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Evolution of quaternionic curve in the semi-Euclidean space $\mathbb{E}_2^4$

Alperen Kızılay[1,2] | **Önder Gökmen Yıldız**[2] | **Osman Zeki Okuyucu**[2]

[1]Department of Mathematics, Faculty of Arts and Sciences, Niğde Ömer Halisdemir University, Niğde,Turkey
[2]Department of Mathematics, Faculty of Arts and Sciences, Bilecik Şeyh Edebali University, Bilecik, Turkey

Correspondence
Osman Zeki Okuyucu, Department of Mathematics, Faculty of Arts and Sciences, Bilecik Şeyh Edebali University, Adapazarı Bilecik Yolu Bilecik Şeyh Edebali Üniversitesi Rektörlüğü, Bilecik 11230, Turkey.
Email: osman.okuyucu@bilecik.edu.tr

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In this paper, kinematics of semi-real quaternionic curve in semi-Euclidean space $\mathbb{E}_2^4$ is obtained in terms of its curvature functions. The evolution equation of Frenet frame and curvatures of quaternionic curve are obtained. Also, examples of evolution equations of curvatures are given.

KEYWORDS
evolution, inextensible flow, semi-real quaternion, semi-Euclidean space

MSC CLASSIFICATION
53C44; 51B20; 53C50

1 | INTRODUCTION

There are a lot of studies in differential geometry where time related to curved space and shapes is neglected. Recently, geometricians have made great progress in understanding the curved spaces that evolve in time. For example, engineering, computer vision,[1,2] computer animation,[3] and even structural mechanics[4] have many applications of evolving curves. Also, we can see many physical applications in previous studies.[5-7]

A family of curves parameterized by time can be thought as an evolving curve. We refer to curve evolutions as flows throughout this paper for the time evolution of a curve generated by its corresponding flow. We see many studies about flow of curves in literature. First of all, inextensible flows of curves and developable surfaces in Euclidean three-space were studied by Kwon and Park,[8] and then in many different spaces, inextensible flows of curves are also studied (see previous studies[9-12]). Also, inextensible flows of curves studied for quaternionic curves. Some of them have studied inextensibility of flows of quaternionic curves and semi-real quaternionic curves , for example, Körpınar et al.[13] and by Yıldız et al.,[14] respectively.

Our aim is to study the evolution of semi-real quaternionic curve in $\mathbb{E}_2^4$. In this paper, we introduce some necessary and sufficient conditions for inextensible flows of semi-real quaternionic curves in $\mathbb{E}_2^4$. Then, the evolution equation of the Frenet frame is expressed by matrix equation. Also, integrability condition (zero curvature condition) for the considered model is obtained.

2 | PRELIMINARIES

In this section, we introduce a brief summary of the semi-real quaternion in $\mathbb{E}_2^4$.

A semi-real quaternion q is expressed as follows:

$$q = d + ae_1 + be_2 + ce_3,$$

such that

$$\begin{aligned} e_i \times e_i &= -\varepsilon_{e_i}, \qquad (1 \leq i \leq 3), \\ e_i \times e_j &= \varepsilon_{e_i}\varepsilon_{e_j} e_k, \quad \text{in} \quad \mathbb{E}_1^3, \\ e_i \times e_j &= -\varepsilon_{e_i}\varepsilon_{e_j} e_k, \quad \text{in} \quad \mathbb{E}_2^4, \end{aligned}$$

where (ijk) is an even permutation of (123) and $a, b, c, d \in \mathbb{R}$.[15] Further, any quaternion can be written as $q = S_q + V_q$, where $S_q = d$ is the scalar part and $V_q = ae_1 + be_2 + ce_3$ is the vector part of q. Let p and q be two semi-real quaternions; multiplication of p and q, with the help of $<,>$ scalar, and $\wedge$ cross products in $\mathbb{E}_1^3$, is defined as

$$p \times q = S_p S_q + < V_p, V_q > + S_p V_q + S_q V_p + V_p \wedge V_q, \quad \text{for every} \quad p, q \in \mathbb{E}_2^4.$$

The conjugate of q is $\gamma q = S_q - V_q$. For every $p, q \in \mathbb{E}_2^4$, the symmetric, nondegenerate, bilinear form h is defined as follows:

$$h : \mathbb{E}_2^4 \times \quad \mathbb{E}_2^4 \to \mathbb{R},$$

$$h(p, q) = \frac{1}{2}\left[\varepsilon_p \varepsilon_{\gamma q} (p \times \gamma q) + \varepsilon_q \varepsilon_{\gamma p} (q \times \gamma p)\right].$$

The norm of q is given by

$$\|q\|^2 = -a^2 - b^2 + c^2 + d^2.$$

Also, we say q is a semi-real unit quaternion if $\|q\|^2 = 1$. If $q \times \gamma q = 0$, then q is called a semi-real spatial quaternion. Let $p, q \in \mathbb{E}_2^4$, then p and q are h orthogonal if and only if $h(p, q) = 0$.[15]

Now, we give the definition of semi-real quaternionic curve and Frenet formulas. $\mathbb{E}_2^4$ is identified with the space of unit semi-quaternions and is denoted by $\mathbb{Q}_v$. Let

$$\beta : I \subset \mathbb{R} \to \mathbb{Q}_v, \quad \beta(s) = \sum_{i=1}^{4} \gamma_i(s) e_i, \quad e_4 = 1,$$

be a smooth curve β with nonzero curvatures $\{\kappa, k, (r - \varepsilon_t \varepsilon_T \varepsilon_N K)\}$ defined over the interval $I = [0, 1]$. Let the parameter s be chosen such that the tangent $T = \beta'(s) = \sum_{i=1}^{4} \gamma_i'(s) e_i$ has unit magnitude and Frenet frame of β is $\{T, N_1, N_2, N_3\}$. The Frenet formulas are

$$\begin{aligned} T'(s) &= \varepsilon_N \kappa N_1(s), \\ N_1'(s) &= -\varepsilon_t \varepsilon_N \kappa T(s) + \varepsilon_n k N_2(s), \\ N_2'(s) &= -\varepsilon_t k N_1(s) + \varepsilon_n (r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa) N_3(s), \\ N_3'(s) &= -\varepsilon_b (r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa) N_2, \end{aligned} \tag{1}$$

where $\kappa = \varepsilon_N \|T'\|$, $N_1 = \varepsilon_T (t \times T)$, $N_2 = \varepsilon_T (n \times T)$, $N_3 = \varepsilon_T (b \times T)$, $h(T, T) = \varepsilon_T$, $h(N_1, N_1) = \varepsilon_N$, $h(N_2, N_2) = \varepsilon_n \varepsilon_T$, and $h(N_3, N_3) = \varepsilon_b \varepsilon_T$. [15]

Frenet formulas of β are obtained by using Frenet formulas of the curve γ in $\mathbb{E}_1^3$. Also, we have relations, which are only determined for quaternions, between curvatures of curves β and γ. Frenet apparatus of γ is $\{t, n, b, k, r\}$. That is, torsion of β is the principal curvature of the curve γ and the bitorsion of β is $(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa)$.[15] Moreover, these formulas can be stated as the following matrix:

$$V_s = QV, \tag{2}$$

where

$$V=[T,N_1,N_2,N_3]^t,\quad Q=\begin{bmatrix}0 & \varepsilon_N\kappa & 0 & 0\\ -\varepsilon_t\varepsilon_N\kappa & 0 & \varepsilon_n k & 0\\ 0 & -\varepsilon_t k & 0 & \varepsilon_n(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)\\ 0 & 0 & -\varepsilon_b(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa) & 0\end{bmatrix}. \tag{3}$$

For further quaternions concepts, see previous studies.[16-18]

Unless otherwise stated, we assume that

$$\beta:\ \left[0,l\right]\times[0,w)\to\mathbb{Q}_v$$

is a one parameter family of smooth semi-real quaternionic curves in $\mathbb{Q}_v$, where l is the arc length of the initial curve. u is the curve parameterization variable, $0\le u\le l$, and t is time parameter. If the metric on the semi-real quaternionic curve β is given by $v(u,t)=h\left(\frac{\partial\beta}{\partial u},\frac{\partial\beta}{\partial u}\right)$, then the arc length variation of $\beta(u,t)$ is given by

$$s(u,t)=\int_0^u\left\|\frac{\partial\beta}{\partial u}\right\|du=\int_0^u\sqrt{v(u,t)}du.$$

The operator $\frac{\partial}{\partial s}$ is given by

$$\frac{\partial}{\partial s}=\frac{1}{\sqrt{v}}\frac{\partial}{\partial u}, \tag{4}$$

that is, the arc length parameter is $ds=\sqrt{v}du.$

3 | EVOLUTION OF SEMI-REAL QUATERNIONIC CURVES BY FLOW IN QV

Definition 1. Let β be a differentiable semi-real quaternionic curve with the Frenet frame $\{T,N_1,N_2,N_3\}$in $\mathbb{Q}_v$. Any flow of the semi-real quaternionic curve can be obtained as follows:

$$\frac{\partial\beta}{\partial t}=f_1T+f_2N_1+f_3N_2+f_4N_3, \tag{5}$$

where, f_1,f_2,f_3, and f_4 are scalar speed functions of flow.[14]

In $\mathbb{Q}_v$, the condition that a semi-real quaternionic curve is not subjected to any elongation or compression can be given as follows:

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u\frac{\partial\sqrt{v}}{\partial t}du=0,u\in\left[0,l\right], \tag{6}$$

where $u\in\left[0,l\right]$.

Definition 2. Let β be a semi-real quaternionic curve in $\mathbb{Q}_v$. A semi-real quaternionic curve evolution $\beta(u,t)$ and its flow $\frac{\partial\beta}{\partial t}$ are said to be inextensible if

$$\frac{\partial}{\partial t}\left\|\frac{\partial\beta}{\partial u}\right\|=0.$$

We need the following lemma[14] before explaining the necessary and sufficient condition for inelastic semi-real quaternionic curve flow.

Lemma 1. *Let β be the semi-real quaternionic curve with the Frenet frame $\{T,N_1,N_2,N_3\}$ and $\frac{\partial\beta}{\partial t}=f_1T+f_2N_1+f_3N_2+f_4N_3$ be a smooth flow of β in $\mathbb{Q}_v$. So, the evolution equation of v is as follows:*[14]

$$\frac{\partial v}{\partial t}=v_t=2v\varepsilon_T\left(\frac{\partial f_1}{\partial s}-\varepsilon_t\varepsilon_N f_2\kappa\right). \tag{7}$$

Proof. We know that, $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ are commutative. By using (4), we have

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t} h\left(\frac{\partial \beta}{\partial u}, \frac{\partial \beta}{\partial u}\right) \\ &= 2h\left(\frac{\partial \beta}{\partial u}, \frac{\partial}{\partial u}\left(f_1 T + f_2 N_1 + f_3 N_2 + f_4 N_3\right)\right) \\ &= 2vh\left(T, \lambda T + \sum_{i=1}^{3} A_i N_i\right),\end{aligned}$$

where

$$\begin{aligned}\lambda &= \left(\frac{\partial f_1}{\partial s} - \varepsilon_t \varepsilon_N f_2 \kappa\right), \\ A_1 &= \varepsilon_N f_1 \kappa + \frac{\partial f_2}{\partial s} - \varepsilon_t f_3 k, \\ A_2 &= \varepsilon_n f_2 k + \frac{\partial f_3}{\partial s} - \varepsilon_b f_4 (r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa), \\ A_3 &= \varepsilon_n f_3 (r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa) + \frac{\partial f_4}{\partial s}.\end{aligned} \tag{8}$$

Then,

$$\frac{\partial v}{\partial t} = 2v\lambda.$$

□

Theorem 1. *The flow of the semi-real quaternionic curve is inextensible if and only if*

$$\frac{\partial f_1}{\partial s} = \varepsilon_t \varepsilon_N f_2 \kappa. \tag{9}$$

Proof. Assume that the semi-real quaternionic curve flow is inextensible. From Equations (6) and (7), it follows that

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{v_t}{2\sqrt{v}} du = 0, u \in \left[0, l\right].$$

This clearly forces

$$\frac{\partial f_1}{\partial s} - \varepsilon_n \varepsilon_T f_2 \kappa = 0 \Rightarrow \frac{\partial f_1}{\partial s} = \varepsilon_n \varepsilon_T f_2 \kappa.$$

On the contrary, assume that $\frac{\partial f_1}{\partial s} = \varepsilon_n \varepsilon_T f_2 \kappa$. By applying $\frac{\partial f_1}{\partial s}$ into (7), we get $v_t = 0$, then $s_t = 0$. This means that the flow is inextensible. □

Theorem 2. *Let the flow of $\beta(u,t)$ be an inextensible. Then, the following statements hold:*

(i) The evolution of the elements of Frenet frame with respect to time parameter t can be given as

$$V_t = MV,$$

where

$$M=\begin{bmatrix} 0 & A_1 & A_2 & A_3 \\ -\varepsilon_T\varepsilon_N A_1 & 0 & B_2 & B_3 \\ -\varepsilon_n A_2 & -\varepsilon_t\varepsilon_n B_2 & 0 & C \\ -\varepsilon_b A_3 & -\varepsilon_b\varepsilon_T\varepsilon_N B_3 & -\varepsilon_n\varepsilon_b C & 0 \end{bmatrix}, \tag{10}$$

$$B_2=\frac{\varepsilon_N}{\kappa}\left(\varepsilon_n kA_1+A_{2,s}-\varepsilon_b(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)A_3\right), B_3=\frac{\varepsilon_N}{\kappa}\left(\varepsilon_n(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)A_2+A_{3,s})\right),$$

$$C=\frac{\varepsilon_n}{k}\left(\varepsilon_t\varepsilon_N\kappa A_3+\varepsilon_n(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_2+B_{3,s}\right).$$

(ii) The evolutions of the curvatures are

$$\begin{aligned} \kappa_t &= -\lambda\kappa+\varepsilon_N A_{1,s}-\varepsilon_t\varepsilon_N kA_2, \\ k_t &= -\lambda k+\varepsilon_t\varepsilon_n\varepsilon_N\kappa A_2+\varepsilon_n B_{2,s}-\varepsilon_n\varepsilon_b(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_3, \\ (r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)_t &= -\lambda(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)+\varepsilon_t\varepsilon_n kB_3+\varepsilon_n C_s. \end{aligned} \tag{11}$$

Proof. Let the semi-real quaternionic curve flow be inextensible. By taking the derivative of (5) with respect to u, we get

$$\beta_{tu}=\sqrt{v}\beta_{ts}=\sqrt{v}\left(\lambda T+\sum_{i=1}^{3}A_iN_i\right). \tag{12}$$

Because $\beta_u=\sqrt{v}\beta_s=\sqrt{v}T$, by taking the derivative of β_u with respect to t, we have

$$\beta_{ut}=\sqrt{v}\left(\frac{v_t}{2v}T+T_t\right). \tag{13}$$

By using (3), (12), (13), and $\beta_{tu}=\beta_{ut}$, we have

$$T_t=\sum_{i=2}^{3}A_iN_i. \tag{14}$$

By taking the derivative of (14) with respect to u, we get

$$\begin{aligned} T_{tu}=\sqrt{v}\,\big((-\varepsilon_t\varepsilon_N\kappa A_1)\,T+\left(A_{1,s}-\varepsilon_t kA_2\right)N_1+\left(\varepsilon_n kA_1+A_{2,s}-\varepsilon_b(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)A_3\right)N_2 \\ \left(\varepsilon_n(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)A_2+A_{3,s}\right)N_3\big), \end{aligned} \tag{15}$$

and we have from the derivative of T_u with respect to t the following equations:

$$T_{ut}=\sqrt{v}\left(\left(\varepsilon_N\frac{v_t}{2v}\kappa+\varepsilon_N\kappa_t\right)N_1+\varepsilon_N\kappa N_{1,t}\right). \tag{16}$$

By using (3), (16), and $T_{tu}=T_{ut}$, we get

$$\begin{aligned} \kappa_t &= -\lambda\kappa+\varepsilon_N A_{1,s}-\varepsilon_t\varepsilon_N kA_2, \\ N_{1,t} &= -\varepsilon_t A_1 T+B_2N_2+B_3N_3, \\ B_2 &= \frac{\varepsilon_N}{\kappa}\left(\varepsilon_n kA_1+A_{2,s}-\varepsilon_b(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)A_3\right), \\ B_3 &= \frac{\varepsilon_N}{\kappa}\left(\varepsilon_n(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)A_2+A_{3,s})\right). \end{aligned} \tag{17}$$

By the way, differentiating Equation (3) with respect to u, we get

$$N_{1,tu} = \sqrt{v}\left(\left(-\varepsilon_t A_{1,s}\right)T + \left(-\varepsilon_t\varepsilon_N\kappa A_1 - \varepsilon_t k B_2\right)N_1 + \left(B_{2,s} - \varepsilon_b(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_3\right)N_2 \right. \\ \left. + \left(\varepsilon_n(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_2 + B_{3,s}\right)N_3\right, \tag{18}$$

and differentiating of $N_{1,u}$ with respect to t, we get

$$N_{1,ut} = \sqrt{v}\left((-\varepsilon_t\varepsilon_N\kappa\lambda - \varepsilon_t\varepsilon_N\kappa_t)T + (-\varepsilon_t\varepsilon_N\kappa A_1)N_1 \right. \\ + (\varepsilon_n k\lambda + \varepsilon_n k_t - \varepsilon_t\varepsilon_N\kappa A_2)N_2 \\ \left. + \varepsilon_n k N_{2,t} - \varepsilon_t\varepsilon_N\kappa A_3 N_3\right). \tag{19}$$

By using (3), (3), and $N_{1,tu} = N_{1,ut}$, we have

$$k_t = -\lambda k + \varepsilon_t\varepsilon_n\varepsilon_N\kappa A_2 + \varepsilon_n B_{2,s} - \varepsilon_n\varepsilon_b(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_3, \\ N_{2,t} = -\varepsilon_n A_2 T - \varepsilon_t\varepsilon_n B_2 N_1 + C N_3 \\ C = \frac{\varepsilon_n}{k}\left(\varepsilon_t\varepsilon_n\kappa A_3 + \varepsilon_n(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_2 + B_{3,s}\right). \tag{20}$$

Now, differentiating $N_{2,t}$ with respect to u, we get

$$N_{2,tu} = \sqrt{v}\left(\left(-\varepsilon_n A_{2,s} + \varepsilon_n\varepsilon_N\kappa B_2\right)T + \left(-\varepsilon_n\varepsilon_N\kappa A_2 - \varepsilon_t\varepsilon_n B_{2,s}\right)N_1 \right. \\ \left. + (-\varepsilon_n k B_2 - \varepsilon_b(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)C)N_2 + C_s N_3\right, \tag{21}$$

and differentiating $N_{2,u}$ with respect to t, we get

$$N_{2,ut} = \sqrt{v}\left((\kappa A_1)T + (-\varepsilon_t\lambda k - \varepsilon_t k_t)N_1 + (-\varepsilon_t k B_2)N_2 \right. \\ \left. + (\varepsilon_n\lambda(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa) - \varepsilon_t k B_3 + \varepsilon_n(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)_t)N_3 + \varepsilon_n(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)N_{3,t}\right). \tag{22}$$

Because $N_{2,tu} = N_{2,ut}$ and from (3) and (3), we have

$$(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)_t = -\lambda(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa) + \varepsilon_t\varepsilon_n k B_3 + \varepsilon_n C_s, \\ N_{3,t} = -\varepsilon_b A_3 T - \varepsilon_t\varepsilon_b B_3 N_1 - \varepsilon_n\varepsilon_b C N_2. \tag{23}$$

By using the mathematical induction, we obtain

$$V_t = MV,$$

where

$$M = \begin{bmatrix} 0 & A_1 & A_2 & A_3 \\ -\varepsilon_T\varepsilon_N A_1 & 0 & B_2 & B_3 \\ -\varepsilon_n A_2 & -\varepsilon_t\varepsilon_n B_2 & 0 & C \\ -\varepsilon_b A_3 & -\varepsilon_b\varepsilon_T\varepsilon_N B_3 & -\varepsilon_n\varepsilon_b C & 0 \end{bmatrix}.$$

From the first equation of (3), (3), and (3), we get

$$\kappa_t = -\lambda\kappa + \varepsilon_N A_{1,s} - \varepsilon_t\varepsilon_N k A_2, \\ k_t = -\lambda k + \varepsilon_t\varepsilon_n\varepsilon_N\kappa A_2 + \varepsilon_n B_{2,s} - \varepsilon_n\varepsilon_b(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_3, \tag{24}$$

$$(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)_t=-\lambda(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)+\varepsilon_t\varepsilon_n kB_3+\varepsilon_n C_s. \tag{25}$$

□

Lemma 2. *Let the flow of $\beta(u,t)$ be inextensible, so partial differential equations (2) are*

$$\begin{aligned}\kappa_t&=\varepsilon_N A_{1,s}-\varepsilon_t\varepsilon_N kA_2,\\ k_t&=\varepsilon_t\varepsilon_n\varepsilon_N\kappa A_2+\varepsilon_n B_{2,s}-\varepsilon_n\varepsilon_b(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_3,\end{aligned} \tag{26}$$

$$(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)_t=\varepsilon_t\varepsilon_n kB_3+\varepsilon_n C_s. \tag{27}$$

Proof. Let the flow of $\beta(u,t)$ be inextensible, so $v_t=0$, moreover $\lambda=0$. The proof of the lemma follows from using this result in Equation (2). □

Theorem 3. *The flow of $\beta(u,t)$ is inextensible if and only if the following equation (zero curvature condition) satisfies*

$$Q_t-M_s+[Q,M]=0, \tag{28}$$

where $[Q,M]=QM-MQ$ is the lie bracket.

Proof. In order to prove the theorem, there is a need for some calculations. Considering Equations (3) and (2), by taking the derivative of $V_u=\sqrt{v}QV$ with respect to t, we obtain

$$V_{ut}=\sqrt{v}\left(\frac{v_t}{2v}Q+Q_t+QM\right)V, \tag{29}$$

and taking the derivative of V_t with respect to u, we have

$$V_{tu}=\sqrt{v}\,(M_s+MQ)\,V. \tag{30}$$

By using Equation (29) together with Equation (30), we obtain

$$V_{ut}-V_{tu}=\sqrt{v}\left(\frac{v_t}{2v}Q+Q_t-M_s+[Q,M]\right)V.$$

First, if the flow is inextensible, then $v_t=0$ and $\partial/\partial u$ and $\partial/\partial t$ are commutative, hence

$$Q_t-M_s+[Q,M]=0.$$

On the other hand, suppose the integrability equation is satisfied, that is,

$$Q_t-M_s+[Q,M]=0.$$

From (3) and (2), we get

$$[Q,M]=\begin{bmatrix}0 & \varepsilon_t kA_2 & A_{2,s} & A_{3,s}\\ -kA_2 & 0 & -\varepsilon_t\varepsilon_N\kappa A_2+\varepsilon_b(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_3 & B_{3,s}\\ -\varepsilon_n A_{2,s} & \varepsilon_n\varepsilon_N\kappa A_2-\varepsilon_t\varepsilon_n\varepsilon_b(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_3 & 0 & -\varepsilon_t kB_3\\ -\varepsilon_b A_{3,s} & -\varepsilon_t\varepsilon_b B_{3,s} & \varepsilon_t\varepsilon_n\varepsilon_b kB_3 & 0\end{bmatrix}. \tag{31}$$

By taking the derivative of Q with respect to t and the derivative of M with respect to s with Equations (2) and (31), we obtain

$$\begin{bmatrix}0 & -\varepsilon_N\lambda\kappa & 0 & 0\\ \varepsilon_t\varepsilon_N\lambda\kappa & 0 & -\varepsilon_n\lambda k & 0\\ 0 & \varepsilon_t\lambda k & 0 & -\varepsilon_n\lambda(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa)\\ 0 & 0 & \varepsilon_b\lambda(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa) & 0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

Last equation forces $\lambda = 0$, that is, $v =$ constant. It says that the flow is inextensible. □

Theorem 4. *Let the curve flow be inextensible. If Q and M are abelian, then*

$$A_2 = B_3 = 0.$$

Proof. Let Q and M be abelian, so $[Q, M] = 0$, then (28) is as follows:

$$M_s - Q_t = 0. \tag{32}$$

Because the flow is inextensible, then

$$Q_t - M_s = \begin{bmatrix} 0 & -\varepsilon_t k A_2 & -A_{2,s} & -A_{3,s} \\ kA_2 & 0 & \varepsilon_t\varepsilon_N\kappa A_2 - \varepsilon_b(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_3 & -B_{3,s} \\ -\varepsilon_n A_{2,s} & -\varepsilon_n\varepsilon_N\kappa A_2 + \varepsilon_t\varepsilon_n\varepsilon_b(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa)B_3 & 0 & \varepsilon_t k B_3 \\ -\varepsilon_b A_{3,s} & \varepsilon_t\varepsilon_b B_{3,s} & -\varepsilon_t\varepsilon_n\varepsilon_b k B_3 & 0 \end{bmatrix}. \tag{33}$$

By using (32) and (33), we have

$$A_2 = B_3 = 0.$$ □

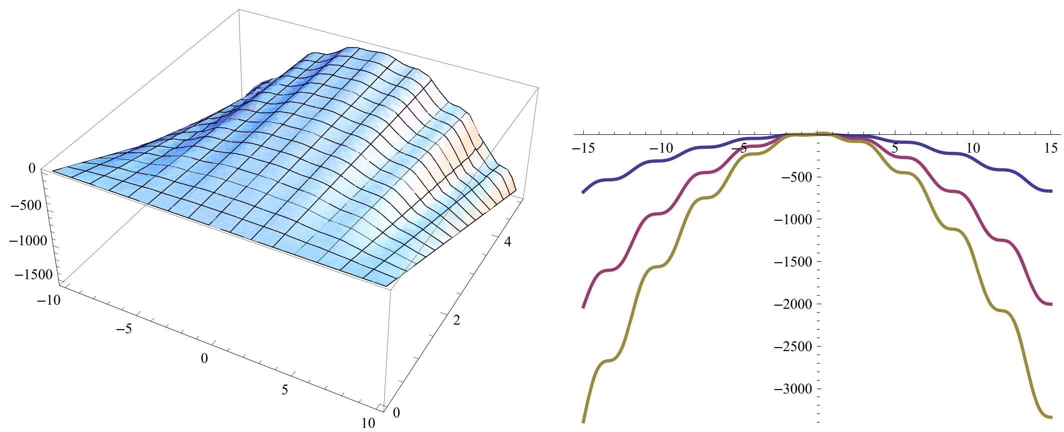

FIGURE 1 The evolution of $\kappa(s, t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

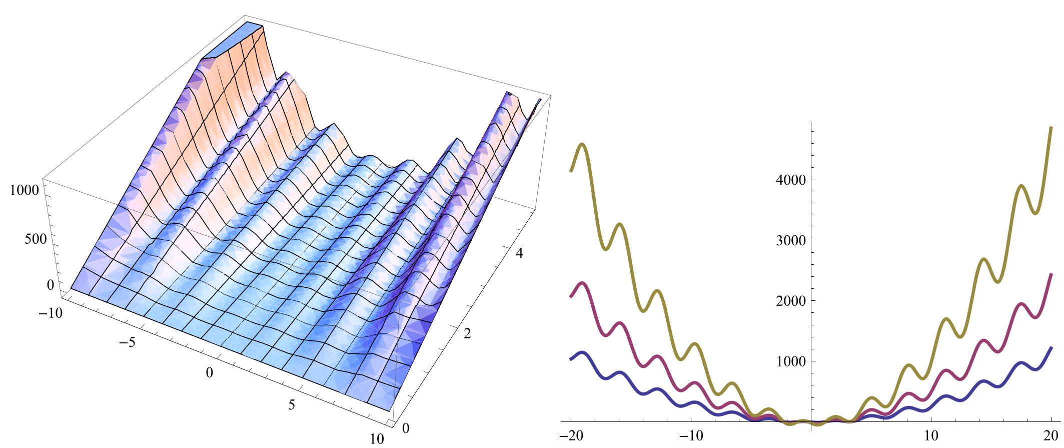

FIGURE 2 The evolution of $k(s, t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

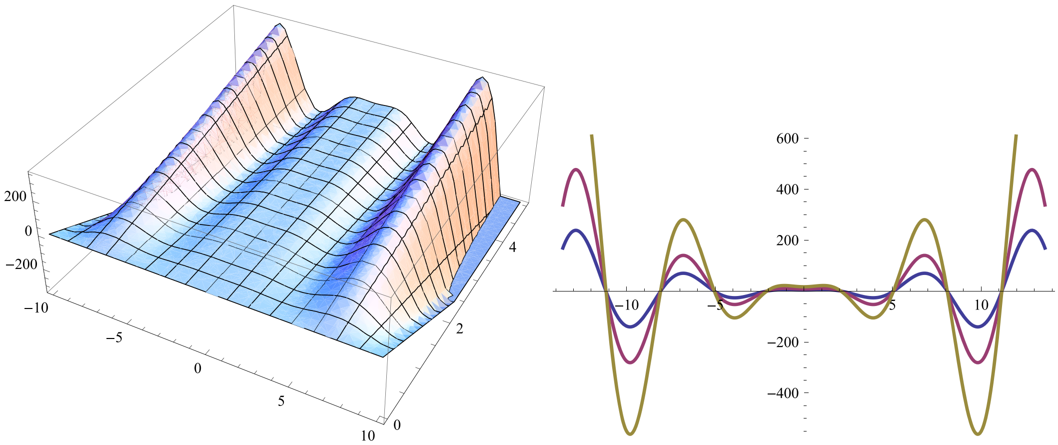

FIGURE 3 The evolution of $(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa)(s, t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

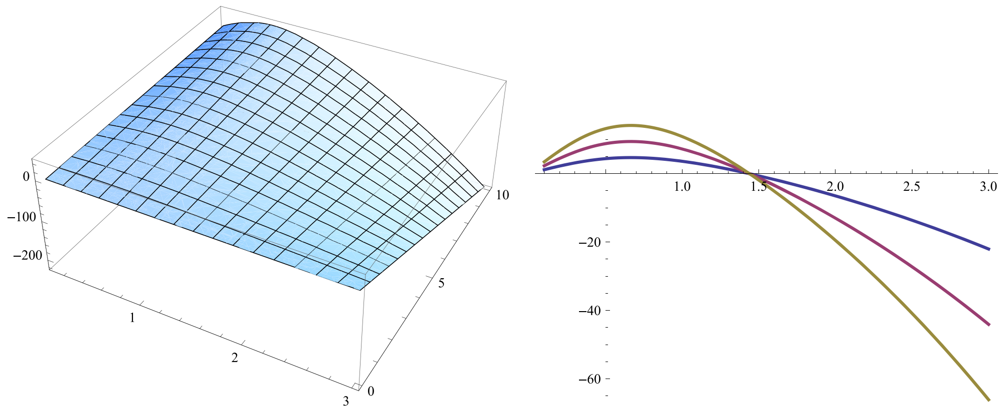

FIGURE 4 The evolution of $\kappa(s, t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

Example 1. Let quaternionic curve $\beta(s)$ be

$$\beta(s) = \left(\cosh s, \sin \sqrt{2}s, \sinh s, \sin \sqrt{2}\right),$$

for all $s \in I$. Curvatures of β are as follows:

$$\kappa = -1, k = \sqrt{2}, (r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa) = 0.$$

If $f_1 = s^2$, $f_2 = s^2$, $f_3 = s \sin s \cos s$, and $f_4 = s^2 \cos s$, then graphs of evolution of the curvatures are in the domain

$$D : \begin{cases} -10 < s < 10 \\ 0 < t \leq 5 \end{cases},$$

and for different values of t, the curvatures are plotted in Figures 1–3.

If $f_1 = s^2 \cosh^2 s$, $f_2 = s^2 \tanh s$, $f_3 = s \tanh s$, and $f_4 = s^4 + s^2$, then graphs of evolution of the curvatures are in the domain

$$D : \begin{cases} 0 < s < 3 \\ 0 < t \leq 3 \end{cases},$$

and for different values of t, the curvatures are plotted in Figures 4–6.

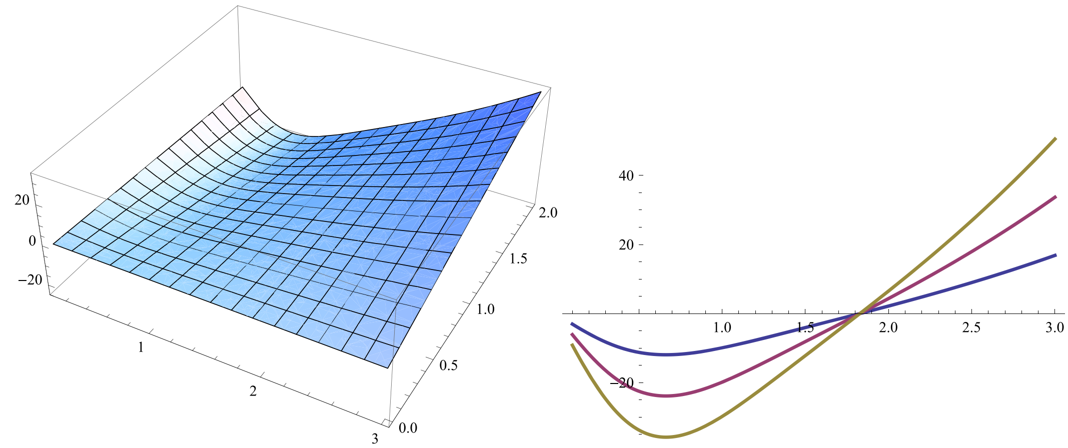

FIGURE 5 The evolution of $k(s, t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

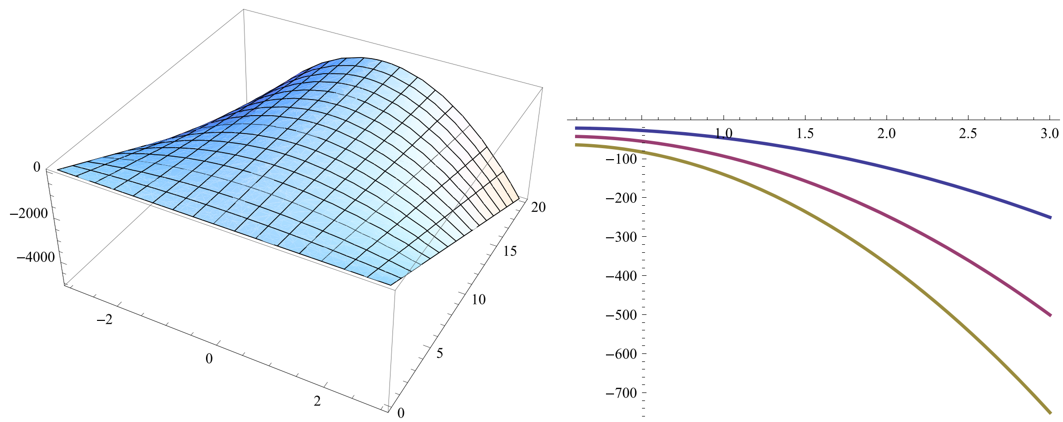

FIGURE 6 The evolution of $(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa)(s, t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

ORCID

Alperen Kızılay https://orcid.org/0000-0002-8612-5351
Önder Gökmen Yıldız https://orcid.org/0000-0002-2760-1223
Osman Zeki Okuyucu https://orcid.org/0000-0003-4379-0252

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