \tilde{s} \in B_{\tilde{\epsilon}}(\tilde{r})$ and hence $\tilde{s} \tilde{\in} SE(H, P)$. Since this is valid for every $(H, P) \in \tau[p]$, we have $\tilde{r} \tilde{\sqsubseteq} \tilde{s}$. □

Example 4 In the soft partial metric space $(\mathbb{R}(P)^*, p)$, $\tilde{r} \tilde{\sqsubseteq} \tilde{s} \Leftrightarrow \tilde{r} \tilde{\geq} \tilde{s}$ and in $(\mathbb{R}(P)^-, p)$, $\tilde{r} \tilde{\sqsubseteq} \tilde{s} \Leftrightarrow \tilde{r} \tilde{\leq} \tilde{s}$. In fact, we obtain that, $p(\tilde{r}, \tilde{r}) = p(\tilde{r}, \tilde{s}) \Leftrightarrow \tilde{r} = max\{\tilde{r}, \tilde{s}\} \Leftrightarrow \tilde{r} \tilde{\geq} \tilde{s}$ in $(\mathbb{R}(P)^-, p)$. $p(\tilde{r}, \tilde{r}) = p(\tilde{r}, \tilde{s}) \Leftrightarrow -\tilde{r} = -min\{\tilde{r}, \tilde{s}\} \Leftrightarrow \tilde{r} \tilde{\leq} \tilde{s}$.

3.2 Soft partial metric and classical partial metric

In this section, we discuss the relationships between a soft partial metric and a classical partial metric to better understand soft partial metric spaces.

Theorem 4 *Each parameterized class of partial metrics $\{p_\alpha : \alpha \in P\}$ on a set X is a soft partial metric on $\tilde{X}$.*

Proof Give $\tilde{r}, \tilde{s} \in SE(\tilde{X})$. Then, $\tilde{r}(\alpha), \tilde{s}(\alpha) \in X$ for each $\alpha \in P$. Let $p : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$ be a mapping defined by $p(\tilde{r}, \tilde{s})(\alpha) = p_\alpha(\tilde{r}(\alpha), \tilde{s}(\alpha))$. Then, p is a soft partial metric on $\tilde{X}$. We now verify the axioms P1–P4 for soft partial metric. For each $\alpha \in P$ and $\tilde{r}, \tilde{s}, \tilde{t} \tilde{\in} \tilde{X}$;

P1. We get $p(\tilde{r}, \tilde{s})(\alpha) = p_\alpha(\tilde{r}(\alpha), \tilde{s}(\alpha)) \geq p_\alpha(\tilde{r}(\alpha), \tilde{r}(\alpha)) = p(\tilde{r}, \tilde{r})(\alpha) \geq \bar{0}(\alpha)$. Thus, $\bar{0} \tilde{\leq} p(\tilde{r}, \tilde{r}) \tilde{\leq} p(\tilde{r}, \tilde{s})$.

P2. Since

$$\begin{aligned} p(\tilde{r}, \tilde{r})(\alpha) &= p_\alpha(\tilde{r}(\alpha), \tilde{r}(\alpha)) \\ &= p_\alpha(\tilde{r}(\alpha), \tilde{s}(\alpha)) \\ &= p_\alpha(\tilde{s}(\alpha), \tilde{s}(\alpha)) \Leftrightarrow \tilde{r}(\alpha) = \tilde{s}(\alpha), \end{aligned}$$

we have

$$p(\tilde{r}, \tilde{r})(\alpha) = p(\tilde{r}, \tilde{s})(\alpha) = p(\tilde{s}, \tilde{s})(\alpha) \Leftrightarrow \tilde{r}(\alpha) = \tilde{s}(\alpha).$$

Thus, $p(\tilde{r}, \tilde{r}) = p(\tilde{r}, \tilde{s}) = p(\tilde{s}, \tilde{s}) \Leftrightarrow \tilde{r} = \tilde{s}$.

P3.
$$\begin{aligned} p(\tilde{r}, \tilde{s})(\alpha) &= p_\alpha(\tilde{r}(\alpha), \tilde{s}(\alpha)) \\ &= p_\alpha(\tilde{s}(\alpha), \tilde{r}(\alpha)) \\ &= p(\tilde{s}, \tilde{r})(\alpha) \Leftrightarrow p(\tilde{r}, \tilde{s}) = p(\tilde{s}, \tilde{r}). \end{aligned}$$

P4.
$$\begin{aligned} [p(\tilde{r}, \tilde{s}) + p(\tilde{s}, \tilde{t}) - p(\tilde{s}, \tilde{s})](\alpha) &= p_\alpha(\tilde{r}(\alpha), \tilde{s}(\alpha)) \\ &\quad + p_\alpha(\tilde{s}(\alpha), \tilde{t}(\alpha)) - p_\alpha(\tilde{s}(\alpha), \tilde{s}(\alpha)) \\ &\geq p_\alpha(\tilde{r}(\alpha), \tilde{t}(\alpha)). \end{aligned}$$

So, $p(\tilde{r}, \tilde{t}) \tilde{\leq} p(\tilde{r}, \tilde{s}) + p(\tilde{s}, \tilde{t}) + p(\tilde{s}, \tilde{t}) - p(\tilde{s}, \tilde{s})$. □

Corollary 1 *Each partial metric σ on a set X can be expanded to a soft partial metric on $\tilde{X}$.*

Proof Assume that the mapping

$$p : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$$

is defined by $p(\tilde{r}, \tilde{s})(\alpha) = \sigma((\tilde{r}(\alpha), \tilde{s}(\alpha))$ for each $\alpha \in P$ and $\tilde{r}, \tilde{s} \tilde{\in} \tilde{X}$. Using the proof method of Theorem 4 it can be shown that p is a soft partial metric on $\tilde{X}$. □

We call a soft partial metric defined by a partial metric σ the soft partial metric generated with σ.

Remark 2 The opposite of Theorem 4 is not valid. Although each parameterized class of partial metrics is a soft partial metric, every soft partial metric may not be a parameterized class of partial metrics. Hence, the soft partial metrics are more comprehensive and general than the parameterized class of the partial metrics.

Example 5 Let $X = \{r, s\}$ and $P = \{\alpha, \beta\}$. Then, $SE(\tilde{X}) = \{\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4\}$, where

$$\tilde{r}_1 = \{(\alpha, r), (\beta, r)\}, \qquad \tilde{r}_3 = \{(\alpha, s), (\beta, r)\},$$
$$\tilde{r}_2 = \{(\alpha, r), (\beta, s)\}, \qquad \tilde{r}_4 = \{(\alpha, s), (\beta, s)\}.$$

Suppose that $d : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$ is defined by $d(\tilde{r}, \tilde{s}) = \bar{0}$ if $\tilde{r} = \tilde{s}$ and $d(\tilde{r}, \tilde{s}) = \bar{1}$ if $\tilde{r} \neq \tilde{s}$, for all $\tilde{r}, \tilde{s} \tilde{\in} \tilde{X}$. Then, d is a soft partial metric. Let's parametrize it as $d(\tilde{r}, \tilde{s})(\alpha) = d_\alpha(\tilde{r}(\alpha), \tilde{s}(\alpha))$ for all $\alpha \in P$ and $\tilde{r}, \tilde{s} \tilde{\in} \tilde{X}$. Clearly, $d(\tilde{r}_1, \tilde{r}_1) = \bar{0}$, so $d(\tilde{r}_1, \tilde{r}_1)(\alpha) = d_\alpha(\tilde{r}_1(\alpha), \tilde{r}_1(\alpha)) = d_\alpha(r, r) = 0$ and $d(\tilde{r}_1, \tilde{r}_2) = \bar{1}$, so $d(\tilde{r}_1, \tilde{r}_2)(\alpha) = d_\alpha(\tilde{r}_1(\alpha), \tilde{r}_2(\alpha)) = d_\alpha(r, r) = 0$. Thus, d_α cannot be a partial metric on X.

Theorem 5 *Suppose that p is a soft partial metric on $\tilde{X}$ that meets the following condition (P5). If*

$$p_\alpha : X \times X \to \mathbb{R}^* = [0, \infty)$$

is defined by $p_\alpha(\tilde{r}(\alpha), \tilde{s}(\alpha)) = p(\tilde{r}, \tilde{s})(\alpha)$ for each $\alpha \in P$ and $\tilde{r}, \tilde{s} \tilde{\in} \tilde{X}$, then p_α is a partial metric on X.

P5. For $\alpha \in P$ and $(a, b) \in X \times X$, $\{p(\tilde{r}, \tilde{s})(\alpha) : \tilde{r}(\alpha) = a, \tilde{s}(\alpha) = b\}$ is a singleton set.

Proof It is clear that, for each $\alpha \in P$, $p_\alpha : X \times X \to \mathbb{R}^* = [0, \infty)$ is rule that appoint an ordered pair of X to a non-negative real number. p_α is well-defined from the condition (P5) and the soft partial metric conditions meets the partial metric axioms of p_α. Hence, the soft partial metric that meets (P5) gives a parameterized class of partial metrics. With this viewpoint, it is directly understood that a soft partial metric that meets (P5) is a special soft mapping defined in Majumdar and Samanta (2010), where $p : P \to (\mathbb{R}^*)^{X \times X}$ is a map. □

Proposition 2 *Let $B_{\tilde{\epsilon}}(\tilde{r})$ be an open ball in $(\tilde{X}, p, P)$ that meets the condition (P5). Then, for each $\alpha \in P$, $SS(B_{\tilde{\epsilon}}(\tilde{r}))(\alpha) = B_{\tilde{\epsilon}(\alpha)}(\tilde{r}(\alpha))$ is an open ball in (X, p_α).*

Proof If $B_{\tilde{\epsilon}}(\tilde{r})$ is an open ball in $(\tilde{X}, p, P)$ and $y \in SS(B_{\tilde{\epsilon}}(\tilde{r}))(\alpha)$, there is a soft element $\tilde{s} \in B_{\tilde{\epsilon}}(\tilde{r})$ with $\tilde{s}(\alpha) = y$ and so $p(\tilde{s}, \tilde{r}) \tilde{\leq} p(\tilde{r}, \tilde{r}) + \tilde{\epsilon}$ since $\tilde{s} \in B_{\tilde{\epsilon}}(\tilde{r})$. Thus, $p(\tilde{s}, \tilde{r})(\alpha) < p_\alpha(\tilde{r}, \tilde{r})(\alpha) + \tilde{\epsilon}(\alpha) \Rightarrow p_\alpha(y, \tilde{r}(\alpha)) < p_\alpha(\tilde{r}(\alpha)), \tilde{r}(\alpha))) + \tilde{\epsilon}(\alpha)$. Since for each $y \in SS(B_{\tilde{\epsilon}}(\tilde{r}))(\alpha)$, this relation is true, $SS(B_{\tilde{\epsilon}}(\tilde{r}))(\alpha) \subset B_{\tilde{\epsilon}(\alpha)}(\tilde{r}(\alpha))$.

Inversely, let us take a $t \in X$ such that $p_\alpha(t, \tilde{r}(\alpha)) < p_\alpha(\tilde{r}(\alpha)), \tilde{r}(\alpha))) + \tilde{\epsilon}(\alpha)$ and choose $\tilde{t} \tilde{\in} \tilde{X}$ such that if $\beta = \alpha$, $\tilde{t}(\beta) = t$ and if $\beta \neq \alpha$, $\tilde{t}(\beta) = \tilde{s}(\alpha)$ for $\tilde{s} \in B_{\tilde{\epsilon}}(\tilde{r})$, $\beta \in P$. Then, $p(\tilde{t}, \tilde{r}) \tilde{\leq} p(\tilde{r}, \tilde{r}) + \tilde{\epsilon}$ from (P5). Hence, $\tilde{r} \in B_{\tilde{\epsilon}}(\tilde{r})$ and $\tilde{t}(\beta) = t$ implies that $B_{\tilde{\epsilon}(\alpha)}(\tilde{r}(\alpha)) \subset SS(B_{\tilde{\epsilon}}(\tilde{r}))(\alpha)$. So $B_{\tilde{\epsilon}(\alpha)}(\tilde{r}(\alpha)) = SS(B_{\tilde{\epsilon}}(\tilde{r}))(\alpha)$. Since this is true for each $\alpha \in P$, we obtain $SS(B_{\tilde{\epsilon}}(\tilde{r}))(\alpha)$ is an open ball in (X, p_α). □

Theorem 6 *Assume that $(\tilde{X}, p, P)$ is a soft partial metric space that meets the condition (P5). Then, the class of all soft open balls forms a basis for T_0-elementary soft topology on $\tilde{X}$. We call this topology the soft elementary partial metric topology and denote by $\tau_E[p]$.*

Proof Since all soft open ball $SS(B_{p(\tilde{r},\tilde{r})+\bar{1}}(\tilde{r}))$ contains $\tilde{r}$, obviously $\tilde{r} \tilde{\in} SS(B_{p(\tilde{r},\tilde{r})+\bar{1}}(\tilde{r})) \tilde{\subset} \tilde{X}$ for each $\tilde{r} \tilde{\in} \tilde{X}$. Assume that $SS(B_{\tilde{\epsilon}}(\tilde{r}))$ and $SS(B_{\tilde{\rho}}(\tilde{s}))$ is two soft open balls in $(\tilde{X}, p, P)$ and $\tilde{t} \tilde{\in} SS(B_{\tilde{\epsilon}}(\tilde{r})) \Cap SS(B_{\tilde{\rho}}(\tilde{s}))$. Put $\tilde{\sigma} = p(\tilde{t}, \tilde{t}) + min\{\tilde{\epsilon} - p(\tilde{r}, \tilde{t}), \tilde{\rho} - p(\tilde{s}, \tilde{t})\}$. Since $\tilde{\epsilon} - p(\tilde{r}, \tilde{t}) \geq \bar{0}$, $\tilde{\rho} - p(\tilde{s}, \tilde{t}) \geq \bar{0}$, $\tilde{t} \tilde{\in} SS(B_{\tilde{\rho}}(\tilde{r}))$. So $p(\tilde{t}, \tilde{t}) \tilde{\leq} \tilde{\sigma}$. We prove that $SS(B_{\tilde{\sigma}}(\tilde{r})) \tilde{\subset} SS(B_{\tilde{\epsilon}}(\tilde{r})) \Cap SS(B_{\tilde{\rho}}(\tilde{s}))$. Let $\tilde{t}_0 \tilde{\in} SS(B_{\tilde{\sigma}}(\tilde{r}))$. We show that $\tilde{t}_0 \tilde{\in} SS(B_{\tilde{\epsilon}}(\tilde{r})) \Cap SS(B_{\tilde{\sigma}}(\tilde{s}))$. From the condition (P4), we get

$$\begin{aligned} p(\tilde{r}, \tilde{t}_0) &\tilde{\leq} p(\tilde{r}, \tilde{t}) + p(\tilde{t}, \tilde{t}_0) - p(\tilde{t}, \tilde{t}) \\ &\tilde{<} \tilde{\epsilon} - p(\tilde{t}, \tilde{t}) + p(\tilde{r}, \tilde{t}) \\ &\tilde{<} \tilde{\epsilon} - p(\tilde{r}, \tilde{t}) + p(\tilde{r}, \tilde{t}) = \tilde{\epsilon}. \end{aligned}$$

So, $\tilde{t}_0 \in B_{\tilde{\epsilon}}(\tilde{r})$ and $\tilde{t}_0 \tilde{\in} SS(B_{\tilde{\epsilon}}(\tilde{r}))$. Also, $\tilde{t}_0 \tilde{\in} SS(B_{\tilde{\rho}}(\tilde{r}))$ since

$$\begin{aligned} p(\tilde{s}, \tilde{t}_0) &\tilde{\leq} p(\tilde{s}, \tilde{t}) + p(\tilde{t}, \tilde{t}_0) - p(\tilde{t}, \tilde{t}) \\ &\tilde{<} \tilde{\rho} - p(\tilde{t}, \tilde{t}) + p(\tilde{s}, \tilde{t}) \\ &\tilde{<} \tilde{\rho} - p(\tilde{s}, \tilde{t}) + p(\tilde{s}, \tilde{t}) = \tilde{\rho}. \end{aligned}$$

Thus, $\tilde{t}_0 \tilde{\in} SS(B_{\tilde{\epsilon}}(\tilde{r})) \Cap SS(B_{\tilde{\rho}}(\tilde{s}))$ and so

$$SS(B_{\tilde{\sigma}}(\tilde{r})) \tilde{\subset} SS(B_{\tilde{\epsilon}}(\tilde{r})) \Cap SS(B_{\tilde{\rho}}(\tilde{s})).$$

Then, the class of soft open balls is a basis for an elementary soft topology $\tau_E[p]$ on $\tilde{X}$. Now, we prove that the elementary soft topology $\tau_E[p]$ is T_0. Given two distinct soft elements $\tilde{r}, \tilde{s} \tilde{\in} \tilde{X}$ and $p(\tilde{r}, \tilde{r}) \tilde{<} p(\tilde{r}, \tilde{s})$. Since

$$SS(B_{\tilde{\epsilon}}(\tilde{r})) = SS\left(\{\tilde{t} \in SE(\tilde{X}) : p(\tilde{r}, \tilde{t}) \tilde{<} \tilde{\epsilon}\}\right)$$

with $\tilde{\epsilon} = (p(\tilde{r}, \tilde{r}) + p(\tilde{r}, \tilde{s}))/\bar{2} > \bar{0}$, we have $\tilde{r} \tilde{\in} SS(B_{\tilde{\epsilon}}(\tilde{r}))$ and $\tilde{s} \tilde{\notin} SS(B_{\tilde{\epsilon}}(\tilde{r}))$. Thus, $2p(\tilde{r}, \tilde{r}) \tilde{<} p(\tilde{r}, \tilde{r}) + p(\tilde{r}, \tilde{s})$ and $2p(\tilde{r}, \tilde{s}) \tilde{\not<} p(\tilde{r}, \tilde{r}) + p(\tilde{r}, \tilde{s})$. □

3.3 Soft partial metric and weightable soft quasi-metric

In this section, we introduce weightable soft quasi-metric and discuss the relationships between a soft partial metric and a weightable soft quasi-metric.

Definition 13 A soft quasi-metric on an absolute soft set $\tilde{X}$ with a parameters set P is a mapping $q : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$ that meets the following axioms. For each $\tilde{r}, \tilde{s}, \tilde{t} \in SE(\tilde{X})$,

Q1. $\tilde{r} = \tilde{s} \Leftrightarrow q(\tilde{r}, \tilde{s}) = \bar{0}, q(\tilde{s}, \tilde{r}) = \bar{0}$,
Q2. $q(\tilde{r}, \tilde{s}) \tilde{\leq} q(\tilde{r}, \tilde{t}) + q(\tilde{t}, \tilde{s})$.

We call the soft set $\tilde{X}$ with a soft quasi-metric q on $\tilde{X}$ a soft quasi-metric space and denote by $(\tilde{X}, q, P)$.

Definition 14 Assume that $(\tilde{X}, q, P)$ is a soft quasi-metric space, $\tilde{r} \in SE(\tilde{X})$ and $\tilde{\epsilon} \tilde{>} \bar{0}$ is a soft real number. We call the class

$$B^q_{\tilde{\epsilon}}(\tilde{r}) = \left\{ \tilde{s} \in SE(\tilde{X}) : \ q(\tilde{s}, \tilde{r}) \tilde{<} \tilde{\epsilon} \right\} \subset SE(\tilde{X}).$$

an open ball and call $SS\left(B^q_{\tilde{\epsilon}}(\tilde{r})\right)$ a soft open ball.

The induced natural soft order on an absolute soft set $\tilde{X}$ generated by q is a relation $\tilde{\sqsubseteq}_q$ such that $\tilde{r} \tilde{\sqsubseteq}_q \tilde{s} \Leftrightarrow q(\tilde{r}, \tilde{s}) = \bar{0}$, $\forall \tilde{r}, \tilde{s} \in SE(\tilde{X})$.

Definition 15 Assume that $(\tilde{X}, q, P)$ is a soft quasi-metric space. A weighted soft quasi-metric on $\tilde{X}$ is a pair (q, ω) consisting of a soft weight mapping $\omega : SE(\tilde{X}) \to \mathbb{R}(P)^*$ and a soft quasi-metric q on $\tilde{X}$ with $q(\tilde{r}, \tilde{s}) + \omega(\tilde{r}) = q(\tilde{s}, \tilde{r}) + \omega(\tilde{s})$ for each $\tilde{r}, \tilde{s} \in SE(\tilde{X})$. We call the quadruple $(\tilde{X}, q, \omega, P)$ a weighted soft quasi-metric space and call $(\tilde{X}, q, P)$ weightable if there is a soft mapping $\omega : SE(\tilde{X}) \to \mathbb{R}(P)^*$ provided that (q, ω) is a weighted soft quasi-metric.

By assuming the soft weight mapping is soft zero, any soft metric can be considered like a weighted soft quasi-metric. The relations between soft partial metrics and weighted soft quasi-metrics are given in the following theorems.

Theorem 7 *Suppose that $(\tilde{X}, q, \omega, P)$ is a weighted soft quasi-metric space and*

$$p : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$$

is a mapping defined by $p(\tilde{r}, \tilde{s}) = \omega(\tilde{r}) + q(\tilde{r}, \tilde{s})$ for each $\tilde{r}, \tilde{s} \in SE(\tilde{X})$. Then, p is a soft partial metric.

Proof Let us show that p provides the soft partial metric axioms.

P1. Let $p(\tilde{r}, \tilde{r}) = p(\tilde{r}, \tilde{s}) = p(\tilde{s}, \tilde{s})$. Then, $\omega(\tilde{r}) = \omega(\tilde{r}) + q(\tilde{r}, \tilde{s})$. Hence, $q(\tilde{r}, \tilde{s}) = \bar{0}$ implies $\tilde{r} = \tilde{s}$. The opposite is clear.

P2. Since (q, ω) is a weighted soft quasi-metric on $\tilde{X}$, from Definition 15, $q(\tilde{r}, \tilde{s}) + \omega(\tilde{r}) = q(\tilde{s}, \tilde{r}) + \omega(\tilde{s})$ for each $\tilde{r}, \tilde{s} \tilde{\in} \tilde{X}$. Hence, we have

$$p(\tilde{r}, \tilde{s}) - p(\tilde{r}, \tilde{r}) = q(\tilde{r}, \tilde{s}) \tilde{\geq} \bar{0}.$$

P3. It is clear that, $p(\tilde{r}, \tilde{s}) = p(\tilde{s}, \tilde{r})$ from the definition of weighted soft quasi-metric for each $\tilde{r}, \tilde{s} \in SE(\tilde{X})$.

P4. For $\tilde{r}, \tilde{s}, \tilde{t} \in SE(\tilde{X})$,

$$\begin{aligned} p(\tilde{r}, \tilde{t}) &= \omega(\tilde{r}) + q(\tilde{r}, \tilde{t}) \\ &\tilde{\leq} \omega(\tilde{r}) + q(\tilde{r}, \tilde{s}) + q(\tilde{s}, \tilde{t}) \\ &= \omega(\tilde{r}) + q(\tilde{r}, \tilde{s}) + \omega(\tilde{t}) + q(\tilde{s}, \tilde{t}) - \omega(\tilde{t}) \\ &= p(\tilde{r}, \tilde{s}) + p(\tilde{s}, \tilde{t}) - p(\tilde{s}, \tilde{s}). \end{aligned}$$

□

Notice that, as a result of Definition 14 and Theorem 7 we provide a soft topology on a soft quasi-metric space by using Theorem 3. We call this topology the soft quasi-metric topology and denote by $\tau[q]$.

Theorem 8 *Suppose that*

$$p : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$$

is a partial metric and

$$q : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$$

is a mapping defined by $q(\tilde{r}, \tilde{s}) = p(\tilde{r}, \tilde{s}) - p(\tilde{r}, \tilde{r})$ and

$$\omega : SE(\tilde{X}) \to \mathbb{R}(P)^*$$

is a mapping with $\omega(\tilde{r}) = p(\tilde{r}, \tilde{r})$ for each $\tilde{r}, \tilde{s} \in SE(\tilde{X})$. Then, q is a weighted soft quasi-metric with soft weight mapping ω. Also $\tilde{r} \tilde{\sqsubseteq}_q \tilde{s} \Leftrightarrow \tilde{r} \tilde{\sqsubseteq}_p \tilde{s}$ for each $\tilde{r}, \tilde{s} \tilde{\in} \tilde{X}$ and $\tau[q] = \tau[p]$.

Proof Obviously, $q(\tilde{r}, \tilde{s}) \tilde{\geq} \bar{0}$ and $q(\tilde{r}, \tilde{r}) = \bar{0}$ for each $\tilde{r}, \tilde{s} \in SE(\tilde{X})$. We show first that q meets the soft quasi-metric axioms.

Q1. Let $q(\tilde{r}, \tilde{s}) = q(\tilde{s}, \tilde{r}) = \bar{0}$. Then,

$$p(\tilde{r}, \tilde{s}) - p(\tilde{r}, \tilde{r}) = p(\tilde{s}, \tilde{r}) - p(\tilde{s}, \tilde{s}) = \bar{0}.$$

Thus, $p(\tilde{r}, \tilde{s}) = p(\tilde{r}, \tilde{r}) = p(\tilde{s}, \tilde{s}) = \bar{0}$ so $\tilde{r} = \tilde{s}$.

Q2. Since p is a soft partial metric, we have

$$p(\tilde{r},\tilde{t}) - p(\tilde{r},\tilde{r}) \tilde{\leq} p(\tilde{r},\tilde{s}) - p(\tilde{r},\tilde{r}) + p(\tilde{s},\tilde{t}) - p(\tilde{s},\tilde{s})$$

for each $\tilde{r},\tilde{s},\tilde{t} \in SE(\tilde{X})$. So $q(\tilde{r},\tilde{t}) \tilde{\leq} p(\tilde{r},\tilde{s}) + p(\tilde{s},\tilde{t})$. Thus, q is a soft quasi-metric.

Let us show that (q,ω) is a weighted soft quasi-metric on $\tilde{X}$. Since $q(\tilde{r},\tilde{s}) = p(\tilde{r},\tilde{s}) - \omega(\tilde{r})$ and $q(\tilde{s},\tilde{r}) = p(\tilde{s},\tilde{r}) - \omega(\tilde{s})$ for each $\tilde{r},\tilde{s} \in SE(\tilde{X})$, we have $q(\tilde{r},\tilde{s}) + \omega(\tilde{r}) = q(\tilde{s},\tilde{r}) + \omega(\tilde{s})$ with $p(\tilde{r},\tilde{s}) = p(\tilde{s},\tilde{r})$. Hence, (q,ω) is a weighted soft quasi-metric.

Now, let $\tilde{r} \tilde{\in} SS\left(B^p_{\tilde{\epsilon}}(\tilde{r})\right)$ for $\tilde{\epsilon} \tilde{>} \bar{0}$ and $\tilde{r} \in SE(\tilde{X})$. Then, $\tilde{r} \in B^p_{\tilde{\epsilon}}(\tilde{r})$. If $\tilde{s} \in B^p_{\tilde{\epsilon}}(\tilde{r})$, $q(\tilde{r},\tilde{s}) = p(\tilde{r},\tilde{s}) - p(\tilde{r},\tilde{r}) \tilde{<} \tilde{\epsilon} - p(\tilde{r},\tilde{r})$ implies $\tilde{s} \in B^q_{\tilde{\epsilon}-p(\tilde{r},\tilde{r})}(\tilde{r})$. Hence, $\tilde{s} \tilde{\in} SS\left(B^q_{\tilde{\epsilon}-p(\tilde{r},\tilde{r})}(\tilde{r})\right)$. Conversely, let $\tilde{s} \tilde{\in} SS\left(B^q_{\tilde{\epsilon}-p(\tilde{r},\tilde{r})}(\tilde{r})\right)$. Then, $\tilde{s} \in B^q_{\tilde{\epsilon}-p(\tilde{r},\tilde{r})}(\tilde{r})$. Hence, $\tilde{s} \in B^p_{\tilde{\epsilon}}(\tilde{r})$ and so $\tilde{s} \tilde{\in} SS\left(B^p_{\tilde{\epsilon}}(\tilde{r})\right)$. Thus,

$$SS\left(B^p_{\tilde{\epsilon}}(\tilde{r})\right) = SS\left(B^q_{\tilde{\epsilon}-p(\tilde{r},\tilde{r})}(\tilde{r})\right)$$

for $\tilde{\epsilon} \tilde{>} p(\tilde{r},\tilde{r})$. From definition of a soft open ball, we have $B^p_{\tilde{\epsilon}}(\tilde{r}) = \emptyset$ and $SS\left(B^p_{\tilde{\epsilon}}(\tilde{r})\right) = \Phi$ for $\bar{0} \tilde{<} \tilde{\epsilon} \tilde{<} p(\tilde{r},\tilde{r})$. For $\tilde{\epsilon} \tilde{>} \bar{0}$ and $\tilde{r} \in SE(\tilde{X})$, if $\tilde{s} \in SE(\tilde{X})$, $\tilde{s} \tilde{\in} SS\left(B^q_{\tilde{\epsilon}}(\tilde{r})\right)$ hence $\tilde{s} \in B^q_{\tilde{\epsilon}}(\tilde{r})$, and $p(\tilde{r},\tilde{s}) - p(\tilde{r},\tilde{r}) \tilde{<} \tilde{\epsilon}$ implies $\tilde{s} \in B^p_{\tilde{\epsilon}+p(\tilde{r},\tilde{r})}(\tilde{r})$ and so $\tilde{s} \tilde{\in} SS\left(B^p_{\tilde{\epsilon}+p(\tilde{r},\tilde{r})}(\tilde{r})\right)$. Conversely, let $\tilde{s} \tilde{\in} SS\left(B^p_{\tilde{\epsilon}+p(\tilde{r},\tilde{r})}(\tilde{r})\right)$. Then, $\tilde{s} \in B^p_{\tilde{\epsilon}+p(\tilde{r},\tilde{r})}(\tilde{r})$. Since $q(\tilde{r},\tilde{s}) \tilde{<} \tilde{\epsilon}$, $\tilde{s} \in B^q_{\tilde{\epsilon}}(\tilde{r})$ and hence $\tilde{s} \tilde{\in} SS\left(B^q_{\tilde{\epsilon}}(\tilde{r})\right)$. Thus,

$$SS(B^q_{\tilde{\epsilon}}(\tilde{r})) = SS\left(B^p_{\tilde{\epsilon}+p(\tilde{r},\tilde{r})}(\tilde{r})\right)$$

for $\tilde{\epsilon} \tilde{>} \bar{0}$ and each $\tilde{r} \in SE(\tilde{X})$. Therefore, $\tau[q] = \tau[p]$.

Lastly, for each $\tilde{r},\tilde{s} \in SE(\tilde{X})$, $p(\tilde{r},\tilde{r}) = p(\tilde{r},\tilde{s})$ if and only if $q(\tilde{r},\tilde{s}) = \bar{0}$. Thus, $\tilde{r} \tilde{\sqsubseteq}_q \tilde{s} \Leftrightarrow \tilde{r} \tilde{\sqsubseteq}_p \tilde{s}$. □

3.4 Soft partial metric and soft metric

Definition 16 Assume that P is a parameters set and X is a non-empty set. We call a pair (d,ω) a weighted soft metric on $\tilde{X}$ if $d : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$ is a soft metric and $\omega : SE(\tilde{X}) \to \mathbb{R}(P)^*$ a soft weight mapping with $d(\tilde{r},\tilde{s}) \tilde{\geq} \omega(\tilde{r}) - \omega(\tilde{s})$ for each $\tilde{r},\tilde{s} \in SE(\tilde{X})$. Then, $(\tilde{X},d,\omega,P)$ is called a weighted soft metric space.

A soft metric d on $\tilde{X}$ is weightable if there is a soft weight mapping $\omega : SE(\tilde{X}) \to \mathbb{R}(P)^*$ with a weighted soft metric (d,ω). Then, $(\tilde{X},d,\omega,P)$ is weightable.

Remark 3 In the weighted soft metric space $(\tilde{X},d,\omega,P)$, $\omega(\tilde{r}) \tilde{\geq} \bar{0}$ for each $\tilde{r} \tilde{\in} \tilde{X}$. If the soft ordering can be defined by $\tilde{r} \tilde{\sqsubseteq} \tilde{s} \Leftrightarrow \omega(\tilde{r}) = d(\tilde{r},\tilde{s}) + \omega(\tilde{s})$ for each $\tilde{r},\tilde{s} \tilde{\in} \tilde{X}$, then $\omega(\tilde{r}) \tilde{\geq} \omega(\tilde{s})$ and $\omega(\tilde{r}) - \omega(\tilde{s}) \tilde{\geq} \bar{0}$.

The relations between soft partial metric and their induced weighted soft metric are given in the following theorems.

Theorem 9 *Suppose that (d,ω) is a weighted soft metric on $\tilde{X}$. Then, the mapping $p : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$ defined by $p(\tilde{r},\tilde{s}) = \frac{d(\tilde{r},\tilde{s})+\omega(\tilde{r})+\omega(\tilde{s})}{\bar{2}}$ is a soft partial metric on $\tilde{X}$.*

Proof Let us show that the partial metric axioms are provided.

P1. Let $p(\tilde{r},\tilde{r}) = p(\tilde{r},\tilde{s}) = p(\tilde{s},\tilde{s})$. Then, $\omega(\tilde{r}) = \omega(\tilde{r}) + d(\tilde{r},\tilde{s})$. Hence, $d(\tilde{r},\tilde{s}) = \bar{0}$ implies $\tilde{r} = \tilde{s}$. The opposite is clear.

P2. Since (d,ω) is a weighted soft metric on $\tilde{X}$, from Definition 16, $d(\tilde{r},\tilde{s}) \tilde{\geq} \omega(\tilde{r}) - \omega(\tilde{s})$ for each $\tilde{r},\tilde{s} \tilde{\in} \tilde{X}$. Hence, we have

$$p(\tilde{r},\tilde{s}) - p(\tilde{r},\tilde{r}) = \frac{d(\tilde{r},\tilde{s}) - \omega(\tilde{r}) + \omega(\tilde{s})}{\bar{2}} \tilde{\geq} \bar{0}.$$

P3. It is clear that, $p(\tilde{r},\tilde{s}) = p(\tilde{s},\tilde{r})$ for each $\tilde{r},\tilde{s} \in SE(\tilde{X})$.

P4. For $\tilde{r},\tilde{s},\tilde{t} \in SE(\tilde{X})$,

$$\begin{aligned} p(\tilde{r},\tilde{t}) &= \left(d(\tilde{r},\tilde{t}) + \omega(\tilde{r}) + \omega(\tilde{t})\right)/2 \\ &\tilde{\leq} \left(d(\tilde{r},\tilde{s}) + d(\tilde{s},\tilde{t}) + \omega(\tilde{r}) + \omega(\tilde{t})\right)/2 \\ &\tilde{\leq} \left(d(\tilde{r},\tilde{s}) + \omega(\tilde{r}) + \omega(\tilde{s})\right)/2 \\ &\quad + \left(d(\tilde{s},\tilde{t}) + \omega(\tilde{r}) + \omega(\tilde{t})\right)/2 - \omega(\tilde{s}) \\ &\tilde{\leq} p(\tilde{r},\tilde{s}) + p(\tilde{s},\tilde{t}) - p(\tilde{s},\tilde{s}). \end{aligned}$$

□

Theorem 10 *Suppose that*

$$p : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$$

is a soft partial metric on $\tilde{X}$. Then, the mapping

$$d^s : SE(\tilde{X}) \times SE(\tilde{X}) \to \mathbb{R}(P)^*$$

defined by $d^s(\tilde{r},\tilde{s}) = 2p(\tilde{r},\tilde{s}) - p(\tilde{r},\tilde{r}) - p(\tilde{s},\tilde{s})$ is a weighted soft metric with soft weight mapping

$$\omega : SE(\tilde{X}) \to \mathbb{R}(P)^*$$

defined by $\omega(\tilde{r}) = p(\tilde{r},\tilde{r})$. Furthermore, $\tau[p] \subset \tau[d^s]$. We call d^s the induced soft metric of the soft partial metric p.

Proof Let us show first that the soft metric axioms are provided. For $\tilde{r},\tilde{s},\tilde{t} \in SE(\tilde{X})$;

M1. We have

$$p(\tilde{r},\tilde{r}) + p(\tilde{s},\tilde{s}) \tilde{\leq} 2p(\tilde{r},\tilde{s})$$

since $p(\tilde{r},\tilde{r})\tilde{\leq}p(\tilde{r},\tilde{s})$ and $p(\tilde{s},\tilde{s})\tilde{\leq}p(\tilde{r},\tilde{s})$. Hence,

$$\bar{0}\tilde{\leq}2p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r})-p(\tilde{s},\tilde{s})$$

and so $\bar{0}\tilde{\leq}d^s(\tilde{r},\tilde{s})$.

M2. Let

$$d^s(\tilde{r},\tilde{s})=2p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r})-p(\tilde{s},\tilde{s})=\bar{0}.$$

Then, $2p(\tilde{r},\tilde{s})=p(\tilde{r},\tilde{r})+p(\tilde{s},\tilde{s})$. Hence, $2p(\tilde{r},\tilde{s})+p(\tilde{r},\tilde{r})\tilde{\leq}p(\tilde{r},\tilde{s})+p(\tilde{r},\tilde{r})+p(\tilde{s},\tilde{s})$ since $p(\tilde{r},\tilde{r})\tilde{\leq}p(\tilde{r},\tilde{s})$ and from this, $p(\tilde{r},\tilde{s})\tilde{\leq}p(\tilde{s},\tilde{s})$. So, we have $p(\tilde{s},\tilde{s})=p(\tilde{r},\tilde{s})$ since $p(\tilde{s},\tilde{s})\tilde{\leq}p(\tilde{r},\tilde{s})$. Similarly, we have $2p(\tilde{r},\tilde{s})+p(\tilde{s},\tilde{s})\tilde{\leq}p(\tilde{r},\tilde{s})+p(\tilde{r},\tilde{r})+p(\tilde{s},\tilde{s})$ since $p(\tilde{s},\tilde{s})\tilde{\leq}p(\tilde{r},\tilde{s})$. Thus, we acquire $p(\tilde{r},\tilde{s})\tilde{\leq}p(\tilde{r},\tilde{r})$ and so $p(\tilde{r},\tilde{r})=p(\tilde{r},\tilde{s})$ since $p(\tilde{r},\tilde{r})\tilde{\leq}p(\tilde{r},\tilde{s})$. As a result, $p(\tilde{r},\tilde{r})=p(\tilde{r},\tilde{s})=p(\tilde{s},\tilde{s})$ and so $\tilde{r}=\tilde{s}$.
Conversely, if $\tilde{r}=\tilde{s}$, then $2p(\tilde{r},\tilde{r})-p(\tilde{r},\tilde{r})-p(\tilde{r},\tilde{r})=\bar{0}$ and so $d^s(\tilde{r},\tilde{s})=\bar{0}$.

M3. $d^s(\tilde{r},\tilde{s})=2p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r})-p(\tilde{s},\tilde{s})=2p(\tilde{s},\tilde{r})-p(\tilde{s},\tilde{s})-p(\tilde{r},\tilde{r})=d^s(\tilde{s},\tilde{r})$.

M4.

$$\begin{aligned}d^s(\tilde{r},\tilde{s})&=2p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r})-p(\tilde{s},\tilde{s})\\&\tilde{\leq}2(p(\tilde{r},\tilde{t})-p(\tilde{t},\tilde{s})-p(\tilde{t},\tilde{t}))-p(\tilde{r},\tilde{r})-p(\tilde{s},\tilde{s})\\&=2p(\tilde{r},\tilde{t})-p(\tilde{r},\tilde{r})-p(\tilde{t},\tilde{t})+2p(\tilde{t},\tilde{s})-p(\tilde{t},\tilde{t})\\&-p(\tilde{s},\tilde{s})\\&=d^s(\tilde{r},\tilde{t})+d^s(\tilde{t},\tilde{s}).\end{aligned}$$

Thus, $(\tilde{X},d^s,P)$ is a soft metric space.

Let us show now that $\tau[p]\subset\tau[d^s]$. By Theorems 8 and 7, the mapping $q:SE(\tilde{X})\times SE(\tilde{X})\to\mathbb{R}(P)^*$ defined by $q(\tilde{r},\tilde{s})=p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r})$ is a weighted soft quasi-metric with soft weight mapping $\omega(\tilde{r})=p(\tilde{r},\tilde{r})$ and hence $\tau[q]=\tau[p]$. Thus, we obtain $d^s(\tilde{r},\tilde{s})=q(\tilde{r},\tilde{s})+q(\tilde{s},\tilde{r})$. Since $q(\tilde{r},\tilde{s})\tilde{\leq}d^s(\tilde{r},\tilde{s})$, we have $B_{\tilde{\epsilon}}\tilde{>}\bar{0}$. Hence,

$$SS(B^q_{\tilde{\epsilon}}(\tilde{r}))\tilde{\subseteq}SS\left(B^{d^s}_{\tilde{\epsilon}}(\tilde{r})\right).$$

Consequently, we get $SS\left(B^q_{\tilde{\epsilon}}(\tilde{r})\right)\in\tau[d^s]$ and so $\tau[q]=\tau[p]\subset\tau[d^s]$.

Finally, since $p(\tilde{r},\tilde{r})\tilde{\leq}p(\tilde{r},\tilde{s})$, we obtain $2p(\tilde{r},\tilde{r})-p(\tilde{r},\tilde{r})-p(\tilde{s},\tilde{s})\tilde{\leq}2p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r})-p(\tilde{s},\tilde{s})$. Thus, $\omega(\tilde{r})-\omega(\tilde{s})\tilde{\leq}d^s(\tilde{r},\tilde{s})$. □

Example 6 Let $(\mathbb{R}(P)^-,p,P)$ be usual soft partial metric space. Since $d^s(\tilde{r},\tilde{s})=2p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r})-p(\tilde{s},\tilde{s})=\tilde{r}+\tilde{s}-2\min\{\tilde{r},\tilde{s}\}=|\tilde{r}-\tilde{s}|$ for each $\tilde{r},\tilde{s}\in\mathbb{R}(P)^-$, we have the usual soft distance in $\mathbb{R}(P)^-$.

Theorem 11 *Assume that $(\tilde{X},p,P)$ is a soft partial metric space. For each $\tilde{r},\tilde{s}\in SE(\tilde{X})$ we define $d^w(\tilde{r},\tilde{s})=\max\{p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r}),p(\tilde{r},\tilde{s})-p(\tilde{s},\tilde{s})\}$. Then, $(\tilde{X},d^\omega,P)$ is a soft metric space.*

Proof Let us prove that the metric axioms are provided. Let $\tilde{r},\tilde{s},\tilde{t}\in SE(\tilde{X})$.

M1. Since $(\tilde{X},p,P)$ is a soft metric space, $\bar{0}\tilde{\leq}p(\tilde{r},\tilde{r})\tilde{\leq}p(\tilde{r},\tilde{s})$ and $\bar{0}\tilde{\leq}p(\tilde{s},\tilde{s})\tilde{\leq}p(\tilde{r},\tilde{s})$. Hence, $\bar{0}\tilde{\leq}d^w(\tilde{r},\tilde{s})$.

M2.

$$\begin{aligned}d^w(\tilde{r},\tilde{s})=\bar{0}&\Leftrightarrow\max\{p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r}),p(\tilde{r},\tilde{s})\\&\quad-p(\tilde{s},\tilde{s})\}=\bar{0}\\&\Leftrightarrow p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r})=\bar{0},\ \ p(\tilde{r},\tilde{s})\\&\quad-p(\tilde{s},\tilde{s})=\bar{0},\\&\quad\bar{0}\tilde{\leq}p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r}),\bar{0}\tilde{\leq}p(\tilde{r},\tilde{s})-p(\tilde{s},\tilde{s})\\&\Leftrightarrow p(\tilde{r},\tilde{s})=p(\tilde{r},\tilde{r})\text{ and }p(\tilde{r},\tilde{s})=p(\tilde{s},\tilde{s})\\&\Leftrightarrow\tilde{r}=\tilde{s}.\end{aligned}$$

M3. Since $(\tilde{X},p,P)$ is a soft metric space, $p(\tilde{r},\tilde{s})=p(\tilde{s},\tilde{r})$. Hence,

$$\begin{aligned}d^w(\tilde{r},\tilde{s})&=\max\{p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r}),p(\tilde{r},\tilde{s})-p(\tilde{s},\tilde{s})\}\\&=\max\{p(\tilde{r},\tilde{s})-p(\tilde{s},\tilde{s}),p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r})\}\\&=\max\{p(\tilde{s},\tilde{r})-p(\tilde{s},\tilde{s}),p(\tilde{s},\tilde{r})-p(\tilde{r},\tilde{r})\}\\&=d^w(\tilde{s},\tilde{r}).\end{aligned}$$

M4. Let $\max\{p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r}),p(\tilde{r},\tilde{s})-p(\tilde{s},\tilde{s})\}=p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r})$, $\max\{p(\tilde{r},\tilde{t})-p(\tilde{r},\tilde{r}),p(\tilde{r},\tilde{t})-p(\tilde{t},\tilde{t})\}=p(\tilde{r},\tilde{t})-p(\tilde{r},\tilde{r})$, $\max\{p(\tilde{t},\tilde{s})-p(\tilde{t},\tilde{t}),p(\tilde{t},\tilde{s})-p(\tilde{s},\tilde{s})\}=p(\tilde{t},\tilde{s})-p(\tilde{t},\tilde{t})$. Then,

$$\begin{aligned}d^w(\tilde{r},\tilde{s})=p(\tilde{r},\tilde{s})-p(\tilde{r},\tilde{r})&\tilde{\leq}p(\tilde{r},\tilde{t})+p(\tilde{t},\tilde{s})-p(\tilde{t},\tilde{t})\\&-p(\tilde{r},\tilde{r})\\&=p(\tilde{r},\tilde{t})-p(\tilde{r},\tilde{r})\\&+p(\tilde{t},\tilde{s})-p(\tilde{t},\tilde{t})\\&=d^w(\tilde{r},\tilde{t})+d^w(\tilde{t},\tilde{s}).\end{aligned}$$

Thus, $(\tilde{X},d^w,P)$ is a soft metric space. □

3.5 Completeness of soft partial metric spaces

Definition 17 Assume that $(\tilde{X},p,P)$ is a soft partial metric space. We call a sequence $\{\tilde{r}_n\}$ of soft elements of $\tilde{X}$ convergent to a soft element $\tilde{r}\tilde{\in}\tilde{X}$ if there is a natural number N for all $\tilde{\epsilon}\tilde{>}\bar{0}$ such that $\tilde{r}_n\in B^p_{\tilde{\epsilon}}(\tilde{r})$ for every natural number $n\geq N$ and denote by $\lim_{n\to\infty}\tilde{r}_n=\tilde{r}$ or $\tilde{r}_n\to\tilde{r}$ as $n\to\infty$.

Theorem 12 *Suppose that $\{\tilde{r}_n\}$ is a sequence of soft elements in $(\tilde{X},p,P)$ and $\tilde{r}\tilde{\in}\tilde{X}$. Then, $\lim_{n\to\infty}\tilde{r}_n=\tilde{r}$ if and only if $\lim_{n\to\infty}p(\tilde{r}_n,\tilde{r})=p(\tilde{r},\tilde{r})$.*

Proof Let $\lim_{n\to\infty}\tilde{r}_n=\tilde{r}$. Then, there is a natural number N for all $\tilde{\epsilon}\tilde{>}\bar{0}$ such that $\tilde{r}_n\in B^p_{\tilde{\epsilon}}(\tilde{r})$ for every $n\geq N$. From here we obtain $\bar{0}\tilde{\leq}p(\tilde{r}_n,\tilde{r})\tilde{<}\tilde{\epsilon}$. So we have $p(\tilde{r}_n,\tilde{r})-p(\tilde{r},\tilde{r})\tilde{<}\tilde{\epsilon}$ since $\bar{0}\tilde{\leq}p(\tilde{r}_n,\tilde{r})\tilde{<}\tilde{\epsilon}\tilde{<}\tilde{\epsilon}+p(\tilde{r},\tilde{r})$. For all $\tilde{\epsilon}\tilde{>}\bar{0}$, when $n\to\infty$, $\lim_{n\to\infty} p(\tilde{r}_n,\tilde{r})-p(\tilde{r},\tilde{r})\tilde{<}\tilde{\epsilon}$. Thus,

$$\lim_{n\to\infty} p(\tilde{r}_n,\tilde{r}) = p(\tilde{r},\tilde{r}).$$

On the contrary, let $\lim_{n\to\infty} p(\tilde{r}_n,\tilde{r}) = p(\tilde{r},\tilde{r})$. Then, there is a natural number N for all $\tilde{\epsilon}_1\tilde{>}\bar{0}$ such that $p(\tilde{r}_n,\tilde{r})\tilde{<}\tilde{\epsilon}_1 + p(\tilde{r},\tilde{r})$ for all $n\geq N$. Hence, for an $\tilde{\epsilon}\tilde{>}\bar{0}$ such that $\tilde{\epsilon}_1 = \tilde{\epsilon} - p(\tilde{r},\tilde{r},\tilde{r}_n \in B^p_{\tilde{\epsilon}}(\tilde{r})$, i.e., $\tilde{r}_n\to\tilde{r}$. ◻

Theorem 13 *Suppose that $\{\tilde{r}_n\}$ is a sequence of soft elements in $(\tilde{X},p,P)$ and $L(\tilde{r}_n)$ is the class of soft limit elements of $\{\tilde{r}_n\}$. If a soft element point $\tilde{r}\in L(\tilde{r}_n)$ and $\tilde{r}'\tilde{\sqsubseteq}\tilde{r}$, then $\tilde{r}'\in L(\tilde{r}_n)$.*

Proof From Theorem 12, $\lim_{n\to\infty}\tilde{r}_n = \tilde{r}'$ if and only if $\lim_{n\to\infty} p(\tilde{r}_n,\tilde{r}') = p(\tilde{r}',\tilde{r}')$ and from Proposition 1, $\tilde{r}'\tilde{\sqsubseteq}_p\tilde{r}$ $\Leftrightarrow$ $p(\tilde{r}',\tilde{r}') = p(\tilde{r}',\tilde{r})$. Let $p(\tilde{r}',\tilde{r}') = p(\tilde{r}',\tilde{r})$. For every $n\in\mathbb{N}$, when $n\to+\infty$, the terms in the right approaches $p(\tilde{r}',\tilde{r}')$ since $p(\tilde{r}_n,\tilde{r}')\tilde{\leq}p(\tilde{r}_n,\tilde{r})+p(\tilde{r},\tilde{r}') - p(\tilde{r},\tilde{r})$. This implies that $\lim_{n\to\infty} p(\tilde{r}_n,\tilde{r}')\tilde{\leq}p(\tilde{r}',\tilde{r}')$. Hence, we obtain $\lim_{r\to\infty} p(\tilde{r}_n,\tilde{r}')\tilde{\geq}p(\tilde{r}',\tilde{r}')$ since $p(\tilde{r}_n,\tilde{r}')\tilde{\geq}p(\tilde{r}',\tilde{r}')$ for every $n\in\mathbb{N}$. Finally, $\lim_{n\to\infty} p(\tilde{r}_n,\tilde{r}') = p(\tilde{r}',\tilde{r}')$. ◻

Example 7 For the sequence $\{\tilde{r}_n\} = \left\{-\frac{\bar{1}}{n}\right\}$ in the soft partial metric space $(\mathbb{R}(P)^-,p,P)$, $L(\tilde{r}_n) = \mathbb{R}(P)^-$. Indeed, if $\tilde{r}\tilde{<}\bar{0}$,

$$p\left(-\frac{\bar{1}}{n},\tilde{r}\right) = -\min\left\{-\frac{\bar{1}}{n},\tilde{r}\right\}.$$

If $\tilde{\epsilon}\tilde{>}\bar{0}$ with $\tilde{r}\in B^p_{\tilde{\epsilon}}(\tilde{r})$, then there is natural number N on condition that $-\frac{\bar{1}}{n}\tilde{>}\tilde{r}$ for all $n\geq N$. Thus, $-\min\left\{-\frac{\bar{1}}{n},\tilde{r}\right\} = \tilde{r}$ and $p\left(-\frac{\bar{1}}{n},\tilde{r}\right) = \tilde{r} = p(\tilde{r},\tilde{r})\tilde{<}\tilde{\epsilon}$. As a result $-\frac{\bar{1}}{n}\in B^p_{\tilde{\epsilon}}(\tilde{r})$. Hence, $\{-\frac{\bar{1}}{n}\}$ convergent to $\tilde{r}$.

Definition 18 Assume that $(\tilde{X},p,P)$ is a soft partial metric space and $\{\tilde{r}_n\}$ is a sequence of soft elements. We call a soft element $\tilde{r}\tilde{\in}\tilde{X}$ a proper soft limit of $\{\tilde{r}_n\}$ if $\tilde{r}_n\to\tilde{r}$ in $(\tilde{X},d^s,P)$ and write $\tilde{r}_n\to\tilde{r}$ (properly). We say that a sequence soft elements is properly convergent if it has a proper soft limit element.

Proposition 3 *Assume that $(\tilde{X},p,P)$ is a soft partial metric space, $\{\tilde{r}_n\}$ is a sequence of soft elements and $\tilde{r}\tilde{\in}\tilde{X}$. Then, $\tilde{r}_n\to\tilde{r}$ (properly) if and only if*

$$\lim_{n\to\infty} p(\tilde{r}_n,\tilde{r}) = \lim_{n\to\infty} p(\tilde{r}_n,\tilde{r}_n) = \lim_{n\to\infty} p(\tilde{r},\tilde{r}).$$

Proof For any sequence $\{\tilde{r}_n\}$ in $(\tilde{X},p,P)$ and soft element $\tilde{r}\tilde{\in}\tilde{X}$, we get that

$$\begin{aligned}
&\tilde{r}_n\to\tilde{r}\ (properly) \Leftrightarrow \lim_{n\to\infty} d^s(\tilde{r}_n,\tilde{r}) = \bar{0}\\
&\Leftrightarrow \lim_{n\to\infty}(2p(\tilde{r}_n,\tilde{r}) - p(\tilde{r}_n,\tilde{r}_n) - p(\tilde{r},\tilde{r})) = \bar{0}\\
&\Leftrightarrow \left(\lim_{n\to\infty} p(\tilde{r}_n,\tilde{r}) - \lim_{n\to\infty} p(\tilde{r}_n,\tilde{r}_n)\right)\\
&+\lim_{n\to\infty}(p(\tilde{r}_n,\tilde{r}) - p(\tilde{r},\tilde{r})) = \bar{0}\\
&\Leftrightarrow \lim_{n\to\infty} p(\tilde{r}_n,\tilde{r}) = \lim_{n\to\infty} p(\tilde{r}_n,\tilde{r}_n) = \lim_{n\to\infty} p(\tilde{r},\tilde{r}).
\end{aligned}$$

◻

Definition 19 Assume that $(\tilde{X},p,P)$ is a soft partial metric space. We call a sequence $\{\tilde{r}_n\}$ of soft elements in $\tilde{X}$ a Cauchy sequence if there is $\lim_{n,m\to\infty} p(\tilde{r}_n,\tilde{r}_m)$ and call $(\tilde{X},p,P)$ complete if every Cauchy sequence of soft elements in $\tilde{X}$ convergent to a soft element in $\tilde{X}$.

Theorem 14 *Suppose that $(\tilde{X},p,P)$ is a soft partial metric space and let $\{\tilde{r}_n\}$ is a sequence of soft elements in $\tilde{X}$. Then, $\{\tilde{r}_n\}$ is a Cauchy sequence in $(\tilde{X},p,P)$ if and only if $\{\tilde{r}_n\}$ is a Cauchy sequence in $(\tilde{X},d^s,P)$.*

Proof Assume that $\{\tilde{r}_n\}$ is a Cauchy sequence of soft elements in $(\tilde{X},p,P)$, i.e., there is $\lim_{n,m\to\infty} p(\tilde{r}_n,\tilde{r}_m)$. Let $d^s(\tilde{r}_n,\tilde{r}_m)\tilde{\geq}\tilde{\epsilon}$, $\exists\tilde{\epsilon}\tilde{\geq}\bar{0}$. Then,

$$2p(\tilde{r}_n,\tilde{r}_m) - p(\tilde{r}_n,\tilde{r}_n) - p(\tilde{r}_m,\tilde{r}_m)\tilde{\geq}\tilde{\epsilon}$$

and

$$p(\tilde{r}_n,\tilde{r}_m)\tilde{\geq}\frac{\tilde{\epsilon}+p(\tilde{r}_n,\tilde{r}_n)+p(\tilde{r}_m,\tilde{r}_m)}{2}.$$

For $\frac{\tilde{\epsilon}}{2} = \tilde{\epsilon}_1$, $p(\tilde{r}_n,\tilde{r}_m)\tilde{\geq}\tilde{\epsilon}_1 + \frac{p(\tilde{r}_n,\tilde{r}_n)}{2} + \frac{p(\tilde{r}_m,\tilde{r}_m)}{2}$. Thus, we obtain

$$\lim_{n,m\to\infty} p(\tilde{r}_n,\tilde{r}_m)\tilde{\geq}\tilde{\epsilon}_1 + \lim_{n\to\infty}\frac{p(\tilde{r}_n,\tilde{r}_n)}{2} + \lim_{m\to\infty}\frac{p(\tilde{r}_m,\tilde{r}_m)}{2}.$$

Since $\tilde{\epsilon}$ is arbitrary, $\lim_{n,m\to\infty} p(\tilde{r}_n,\tilde{r}_m)\to\infty$ which is a contradiction. Therefore, for all $\tilde{\epsilon}\tilde{>}\bar{0}$, there is $N\in\mathbb{N}$ on condition that $p(\tilde{r}_n,\tilde{r}_m)\tilde{\leq}\tilde{\epsilon}$ for $n,m\in\mathbb{N}$. Namely, $\{\tilde{r}_n\}$ is a Cauchy sequence in $(\tilde{X},d^s,P)$.

Conversely, $\{\tilde{r}_n\}$ is a Cauchy sequence of soft elements in $(\tilde{X},d^s,P)$. Then, for all $\tilde{\epsilon}\tilde{>}\bar{0}$, there is $N\in\mathbb{N}$ such that $p(\tilde{r}_n,\tilde{r}_m)\tilde{\leq}\tilde{\epsilon}$ for $n,m\geq N$, hence

$$2p(\tilde{r}_n,\tilde{r}_m) - p(\tilde{r}_n,\tilde{r}_n) - p(\tilde{r}_m,\tilde{r}_m)\tilde{\leq}\tilde{\epsilon}.$$

For $\frac{\tilde{\epsilon}}{2} = \tilde{\epsilon}_1$, we have $p(\tilde{r}_n, \tilde{r}_m)\tilde{\leq}\tilde{\epsilon}_1 + \frac{p(\tilde{r}_n, \tilde{r}_n)}{2} + \frac{p(\tilde{r}_m, \tilde{r}_m)}{2}$. Therefore,

$$\lim_{n,m\to\infty} p(\tilde{r}_n, \tilde{r}_m)\tilde{\leq}\tilde{\epsilon}_1 + \lim_{n\to\infty} \frac{p(\tilde{r}_n, \tilde{r}_n)}{2} + \lim_{m\to\infty} \frac{p(\tilde{r}_m, \tilde{r}_m)}{2}.$$

This shows that $\lim_{n,m\to\infty} p(\tilde{r}_n, \tilde{r}_m)$ exists. Thus, $\{\tilde{r}_n\}$ is a Cauchy sequence in $(\tilde{X}, p, P)$. □

Example 8 The usual soft partial metric $(\mathbb{R}(P)^-, p, P)$ is complete. In fact, It can be easily seen that the induced usual soft metric space $(\mathbb{R}(P)^-, d^s, P)$ such that for all $\tilde{r}, \tilde{s} \in \mathbb{R}(P)^-$, $d^s(\tilde{r}, \tilde{s}) = |\tilde{r} - \tilde{s}|$ is complete.

Theorem 15 *A soft partial metric space $(\tilde{X}, p, P)$ is complete if and only if the induced soft metric space $(\tilde{X}, d^s, P)$ is complete.*

Proof We assume that $(\tilde{X}, p, P)$ is complete. Then, each the sequence $\{\tilde{r}_n\}$ of soft elements in $(\tilde{X}, p, P)$ converges an element $\tilde{r} \in SE(\tilde{X})$. From Proposition 3,

$$\lim_{n\to\infty} p(\tilde{r}_n, \tilde{r}) = p(\tilde{r}, \tilde{r}).$$

Since $\{\tilde{r}_n\}$ is a Cauchy sequence in $(\tilde{X}, p, P)$, it is also a Cauchy sequence in $(\tilde{X}, d^s, P)$ by Theorem 14. That is, for each $\tilde{\epsilon}\tilde{>}\bar{0}$, there is $N \in \mathbb{N}$ with $\lim_{n\to\infty} d^s(\tilde{r}_n, \tilde{r}_m)\tilde{<}\tilde{\epsilon}$ when $n, m \geq N$. Hence,

$$\lim_{n,m\to\infty} (2p(\tilde{r}_n, \tilde{r}_m) - p(\tilde{r}_n, \tilde{r}_n) - p(\tilde{r}_m, \tilde{r}_m)) \tilde{\leq}\tilde{\epsilon}.$$

Since $\{\tilde{r}_n\}$ is convergent to $\tilde{r}$ in $(\tilde{X}, p, P)$,

$$\lim_{n,m\to\infty} (2p(\tilde{r}_n, \tilde{r}) - p(\tilde{r}_n, \tilde{r}_n) - p(\tilde{r}, \tilde{r})) \tilde{\leq}\tilde{\epsilon}.$$

That is, $\lim_{n\to\infty} d^s(\tilde{r}_n, \tilde{r})\tilde{<}\tilde{\epsilon}$. Since this inequality is true for every $\tilde{\epsilon}\tilde{>}\bar{0}$, $\lim_{n\to\infty} d^s(\tilde{r}_n, \tilde{r}) = \bar{0}$ and so $\{\tilde{r}_n\}$ is convergent in $(\tilde{X}, d^s, P)$. Thus, $(\tilde{X}, d^s, P)$ is complete.

Conversely, let $(\tilde{X}, d^s, P)$ be complete. Then, for every the Cauchy sequence $\{\tilde{r}_n\}$ of soft elements in $(\tilde{X}, d^s, P)$, $\lim_{n\to\infty} d^s(\tilde{r}_n, \tilde{r}) = \bar{0}$. Hence,

$$\lim_{n\to\infty} (2p(\tilde{r}_n, \tilde{r}) - p(\tilde{r}_n, \tilde{r}_n) - p(\tilde{r}, \tilde{r})) = \bar{0}$$

or

$$\lim_{n\to\infty} 2p(\tilde{r}_n, \tilde{r}) = \lim_{n\to\infty} p(\tilde{r}_n, \tilde{r}_n) + p(\tilde{r}, \tilde{r}).$$

Since $p(\tilde{r}_n, \tilde{r}_n)\tilde{\leq}p(\tilde{r}_n, \tilde{r})$, $\lim_{n\to\infty} p(\tilde{r}_n, \tilde{r}_n)\tilde{\leq} \lim_{n\to\infty} p(\tilde{r}_n, \tilde{r})$. If we write

$$\lim_{n\to\infty} p(\tilde{r}_n, \tilde{r}) + \lim_{n\to\infty} p(\tilde{r}_n, \tilde{r}_n)\tilde{\leq} \lim_{n\to\infty} p(\tilde{r}_n, \tilde{r}_n) + p(\tilde{r}, \tilde{r}),$$

we get $\lim_{n\to\infty} p(\tilde{r}_n, \tilde{r}) \leq p(\tilde{r}, \tilde{r})$. Since $p(\tilde{r}, \tilde{r}) \leq p(\tilde{r}_n, \tilde{r})$, $p(\tilde{r}, \tilde{r}) \leq \lim_{n\to\infty} p(\tilde{r}_n, \tilde{r})$. Thus, we obtain $\lim_{n\to\infty} p(\tilde{r}_n, \tilde{r}) = p(\tilde{r}, \tilde{r})$. Hence, $\{\tilde{r}_n\}$ converges to an element $\tilde{r}$. From Theorem 14, $\{\tilde{r}_n\}$ is a Cauchy sequence in $(\tilde{X}, p, P)$. □

4 Fixed point theory

In the section, Banach fixed point theorem in the complete soft metric space is extended to the complete soft partial metric space with the help of a contraction mapping and a monotone non-decreasing mapping.

Definition 20 Assume that $(\tilde{X}, p, P)$ is a soft partial metric space and $L : SE(\tilde{X}) \to SE(\tilde{X})$ is a soft mapping.

(1). We call a soft element $\tilde{r}_0 \in SE(\tilde{X})$ a fixed soft element of L if $L(\tilde{r}_0) = \tilde{r}_0$ and $p(\tilde{r}_0, \tilde{r}_0) = \bar{0}$.

(2).We call the soft mapping L a contraction if there exists a positive soft real number $\tilde{\mu}$ with $\bar{0}\tilde{\leq}\tilde{\mu}\tilde{<}\bar{1}$ to ensure the inequality $p(L(\tilde{r}), L(\tilde{s}))\tilde{\leq}\tilde{\mu}p(\tilde{r}, \tilde{s})$ for each $\tilde{r}, \tilde{s} \in SE(\tilde{X})$ (We say $\tilde{\mu}$ a soft contraction constant).

Theorem 16 *Suppose that $(\tilde{X}, p, P)$ is a complete soft partial metric space and $L : SE(\tilde{X}) \to SE(\tilde{X})$ is a contraction. Then, there is a unique soft element $\tilde{r}_0 \in SE(\tilde{X})$ with $L(\tilde{r}_0) = \tilde{r}_0$ and $p(\tilde{r}_0, \tilde{r}_0) = \bar{0}$.*

Proof Assume that $\tilde{\mu}$ is the contraction constant of L and $\tilde{v} \in SE(\tilde{X})$. Since

$$p\left(L^{n+m}(\tilde{v}), L^{n+m}(\tilde{v})\right) \tilde{\geq}\bar{0}$$

for each $n, m \in \mathbb{N}$, we get

$$\begin{aligned} p(L^{n+m+1}(\tilde{v}), L^n(\tilde{v}))&\tilde{\leq}p(L^{n+m+1}(\tilde{v}), L^{n+m}(\tilde{v})) \\ &+ p(L^{n+m}(\tilde{v}), L^n(\tilde{v})) - p(L^{n+m}(\tilde{v}), L^{n+m}(\tilde{v})) \\ &\tilde{\leq} \tilde{\mu}^{n+m} p(L^n(\tilde{v}), \tilde{v}) + p(L^{n+m}(\tilde{v}), L^n(\tilde{v})) \\ &\tilde{\leq} (\tilde{\mu}^{n+m} + ... + \tilde{\mu}^n) p(L(\tilde{v}), \tilde{v}) + p(L^n(\tilde{v}), L^n(\tilde{v})) \\ &\tilde{\leq} \tilde{\mu}^n \left(1 + \tilde{\mu} + ... + \tilde{\mu}^m\right) p(L(\tilde{v}), \tilde{v}) + \tilde{\mu}^n p(\tilde{v}, \tilde{v}) \\ &\tilde{\leq} \tilde{\mu}^n \left[\left(\frac{1 - \tilde{\mu}^{m+1}}{1 - \tilde{\mu}}\right) p(L(\tilde{v}), \tilde{v}) + p(\tilde{v}, \tilde{v})\right] \\ &\tilde{\leq}\tilde{\mu}^n \left[\frac{p(L(\tilde{v}), \tilde{v})}{1 - \tilde{\mu}} + p(\tilde{v}, \tilde{v})\right]. \end{aligned}$$

Thus, $\{L^n(\tilde{v})\}$ is a Cauchy sequence in $(\tilde{X}, p, P)$ with $\lim_{n,m\to\infty} p(L^n(\tilde{v}), L^m(\tilde{v})) = \bar{0}$. So, there is $\tilde{r}_0 \in SE(\tilde{X})$ such that $\{L^n(\tilde{v})\}$ converges to $\tilde{r}_0$ and

$$\begin{aligned} p(\tilde{r}_0, \tilde{r}_0) &= \lim_{n\to\infty} p(L^n(\tilde{v}), \tilde{r}_0) \\ &= \lim_{n\to\infty} p(L^n(\tilde{v}), L^n(\tilde{v})) = \bar{0} \end{aligned}$$

since $(\tilde{X}, p, P)$ is complete.

Now, for all $n \in \mathbb{N}$,

$$\begin{aligned} p(L(\tilde{r}_0), \tilde{r}_0) &\tilde{\leq} p(L(\tilde{r}_0), L^{n+1}(\tilde{v})) + p(L^{n+1}(\tilde{v}), \tilde{r}_0) \\ &\quad - p(L^{n+1}(\tilde{v}), L^{n+1}(\tilde{v})) \\ &\tilde{\leq} \tilde{\tilde{\mu}} p(\tilde{r}_0, L^n(\tilde{v})) + p(L^{n+1}(\tilde{v}), \tilde{r}_0). \end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} p(\tilde{r}_0, L^n(\tilde{v})) = \lim_{n\to\infty} p(L^{n+1}(\tilde{v}), \tilde{r}_0) = \bar{0},$$

when $n \to \infty$ and then $p(L(\tilde{r}_0), \tilde{r}_0) = \bar{0}$. Yet we obtain $p(L(\tilde{r}_0), L(\tilde{r}_0)) = p(L(\tilde{r}_0), \tilde{r}_0) = p(\tilde{r}_0, \tilde{r}_0) = \bar{0}$ since $p(L(\tilde{r}_0), L(\tilde{r}_0)) \tilde{\leq} p(L(\tilde{r}_0), \tilde{r}_0)$. This give $L(\tilde{r}_0) = \tilde{r}_0$. If $\tilde{s}_0 \in SE(\tilde{X})$ such that $L(\tilde{s}_0) = \tilde{s}_0$, then $p(\tilde{r}_0, \tilde{s}_0) = p(L(\tilde{r}_0), L(\tilde{s}_0)) \tilde{\leq} \tilde{\mu} p(\tilde{r}_0, \tilde{s}_0)$. So we obtain $p(\tilde{r}_0, \tilde{s}_0) = \bar{0} = p(\tilde{r}_0, \tilde{s}_0) = p(\tilde{s}_0, \tilde{s}_0)$ Since $\tilde{\mu} \tilde{<} \bar{1}$. Hence, $\tilde{r}_0 = \tilde{s}_0$. Thus, the fixed soft element of T is unique. □

Theorem 17 *Suppose that $(\tilde{X}, p, P)$ is a complete soft partial metric space and for every fixed soft real $\tilde{r} \tilde{>} \bar{0}$,*

$$\psi : \mathbb{R}(P)^* \to \mathbb{R}(P)^*$$

is a monotone non-decreasing soft mapping with

$$\lim_{n\to\infty} \psi^n(\tilde{r}) = \bar{0}.$$

Then, the soft mapping $L : SE(\tilde{X}) \to SE(\tilde{X})$ defined by $p(L(\tilde{r}), L(\tilde{s})) \tilde{\leq} \psi(p(\tilde{r}, \tilde{s}))$ for each $\tilde{r}, \tilde{s} \in SE(\tilde{X})$ has a unique fixed soft element $\tilde{r}_0 \in SE(\tilde{X})$ such that $L(\tilde{r}_0) = \tilde{r}_0$ and $p(\tilde{r}_0, \tilde{r}_0) = \bar{0}$.

Proof Since $\tilde{r} \tilde{>} \bar{0}$, $\psi(\tilde{r}) \tilde{<} \tilde{r}$ if $\tilde{r} \tilde{\leq} \psi(\tilde{r})$, $\psi(\tilde{r}) \tilde{\leq} \psi(\psi(\tilde{r}))$ so $\tilde{r} \tilde{\leq} \psi^2(\tilde{r})$. From induction, for $n \geq 1$, we have $\tilde{r} \tilde{\leq} \psi^n(\tilde{r})$. From here we get $\tilde{r} \tilde{\leq} \lim_{n\to\infty} \psi^n(\tilde{r}) = \bar{0}$, which is a contradiction. If $\tilde{r} \in SE(\tilde{X})$, $p(L^n(\tilde{r}), L^n(\tilde{r})) \tilde{\leq} \psi^n(p(\tilde{r}, \tilde{r}))$ and $p(L^n(\tilde{r}), L^{n+1}(\tilde{r})) \tilde{\leq} \psi^n(p(\tilde{r}, L(\tilde{r}))$ for every $n \in \mathbb{N}$. With $\tau[p] = \tau[q]$, $q(\tilde{r}, \tilde{s}) = p(\tilde{r}, \tilde{s}) - p(\tilde{r}, \tilde{r})$ is a soft quasi-metric on $\tilde{X}$. Thus,

$$\begin{aligned} q(L^n(\tilde{r}), L^{n+1}(\tilde{r})) &\tilde{\leq} p(L^n(\tilde{r}), L^{n+1}(\tilde{r})) + p(L^n(\tilde{r}), L^n(\tilde{r})) \\ &\tilde{\leq} \psi^n(p(\tilde{r}, L(\tilde{r}))) + \psi^n(p(\tilde{r}, \tilde{r})) \\ &\tilde{\leq} 2\psi^n(p(\tilde{r}, \tilde{r})). \end{aligned}$$

Now, for $n, m \in \mathbb{N}$,

$$\begin{aligned} &p(L^n(\tilde{r}), L^{n+m+1}(\tilde{r})) \tilde{\leq} p(L^{n+m+1}(\tilde{r}), L^{n+1}(\tilde{r})) \\ &\quad + p(L^{n+m}(\tilde{r}), L^n(\tilde{r})) - p(L^{n+m}(\tilde{r}), L^{n+m}(\tilde{r})) \\ &\quad \tilde{\leq} \psi^{n+m}(p(\tilde{r}, L(\tilde{r}))) + p(L^{n+m}(\tilde{r}), L^n(\tilde{r})) \\ &\quad \tilde{\leq} (\psi^{n+m} + \ldots + \psi^n)(p(\tilde{r}, L(\tilde{r}))) + p(L^n(\tilde{r}), L^n(\tilde{r})). \end{aligned}$$

Hence, we have

$$p(L^n(\tilde{r}), L^{n+m+1}(\tilde{r})) \tilde{\leq} (\psi^{n+m} + \ldots + \psi^n)(p(\tilde{r}, L(\tilde{r})))$$

since the term in the right approaches to $\bar{0}$ when $n, m \to \infty$, $\{L^n(\tilde{r})\}$ is a Cauchy sequence of soft elements in $(\tilde{X}, p, P)$ with $\lim_{n,m\to\infty} p(L^n(\tilde{r}), L^m(\tilde{r})) = \bar{0}$. From the completeness of $(\tilde{X}, p, P)$, there is an $\tilde{r}_0 \in SE(\tilde{X})$ satisfying

$$\begin{aligned} p(\tilde{r}_0, \tilde{r}_0) &= \lim_{n\to\infty} p(L^n(\tilde{r}), \tilde{r}_0) \\ &= \lim_{n\to\infty} p(L^n(\tilde{r}), L^m(\tilde{r})) = \bar{0}. \end{aligned}$$

Furthermore, for every $n \in \mathbb{N}$,

$$\begin{aligned} p(L(\tilde{r}_0), \tilde{r}_0) &\tilde{\leq} p(L(\tilde{r}_0), L^{n+1}(\tilde{r})) + p(L^{n+1}(\tilde{r}), \tilde{r}_0) \\ &\quad - p(L^{n+m}(\tilde{r}), L^{n+m}(\tilde{r})) \\ &\tilde{\leq} \psi(p(\tilde{r}_0, L^n(\tilde{r}))) + p(L^{n+1}(\tilde{r}), \tilde{r}_0) \\ &\tilde{\leq} p(\tilde{r}_0, L^n(\tilde{r}_0)) + p(L^{n+1}(\tilde{r}), \tilde{r}_0). \end{aligned}$$

Taking the limit, we have that

$$\lim_{n\to\infty} p(\tilde{r}_0, L^n(\tilde{r}_0)) = \lim_{n\to\infty} p(L^{n+1}(\tilde{r}), \tilde{r}_0) = \bar{0},$$

hence $p(L^n(\tilde{r}_0), \tilde{r}_0) = \bar{0}$. So we obtain

$$p(L^n(\tilde{r}_0), L^n(\tilde{r}_0)) = p(L^n(\tilde{r}_0), \tilde{r}_0) = p(\tilde{r}_0, \tilde{r}_0) = \bar{0}$$

since $p(L^n(\tilde{r}_0), L^n(\tilde{r}_0)) \tilde{\leq} p(L^n(\tilde{r}_0), \tilde{r}_0)$. Hence, $L^n(\tilde{r}_0) = \tilde{r}_0$.

If $\tilde{s}_0 \in SE(\tilde{X})$ such that $L(\tilde{s}_0) = \tilde{s}_0$, then

$$p(\tilde{r}_0, \tilde{s}_0) = p(L(\tilde{r}_0), L(\tilde{s}_0)) \tilde{\leq} \psi(p(\tilde{r}_0, \tilde{s}_0)).$$

If $p(\tilde{r}_0, \tilde{s}_0) \neq \bar{0}$, then $\psi(p(\tilde{r}_0, \tilde{s}_0)) \tilde{<} p(\tilde{r}_0, \tilde{s}_0)$ which the relation would not hold. Thus, $p(\tilde{r}_0, \tilde{s}_0) = \bar{0}$ and so $p(\tilde{r}_0, \tilde{r}_0) = p(\tilde{r}_0, \tilde{s}_0) = p(\tilde{s}_0, \tilde{s}_0) = \bar{0}$ which implies that $\tilde{r}_0 = \tilde{s}_0$. Then, the fixed soft element of L is unique. □

5 Conclusion

In this paper, we introduced the concept of soft partial metric spaces via soft elements. A soft partial metric is suppler than a soft metric. A soft metric creates partial orderings and its topological properties are more general than that of a soft metric, defending the fact that the self-distance of any soft element does not have to be zero. It is very helpful in partially defined information in the study of semantics and domains in computer science.

Before, mathematical notations of the soft partial metrics that are equivalent to the weightable soft quasi-metrics were developed within the meaning soft partial metrics could have been explained as the weightable soft quasi-metrics, and the weighted soft quasi-metrics could have been interpreted as the soft partial metrics.

Later, Banach fixed point theorem was extended to the complete soft partial metric spaces with the help of a soft monotone non-decreasing mapping or a soft contractive mapping to prove the existence and uniqueness of fixed soft elements.

There exists ample scope for further exploratory in soft partial metric spaces. This paper is a basis to works on the above-mentioned ideas.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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