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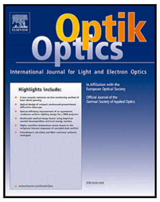

Original research article

Electromagnetic curves and Berry phase construction of a polarized light wave along an optical fiber which is a singular curve on S^2

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ABSTRACT

In this study, we investigate the cases of a linearly polarized light wave along an optical fiber which is accepted as a singular curve on S^2 and the rotation of the polarization planes. Also, we construct a Berry-phase model of polarized light wave along an optical fiber with singular points. In addition, we introduce two different Rytov curves according to the polarization vector **E** being perpendicular to the defined planes along an optical fiber with singular points. Finally, we define the Lorentz force equations for all cases and we examine the electromagnetic trajectories created by the electric field of the light wave traveling in the singular optical fiber.

1. Introduction

Space curves are an important structures because of their application areas in physics and other sciences. Nuclear physics, optics, geometric phase, electromagnetism are examples of these. Berry phase or geometric phase is a quantum phase effect defined by Berry [1]. Important results have been obtained with the movements of quantum systems in space with the Berry phase. These results provide important information about the fundamental relationships between geometry and physics. Classical examples are the motion of the charged particle due to the magnetic vector field and the propagation of a polarized light wave along an optical fiber. Ross, Kugler and Shtrikman consider the optical fiber as a space curve and they study the geometry of the polarization vector along an optical fiber [2,3]. Later, the generalization of this approach was given by Dandoloff and Zakrzewski [4]. In addition, in many studies, Berry phase models create with the help of various frame areas along an optical fiber in the relevant spaces. Geometric phase models along an optical fiber according to Bishop frame in 3D Riemann manifolds in [5], geometric phase model and electromagnetic curves along an optical fiber for an alternative moving frame $\{n, c, w\}$ in [6], Berry phase model according to quasi-adapted frame in [7], the electromagnetic curves of a polarized light wave in a 3D semi-Riemann manifold in [8], and geometric phase of the Heisenberg ferromagnetic version with directional geometric flows of semi-binormal magnetic particles in [9], are given. In [10], the authors give a new characterization for the spherical electric and magnetic phase with Heisenberg spherical ferromagnetic spin with fractional solutions. In addition, comprehensive studies such as Heisenberg magnetic flux and Heisenberg antiferromagnetic flux in Minkowski space are given in [11–17]. In [18], the behavior of a polarized light wave along an optical fiber, which is a spherical curve in de-Sitter 2 space S_1^2 are given. In [19], the authors give transformation equations for electromagnetic fields of polarized light in de-Sitter 2 space. On the other hand, basic definitions and explanations for Rytov's law, which is defined by the rotation of the polarization plane along an optical fiber, are given in [20–22].

Lorentz equations, Lorentz force, magnetic trajectories and magnetic curves are studied thanks to magnetic fields which are accepted as closed 2-forms in Riemann manifold or semi-Riemann manifolds. Fundamental studies for magnetic curves and magnetic

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trajectories in 3D Riemannian manifold and 3D semi-Riemannian manifold are given in [23–28]. When all geometric phase models and electromagnetic curves in the relevant spaces are examined, it is seen that the curve considered as optical fiber is a regular curve. Therefore, geometric phase models and electromagnetic trajectories were created along an optical fiber, which is a regular curve. This is because a non-regular curve does not have a moving frame. However, recently, thanks to the Legendre and spherical Legendre curves, which are defined in 2D and 3D spaces, respectively, a moving frame can be defined even if the curve has singular points [29–35]. Curvatures give from existence and uniqueness conditions as in regular curves, and these curvatures are important since they characterize singular points. The base curve is called front or frontal depending on whether there is Legendre immersion or not. Existence and uniqueness conditions of Legendre curves in [29], evolutes of fronts in Euclidean plane in [30], involutes and evolutes of frontals in [31], pedal curves of frontals in Euclidean plane in [32], and dualities and evolutes of fronts in hyperbolic and de-Sitter space in [33] are given. Roughly, singular curves lying in a unit sphere consider as spherical Legendre curves. The frame construct, evolutes and envelopes for spherical Legendre curves are available in [34–36]. Framed curves are both a generalization of Legendre curves in unit tangent bundles and a generalization of regular curves under linearly independent conditions [37]. Some case studies for framed curves in Euclidean space are available in [37–40].

In this study, we create a Berry phase model according to the cases of the polarization vector **E** of a polarized light wave along an optical fiber, which is considered as a singular curve on S^2. In addition, Rytov curves $\mathbf{E}_\mu$ and $\mathbf{E}_\nu$ are introduced according to Fermi Walker's law. In addition, we give Lorentz force equations, magnetic field equations and electromagnetic curves along an optical fiber, which is a singular curve on S^2. We give some physical interpretations of electromagnetic trajectories according to the behavior of the curvatures of the singular curve on S^2. Examples and figures are given that support the theory of the polarization vector **E** and the associated Rytov curves formed along an singular optical fiber on S^2.

2. Spherical Legendre curves as a singular curve on S^2

Let us assume that $\gamma : I \to S^2$ is a curve with singular points. A three-dimensional manifold is denoted by

$$\Delta = \{(x, y) \in S^2 \times S^2 | \langle x, y\rangle = 0\}, \tag{1}$$

where the unit spherical bundle $T_1 S^2 = S^2 \times S^2$ over S^2. Then, the definition of a spherical Legendre curve is given as follows:

Definition 2.1. $(\gamma, \mu) : I \to \Delta \subset S^2 \times S^2$ is called a spherical Legendre curve if $\langle \gamma'(s), \mu(s)\rangle = 0$ for every $s \in I$. γ is called a frontal and μ a dual of γ. If (γ, μ) is a Legendre immersion, γ is called a front [35].

The $\nu(s) \in S^2$ is denoted by $\nu(s) = \gamma(s)\times\mu(s)$. $\{\gamma(s), \mu(s), \nu(s)\}$ is a moving frame along the frontal or front $\gamma(s)$. The Frenet–Serret type formula as given by:

$$\begin{pmatrix} \gamma'(s) \\ \mu'(s) \\ \nu'(s) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_1(s) \\ 0 & 0 & m_2(s) \\ -m_1(s) & -m_2(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(s) \\ \mu(s) \\ \nu(s) \end{pmatrix}.$$

The functions (m_1, m_2) is said that curvature of the Legendre curve $(\gamma, \mu) : I \to \Delta \subset S^2 \times S^2$. Also, s_0 is a singular point of γ (respectively, μ) if and only if $m_1(s_0) = 0$ (respectively, $m_2(s_0) = 0$). Then, if $(m_1(s_0), m_2(s_0)) \neq (0, 0)$ for every $s_0 \in I$, (γ, μ) is a Legendre immersion. If γ is a regular spherical curve, then the relationship between the geodesic curvature k_g of γ and the curvature (m_1, m_2) of (γ, μ) is defined by $k_g(s) = \frac{m_2(s)}{|m_1(s)|}$ [35].

3. Berry phase construction of polarized light wave along an optical fiber which is a singular curve on S^2

As is known, an optical fiber can be thought of as a space curve in the 3D Riemann manifold. However, when the literature is examined, it is seen that these space curves are considered as regular curves. This is due to the inability to install a moving frame for singular curves. In this section, a singular optical fiber is examined with the help of spherical Legendre curves, which is a singular curve at S^2. The frontal γ of the spherical Legendre curve (γ, μ) is considered as an optical fiber. The direction of polarized light is defined by the direction of the electric field **E**. Thus, the direction of **E** together with the optical fiber can be written as a linear combination of the $\{\gamma, \mu, \nu\}$ frame fields defined for the spherical Legendre curves. Then, the direction of the electric field **E** can be denoted as follows

$$\frac{d\mathbf{E}}{ds} = \lambda_1\gamma(s) + \lambda_2\mu(s) + \lambda_3\nu(s), \tag{2}$$

where $\lambda_1, \lambda_2, \lambda_3$ are smooth functions.

Now, we investigate the direction of polarized light in two different categories: the polarization vector **E** lies in the plane perpendicular to μ, and the polarization vector **E** lies in the plane perpendicular to ν.

3.1. Berry phase construction of polarized light wave is in the plane where $\mathbf{E} \perp \mu$

Let us assume that

$$\langle \mathbf{E}, \mu(s) \rangle = 0. \tag{3}$$

By differentiating Eq. (3), we get

$$\langle \frac{d\mathbf{E}}{ds}, \mu(s) \rangle = -m_2(s)\langle \mathbf{E}, \nu(s) \rangle. \tag{4}$$

If the Eqs. (2) and (4) are combined, we have

$$\lambda_2 = -m_2(s)\langle \mathbf{E}, \nu(s) \rangle.$$

Let us assume that there is no loss of mechanism due to absorption in the optical fiber. So let $\langle \mathbf{E}, \mathbf{E} \rangle = c$, where c is a constant. Therefore, we get

$$\langle \frac{d\mathbf{E}}{ds}, \mathbf{E} \rangle = 0,$$

and since $\mathbf{E} \perp \mu$, we have

$$\lambda_1 \langle \mathbf{E}, \gamma(s) \rangle + \lambda_3 \langle \mathbf{E}, \nu(s) \rangle = 0.$$

If $\langle \mathbf{E}, \gamma(s) \rangle \neq 0$ and $\langle \mathbf{E}, \nu(s) \rangle \neq 0$, we can write

$$\lambda_1 = \lambda \langle \mathbf{E}, \nu(s) \rangle, \lambda_3 = -\lambda \langle \mathbf{E}, \gamma(s) \rangle.$$

As a result, we get

$$\frac{d\mathbf{E}}{ds} = \lambda \langle \mathbf{E}, \nu(s) \rangle \gamma(s) - m_2(s) \langle \mathbf{E}, \nu(s) \rangle \mu(s) - \lambda \langle \mathbf{E}, \gamma(s) \rangle \nu(s). \tag{5}$$

On the other hand, since $\nu(s) \wedge \gamma(s) = \mu(s)$ for moving frame $\{\gamma, \mu, \nu\}$ of spherical Legendre curves, we get

$$\frac{d\mathbf{E}}{ds} = -m_2(s) \langle \mathbf{E}, \nu(s) \rangle \mu(s) + \lambda (\mathbf{E} \wedge \mu(s)). \tag{6}$$

The expression $\mathbf{E} \wedge \mu(s)$ represents the rotation about the unit vector μ. Let us assume that μ is parallel transported. So $\lambda = 0$ and we have

$$\frac{d\mathbf{E}}{ds} = -m_2(s) \langle \mathbf{E}, \nu(s) \rangle \mu(s).$$

Also, we can write

$$\mathbf{E} = \langle \mathbf{E}, \gamma(s) \rangle \gamma(s) + \langle \mathbf{E}, \nu(s) \rangle \nu(s). \tag{7}$$

By differentiating Eq. (7), we get

$$\begin{pmatrix} \langle \mathbf{E}, \gamma(s) \rangle' \\ \langle \mathbf{E}, \nu(s) \rangle' \end{pmatrix} = \begin{pmatrix} 0 & m_1(s) \\ -m_1(s) & 0 \end{pmatrix} \begin{pmatrix} \langle \mathbf{E}, \gamma(s) \rangle \\ \langle \mathbf{E}, \nu(s) \rangle \end{pmatrix}.$$

In addition, since $\langle \mathbf{E}, \mathbf{E} \rangle = c$ where c is a constant, and for convenience, with the help of spherical coordinates, $\mathbf{E}$ can be given as follows:

$$\mathbf{E} = \sin\psi \, \nu(s) + \cos\psi \, \gamma(s). \tag{8}$$

By differentiating Eq. (8), we have

$$\frac{d\mathbf{E}}{ds} = -m_2(s) \sin\psi \, \mu + \left(\frac{d\psi}{ds} + m_1(s) \right) \cos\psi \, \nu(s) - \left(\frac{d\psi}{ds} + m_1(s) \right) \sin\psi \, \gamma(s).$$

Then, by using Eq. (8) we get

$$\frac{d\mathbf{E}}{ds} = -m_2(s) \langle \mathbf{E}, \nu(s) \rangle + \left(\frac{d\psi}{ds} + m_1(s) \right) \mathbf{E} \wedge \mu(s).$$

Therefore, $\frac{d\psi}{ds} = -m_1(s)$ in optical fiber. Finally, from the Eq. (8), the direction of polarization vector in the optical fiber is given as

$$\mathbf{E} = -\sin\left(\int m_1(s) ds \right) \nu(s) + \cos\left(\int m_1(s) ds \right) \gamma(s).$$

Therefore, $\mathbf{E}$ moves the parallel transport along the unit vector μ, and we can express this motion along the unit vector μ with the Fermi–Walker transportation law as follows:

$$\frac{d\mathbf{E}}{ds}_{\mathbf{FW}} = \frac{d\mathbf{E}}{ds} \pm \langle \mathbf{E}, \mu(s) \rangle \frac{d\mu}{ds} + \langle \mathbf{E}, \frac{d\mu}{ds} \rangle \mu(s).$$

Since, $\langle \mathbf{E}, \mu(s)\rangle = 0$, we get

$$\frac{d\mathbf{E}}{ds}_{\mathbf{FW}} = \frac{d\mathbf{E}}{ds} + m_2(s)\langle \mathbf{E}, \nu(s)\rangle \mu(s).$$

Therefore, we have

$$\frac{d\mathbf{E}}{ds} = m_2(s)\langle \mathbf{E}, \nu(s)\rangle \mu(s).$$

From here, we have the following result:

Corollary 3.1. *If* $\mathbf{E}$ *is Fermi–Walker parallel along an optical fiber, then the optical fiber is an* $\mathbf{E}_\mu$*-Rytov curve with the* $\langle \mathbf{E}, \mu(s)\rangle = 0$ *condition. Then, the* $\mathbf{E}_\mu$*-Rytov curve is given as parametric form*

$$\mathbf{E}_\mu = \gamma - \sin\left(\int m_1(s)ds\right)\nu(s) + \cos\left(\int m_1(s)ds\right)\gamma(s).$$

3.2. Berry phase construction of polarized light wave in the plane where $\mathbf{E} \perp \nu$

Consider that

$$\langle \mathbf{E}, \nu(s)\rangle = 0. \tag{9}$$

By differentiating Eq. (9), we have

$$\langle \frac{d\mathbf{E}}{ds}, \nu(s)\rangle = m_1(s)\langle \mathbf{E}, \gamma(s)\rangle + m_2(s)\langle \mathbf{E}, \mu(s)\rangle. \tag{10}$$

Considering the Eqs. (2) and (10), we get

$$\lambda_3 = m_1(s)\langle \mathbf{E}, \gamma(s)\rangle + m_2(s)\langle \mathbf{E}, \mu(s)\rangle.$$

Assume that $\langle \mathbf{E}, \mathbf{E}\rangle = c$, where c is a constant. Therefore, we get

$$\langle \frac{d\mathbf{E}}{ds}, \mathbf{E}\rangle = 0,$$

and since $\mathbf{E} \perp \nu$, we have

$$\lambda_1\langle \mathbf{E}, \gamma(s)\rangle + \lambda_2\langle \mathbf{E}, \mu(s)\rangle = 0.$$

If we consider $\langle \mathbf{E}, \gamma(s)\rangle \neq 0$ and $\langle \mathbf{E}, \mu(s)\rangle \neq 0$, we can write

$$\lambda_1 = \lambda\langle \mathbf{E}, \mu(s)\rangle, \lambda_2 = -\lambda\langle \mathbf{E}, \gamma(s)\rangle.$$

Consequently, the general expression of $\frac{d\mathbf{E}}{ds}$ is given by

$$\frac{d\mathbf{E}}{ds} = \lambda\langle \mathbf{E}, \mu(s)\rangle\gamma(s) - \lambda\langle \mathbf{E}, \gamma(s)\rangle\mu(s) + (m_1(s)\langle \mathbf{E}, \gamma(s)\rangle + m_2(s)\langle \mathbf{E}, \mu(s)\rangle)\nu(s). \tag{11}$$

Also, since $\nu(s) = \gamma(s) \wedge \mu(s)$ for moving frame $\{\gamma, \mu, \nu\}$ of spherical Legendre curves, we get

$$\frac{d\mathbf{E}}{ds} = (m_1(s)\langle \mathbf{E}, \gamma(s)\rangle + m_2(s)\langle \mathbf{E}, \mu(s)\rangle)\nu(s) + \lambda(\mathbf{E} \wedge \nu(s)).$$

Suppose that $\boldsymbol{\nu}$ is parallel transported. So $\lambda = 0$ and we have

$$\frac{d\mathbf{E}}{ds} = (m_1(s)\langle \mathbf{E}, \gamma(s)\rangle + m_2(s)\langle \mathbf{E}, \mu(s)\rangle)\nu(s).$$

Then, we can write

$$\mathbf{E} = \langle \mathbf{E}, \gamma(s)\rangle\gamma(s) + \langle \mathbf{E}, \mu(s)\rangle\mu(s) \tag{12}$$

According to Eq. (12), we get

$$\begin{pmatrix} \langle \mathbf{E}, \gamma(s)\rangle' \\ \langle \mathbf{E}, \mu(s)\rangle' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \langle \mathbf{E}, \gamma(s)\rangle \\ \langle \mathbf{E}, \mu(s)\rangle \end{pmatrix}.$$

In addition, since $\langle \mathbf{E}, \mathbf{E}\rangle = c$ where c is a constant, similar to Case 1, $\mathbf{E}$ can be write as follows:

$$\mathbf{E} = \sin\psi\gamma(s) + \cos\psi\mu(s). \tag{13}$$

By differentiating Eq. (13), we get

$$\frac{d\mathbf{E}}{ds} = (m_1(s)\langle \mathbf{E}, \gamma(s)\rangle + m_2(s)\langle \mathbf{E}, \mu(s)\rangle)\nu(s) + \frac{d\psi}{ds}\mathbf{E} \wedge \nu(s).$$

Then, we have $\frac{d\psi}{ds} = 0$ (i.e $\psi = constant$) for optical fiber. According to Eq. (13), the polarization vector in the optical fiber is denoted as

$$\mathbf{E} = \sin\psi\gamma(s) + \cos\psi\mu(s),$$

where ψ is a constant. From, Fermi–Walker transportation law we can write,

$$\frac{d\mathbf{E}}{ds}_{\mathbf{FW}} = \frac{d\mathbf{E}}{ds} \pm \langle \mathbf{E}, \nu(s)\rangle \frac{d\nu}{ds} + \langle \mathbf{E}, \frac{d\nu}{ds}\rangle \nu(s).$$

Since, $\langle \mathbf{E}, \nu(s)\rangle = 0$, we have

$$\frac{d\mathbf{E}}{ds}_{\mathbf{FW}} = \frac{d\mathbf{E}}{ds} - (m_1(s)\langle \mathbf{E}, \gamma(s)\rangle + m_2(s)\langle \mathbf{E}, \mu(s)\rangle)\nu(s).$$

and thus we have

$$\frac{d\mathbf{E}}{ds} = (m_1(s)\langle \mathbf{E}, \gamma(s)\rangle + m_2(s)\langle \mathbf{E}, \mu(s)\rangle)\nu(s).$$

We can give the following result:

Corollary 3.2. *If* $\mathbf{E}$ *is Fermi–Walker parallel along an optical fiber, then the optical fiber is an* $\mathbf{E}_\nu$*-Rytov curve with the* $\langle \mathbf{E}, \nu(s)\rangle = 0$ *condition. Then, the* $\mathbf{E}_\nu$*-Rytov curve is given as parametric form*

$$\mathbf{E}_\nu = \gamma + \sin\psi\,\gamma(s) + \cos\psi\,\mu(s),$$

where ψ *is a constant.*

4. Electromagnetic curves and trajectories along the polarization plane of a light wave along an optical fiber which is a singular curve on S^2

Physically, a polarized particle enters the electromagnetic field with $\langle \mathbf{E}, \mu\rangle = 0$ and $\langle \mathbf{E}, \nu\rangle = 0$, and the Lorentz force is formed, which satisfies the equation $\Omega(\mathbf{E}) = V \times \mathbf{E} = \frac{d\mathbf{E}}{ds}$ where V is a Killing magnetic vector field [5]. The Lorentz force contributes to the motion of the particle and the trajectory it follows along a singular optical fiber is called the electromagnetic trajectory. These trajectories are $\mathbf{E}_\mu M$-trajectories and $\mathbf{E}_\nu M$-trajectories.

4.1. *Electromagnetic curves of a light wave in a optical fiber which is a singular curve on* S^2 *for* $\mathbf{E} \perp \mu$

Let us assume that $\langle \mathbf{E}, \mu\rangle = 0$. Then, according to Eq. (5), we have

$$\frac{d\mathbf{E}}{ds} = \lambda\langle \mathbf{E}, \nu(s)\rangle\gamma(s) - m_2(s)\langle \mathbf{E}, \nu(s)\rangle\mu(s) - \lambda\langle \mathbf{E}, \gamma(s)\rangle\nu(s). \tag{14}$$

Then, we get

$$\begin{aligned}\langle \Omega(\mathbf{E}), \gamma(s)\rangle &= -\langle \Omega(\gamma(s)), \mathbf{E}\rangle,\\ \langle \Omega(\mathbf{E}), \mu(s)\rangle &= -\langle \Omega(\mu(s)), \mathbf{E}\rangle,\\ \langle \Omega(\mathbf{E}), \nu(s)\rangle &= -\langle \Omega(\nu(s)), \mathbf{E}\rangle.\end{aligned} \tag{15}$$

From Eqs. (14) and (15),

$$\begin{pmatrix} \Omega(\gamma(s)) \\ \Omega(\mu(s)) \\ \Omega(\nu(s)) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\lambda \\ 0 & 0 & m_2(s) \\ \lambda & -m_2(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(s) \\ \mu(s) \\ \nu(s) \end{pmatrix}.$$

The magnetic vector field V is written by

$$V = \lambda_1\gamma(s) + \lambda_2\mu(s) + \lambda_3\nu(s), \tag{16}$$

where

$$\begin{aligned}\Omega(\gamma) &= V \wedge \gamma,\\ \Omega(\mu) &= V \wedge \mu,\\ \Omega(\nu) &= V \wedge \nu.\end{aligned} \tag{17}$$

Consequently, it seen that from Eqs. (16) and (17), the magnetic vector field of the $\mathbf{E}_\mu M-$trajectories are given by

$$V = -m_2(s)\gamma(s) + \lambda\mu(s).$$

If we suppose that $\mathbf{E} \perp \mu$, the matrix form of Lorentz force equations according the $\{\gamma, \mu, \nu\}$ can be obtained by

$$\begin{pmatrix} \Omega(\gamma(s)) \\ \Omega(\mu(s)) \\ \Omega(\nu(s)) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m_2(s) \\ 0 & -m_2(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(s) \\ \mu(s) \\ \nu(s) \end{pmatrix},$$

and $\mathbf{E}_\mu M-$trajectories are magnetic trajectories of the V magnetic field if and only if

$$V = -l(s)\gamma(s).$$

From all this, it is said that, V is the Killing magnetic vector field of the $\mathbf{E}_\mu M-$trajectories of Ω.

4.2. *Electromagnetic curves of a light wave in a optical fiber which is a singular curve on S^2 for $\mathbf{E} \perp \nu$*

Let us assume that $\langle \mathbf{E}, \nu \rangle = 0$. Then, according to Eq. (11), we have

$$\frac{d\mathbf{E}}{ds} = \lambda\langle \mathbf{E}, \mu(s)\rangle\gamma(s) - \lambda\langle \mathbf{E}, \gamma(s)\rangle\mu(s) + (m_1(s)\langle \mathbf{E}, \gamma(s)\rangle + m_2(s)\langle \mathbf{E}, \mu(s)\rangle)\nu(s). \tag{18}$$

Therefore, we find

$$\begin{pmatrix} \Omega(\gamma(s)) \\ \Omega(\mu(s)) \\ \Omega(\nu(s)) \end{pmatrix} = \begin{pmatrix} 0 & -\lambda & m_1(s) \\ \lambda & 0 & m_2(s) \\ -m_1(s) & -m_2(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(s) \\ \mu(s) \\ \nu(s) \end{pmatrix}.$$

By using Eqs. (16), (17) and (18), the magnetic vector field V is given by

$$V = m_2(s)\gamma(s) - m_1(s)\mu(s) - \lambda\nu(s).$$

If we suppose that $\mathbf{E} \perp \nu$, the matrix form of Lorentz force equations according the $\{\gamma, \mu, \nu\}$ can be expressed by

$$\begin{pmatrix} \Omega(\gamma(s)) \\ \Omega(\mu(s)) \\ \Omega(\nu(s)) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_1(s) \\ 0 & 0 & m_2(s) \\ -m_1(s) & -m_2(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma(s) \\ \mu(s) \\ \nu(s) \end{pmatrix},$$

and $\mathbf{E}_\nu M-$trajectories are magnetic trajectories of the V magnetic field if and only if

$$V = m_2(s)\gamma(s) - m_1(s)\mu(s).$$

Therefore, V is the Killing magnetic vector field of the $\mathbf{E}_\nu M-$trajectories of Ω.

5. Physical results of electromagnetic trajectories along a singular optical fiber on S^2

Let the frontal γ on S^2 which is considered an optical fiber have at least one non-singular point. Under this approach, we examine the states of electric field $\mathbf{E}$ and magnetic field V under three cases. As the first case, we can say that

1. If the electric field $\mathbf{E}$ is parallel to the magnetic field V, the Lorentz force on the particle is zero.

For the second case, suppose the electric field $\mathbf{E}$ is perpendicular to the magnetic field V. Therefore, we have two characterizations. Since the optical fiber is a spherical curve and $\{\gamma, \mu, \nu\}$ is a spherical frame, electromagnetic trajectories are obtained as a circle or a great circle on S^2 as follows [38]:

2. Let us assume that $\langle \mathbf{E}, V \rangle = 0$. Then, we have

i. If $\mathbf{E} \perp \mu$, we get $\langle \mathbf{E}, V \rangle = -m_2 \cos\psi = 0$. If $m_2(s) = 0$ and $m_1(s) \neq 0$, $\mathbf{E}_\mu M$-trajectories are a great circle on S^2.

ii. If $\mathbf{E} \perp \nu$, we get $\langle \mathbf{E}, V \rangle = m_2 \sin\psi - m_1 \cos\psi = 0$ where $\psi =$ constant and therefore we have $\frac{m_2}{m_1} =$ constant. Consequently, the $\mathbf{E}_\nu M$-trajectories are a circle.

Finally, let us assume that the electric field $\mathbf{E}$ makes a constant angle with the magnetic field V, so that it is not perpendicular. Therefore, we again come across two characterizations. Then, investigations of electromagnetic trajectories are obtained as follows:

3. Let us assume that $\langle \mathbf{E}, V \rangle =$ constant $\neq 0$. Then, we have

i. If $\mathbf{E} \perp \mu$, the $\mathbf{E}_\mu M$-trajectories are the curves whose curvature satisfies the equation $m_2 \cos\left(\int m_1(s)ds\right) =$ constant where $\int m_1(s)ds \neq \frac{\pi}{2} + k\pi$.

ii. If $\mathbf{E} \perp \nu$, the $\mathbf{E}_\nu M$-trajectories are the curves whose curvature satisfies the equation $m_2 \sin\psi - m_1 \cos\psi = c$ constant where $\psi =$ constant and $c \neq 0$.

Example 5.1. Consider a linear light wave coupled to a singular optical fiber on S^2 where the moving frame $\{\gamma, \mu, \nu\}$. Let the equation of this optical fiber, known as the spherical nephroid curve be

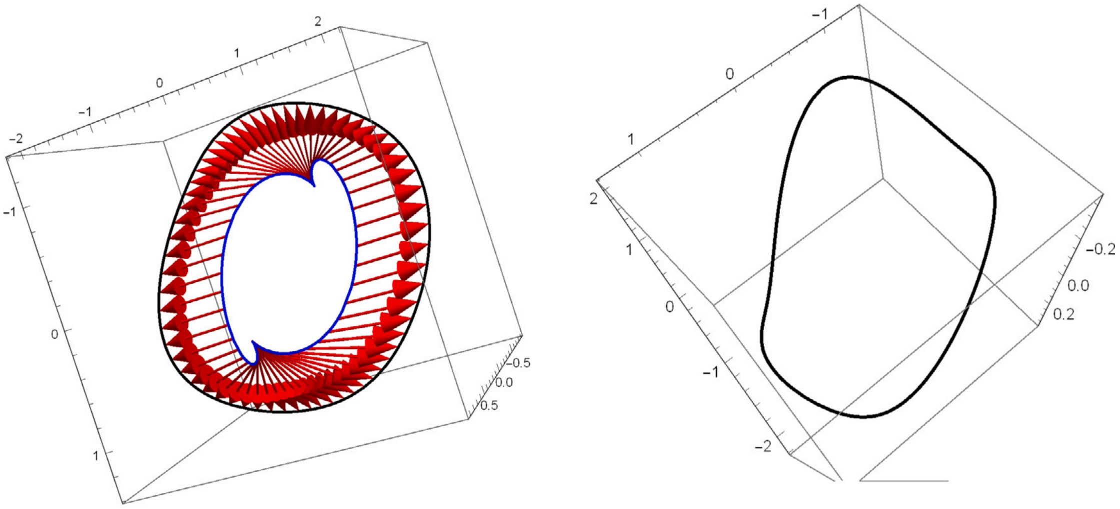
$$\gamma(s) = \left(\frac{3}{4}\cos s - \frac{1}{4}\cos 3s, \frac{3}{4}\sin s - \frac{1}{4}\sin 3s, \frac{\sqrt{3}}{2}\cos s\right). \tag{19}$$

Also, let us consider

$$\mu(s) = \left(\frac{3}{4}\sin s + \frac{1}{4}\sin 3s, -\frac{3}{4}\cos s - \frac{1}{4}\cos 3s, -\frac{\sqrt{3}}{2}\sin s\right). \tag{20}$$

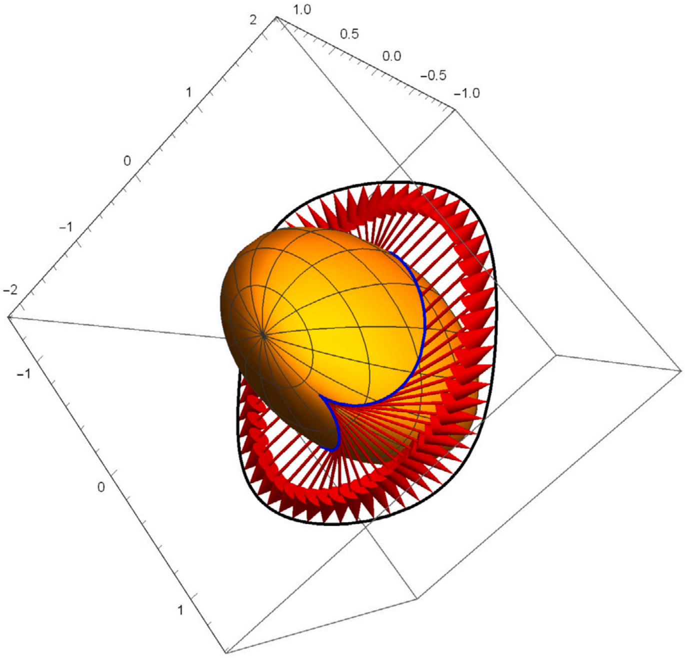
Therefore, $(\gamma, \mu) : [0, 2\pi) : \Delta \subset S^2 \times S^2$ is a spherical Legendre curve where $\langle \gamma(s), \mu(s) \rangle = 0$ and $\langle \gamma'(s), \mu(s) \rangle = 0$. Then, we have

$$\nu(s) = \left(\frac{\sqrt{3}}{2}\cos 2s, \frac{\sqrt{3}}{2}\sin 2s, -\frac{1}{2}\right).$$



(a) Polarization vectors along an optical fiber with a front when $\mathbf{E} \perp \mu$

(b) $\mathbf{E}_{\mu}$-Rytov curve



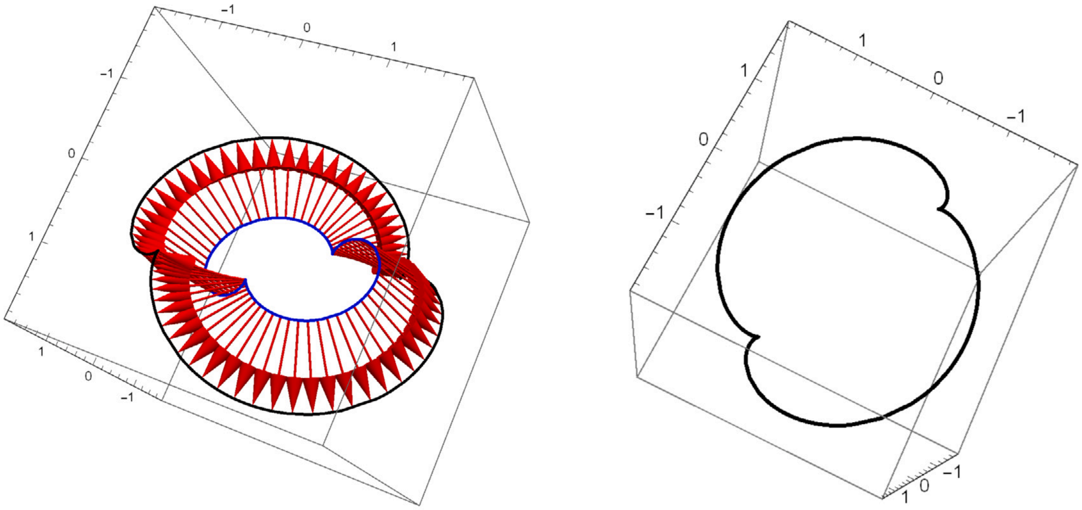
(c) Polarization vectors and Rytov curve along an optical fiber with singular points on the sphere

Fig. 1. The corresponding polarization vectors and Rytov curve along an optical fiber with singular points for the case $\mathbf{E} \perp \mu$.

It is clear from here that (γ, μ) is a Legendre immersion where

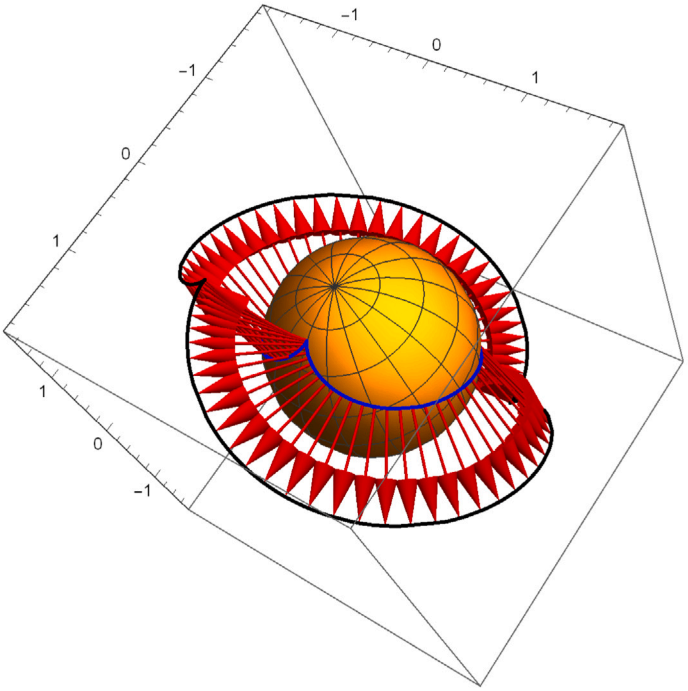
$$m_1(s) = \sqrt{3}\sin s, m_2(s) = \sqrt{3}\cos s.$$

Therefore, the front γ is considered an optical fiber. The two cases of the Berry phase structure of the polarization plane of the light wave moving in the front γ optical fiber and the Rytov curves obtained under the relevant conditions are given in Figs. 1 and 2:



(a) Polarization vectors along an optical fiber with a front when $\mathbf{E} \perp \nu$

(b) $\mathbf{E}_\nu$-Rytov curve



(c) Polarization vectors and Rytov curve along an optical fiber with singular points on the sphere

Fig. 2. The corresponding polarization vectors and Rytov curve along an optical fiber with singular points for the case $\mathbf{E} \perp \nu$.

6. Conclusion

Berry-phase construction of a polarized light wave along an optical fiber, which is considered a space curve, and electromagnetic curves have been a wide field of study. Significant relationships have been obtained between the two disciplines, both physically and geometrically. When the literature is examined, we have seen that space curves are accepted as regular curves. In this study, we created our study along an optical fiber with singular points. With the help of moving frames of fronts or frontals, we constructed a geometric phase and studied electromagnetic curves. We think that this aspect of the study will contribute to the readers. We created our work along a singular optical fiber lying on the sphere. Also, a space curve with general singular points can be thought of as a framed curve, and electromagnetic curves can be studied through framed frame and framed curvature [37]. Also, a moving frame is defined for Legendre curves and framed curves in Minkowski space [41,42] Therefore, a large study can be created in these spaces as well. Also, in the future, inspired by [18], we intend to study the electromagnetic force and Maxwell's equations on the point particle along an optical fiber with a singular curve. Thus, we hope to have contributed to the work done in these fields.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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