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Slant helices in three dimensional Lie groups

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ABSTRACT

In this paper, we define slant helices in three dimensional Lie groups with a bi-invariant metric and obtain a characterization of slant helices. Moreover, we give some relations between slant helices and their involutes, spherical images.

1. Introduction

In differential geometry, we think that curves are geometric set of points of loci. Curves theory is important workframe in the differential geometry studies and we have a lot of special curves such as geodesics, circles, Bertrand curves, circular helices, general helices, slant helices etc. Characterizations of these special curves are heavily studied for a long time and are still studied. We can see helical structures in nature and mechanic tools. In the field of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or design of highways. Also we can see the helix curve or helical structure in fractal geometry, for instance hyperhelices. In differential geometry; a curve of constant slope or general helix in Euclidean 3-space $\mathbb{E}^3$, is defined by the property that its tangent vector field makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 (see [1,2] for details) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion is constant. If both of $\varkappa$ and τ are non-zero constants then the curve is called as a circular helix. It is known that a straight line and a circle are degenerate-helix examples ($\varkappa = 0$, if the curve is straight line and $\tau = 0$, if the curve is a circle).

The Lancret theorem was revisited and solved by Barros [3] in 3-dimensional real space forms by using killing vector fields along curves. Also in the same spaceforms, a characterization of helices and Cornu spirals is given by Arroyo et al. [4].

The degenarete semi-Riemannian geometry of Lie group is studied by Çö ken and Çiftçi [5]. Moreover, they obtained a naturally reductive homogeneous semi-Riemannian space using the Lie group. Then Ç iftçi [6] defined general helices in three dimensional Lie groups with a bi-invariant metric and obtained a generalization of Lancret's theorem and gave a relation between the geodesics of the so-called cylinders and general helices.

Recently, Izumiya and Tkeuchi [7], have introduced the concept of slant helix in Euclidean 3-space. A slant helix in Euclidean space $\mathbb{E}^3$ was defined by the property that its principal normal vector field makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi showed that α is a slant helix if and only if the geodesic curvature of spherical image of principal normal indicatrix (N) of a space curve α

$$\sigma_N(s) = \left(\frac{\varkappa^2}{(\varkappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\varkappa} \right)' \right)(s)$$

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is a constant function. In [8]; Kula and Yayli have studied spherical images of a slant helix and showed that the spherical images of a slant helix are spherical helices. In [9], the authors characterize slant helices by certain differential equations verified for each one of spherical indicatrix in Euclidean 3-space. Ali and Lopez, in [10], have studied slant helix in Minkowski 3-space. They showed that the spherical indicatrix of a slant helix are helices in $\mathbb{E}_1^3$. Then Ali and Turgut studied position vector of a time-like slant helix with respect to standard frame of Minkowski space $\mathbb{E}_1^3$ in terms of Frenet equations (see [11] for details). Also slant helices are used in some applications in quaternion algebra (see [12,13] for details).

In this paper, first of all, we define slant helices in a three dimensional Lie group G with a bi-invariant metric as a curve $\alpha : I \subset \mathbb{R} \to G$ whose normal vector field makes a constant angle with a left invariant vector field (Definition 3.1). And then the main result to this paper is given as (Theorem 3.6): A curve $\alpha : I \subset \mathbb{R} \to G$ with the Frenet apparatus $\{T, N, B, \varkappa, \tau\}$ is a slant helix if and only if

$$\frac{\varkappa(H^2+1)^{\frac{3}{2}}}{H^{|}}$$

is a constant function where H is a harmonic curvature function of the curve α (Definition 3.2).

Then we define the involutes and spherical image of a curve in three dimensional Lie group G. Also we show that the spherical image of a slant helix and the involutes of a slant helix are general helices. Finally, we give characterization of a slant helix if G are Abellian, SO^3 and S^3.

Note that three dimensional Lie groups admitting bi-invariant metrics are $SO(3), SU^2$ and Abellian Lie groups. So we believe that characterizations of slant curves in this study will be useful for curves theory in Lie groups.

2. Preliminaries

Let G be a Lie group with a bi-invariant metric $\langle , \rangle$ and D be the Levi–Civita connection of Lie group G. If $\mathfrak{g}$ denotes the Lie algebra of G then we know that $\mathfrak{g}$ is isomorphic to T_eG where e is neutral element of G. If $\langle , \rangle$ is a bi-invariant metric on G then we have

$$\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$$

and

$$D_X Y = \frac{1}{2}[X, Y]$$

for all X, Y and $Z \in \mathfrak{g}$.

Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed curve and $\{X_1, X_2, \ldots, X_n\}$ be an orthonormal basis of $\mathfrak{g}$. In this case, we write that any two vector fields W and Z along the curve α as $W = \sum_{i=1}^n w_i X_i$ and $Z = \sum_{i=1}^n z_i X_i$ where $w_i : I \to \mathbb{R}$ and $z_i : I \to \mathbb{R}$ are smooth functions. Also the Lie bracket of two vector fields W and Z is given

$$[W, Z] = \sum_{i=1}^n w_i z_i [X_i, X_j]$$

and the covariant derivative of W along the curve α with the notation $D_{\alpha^|} W$ is given as follows

$$D_{\alpha^|} W = \dot{W} + \frac{1}{2}[T, W] \tag{2.1}$$

where $T = \alpha'$ and $\dot{W} = \sum_{i=1}^n \dot{w}_i X_i$ or $\dot{W} = \sum_{i=1}^n \frac{dw}{dt} X_i$. Note that if W is the restriction of a left-invariant vector field to the curve α then $\dot{W} = 0$ (see [14] for details).

Let G be a three dimensional Lie group and $(T, N, B, \varkappa, \tau)$ denote the Frenet apparatus of the curve α, and calculate $\varkappa = \|T^{\cdot}\|$.

Definition 2.1. Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve. Then α is called a general helix if it makes a constant angle with a left-invariant vector field X. That is,

$$\langle T(s), X \rangle = \cos\theta \quad \text{for all } s \in I,$$

for the left-invariant vector field $X \in g$ is unit length and θ is a constant angle between X and T which is the tangent vector field of the curve α (see [6]).

Definition 2.2. Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with the Frenet apparatus $(T, N, B, \varkappa, \tau)$ then

$$\tau_G = \frac{1}{2}\langle [T, N], B \rangle \tag{2.2}$$

or

$$\tau_G = \frac{1}{2\varkappa^2\tau}\left\langle \ddot{T}, \left[T, \dot{T}\right] \right\rangle + \frac{1}{4\varkappa^2\tau}\left\| \left[T, \dot{T}\right] \right\|^2$$

(see [6]).

Theorem 2.3. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with the Frenet apparatus $(T, N, B, \varkappa, \tau)$. If the curve α is a general helix, if and only if,*

$$\tau = c\varkappa + \tau_G,$$

where c is a constant (*see* [6]).

3. Slant helices in a three dimensional Lie group

In this section we define slant helix and its axis in a three dimensional Lie group G with a bi-invariant metric $\langle , \rangle$. Also we give a characterization and some results for characterization of the slant helices in the special cases of G.

Definition 3.1. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve. Then α is called a slant helix if its principal normal vector makes a constant angle with a left-invariant vector field X which is unit length. That is,

$$\langle N(s), X\rangle = \cos\theta \quad \text{for all } s \in I,$$

where $\theta \neq \frac{\pi}{2}$ is a constant angle between X and N which is the principal normal vector field of the curve α.

Definition 3.2. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. Then the harmonic curvature function of the curve α is defined by

$$H = \frac{\tau - \tau_G}{\varkappa},$$

where $\tau_G = \frac{1}{2}\langle [T, N], B\rangle$.

Definition 3.3. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. Then the geodesic curvature of the spherical image of the principal normal indicatrix (N) of the curve α is defined by a constant σ_N given by

$$\sigma_N = \frac{\varkappa(1 + H^2)^{\frac{3}{2}}}{H^{|}},$$

where H is harmonic curvature function of the curve α.

Proposition 3.4. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $\{T, N, B\}$. Then the following equalities*

$$[T, N] = \langle [T, N], B\rangle B = 2\tau_G B,$$
$$[T, B] = \langle [T, B], N\rangle N = -2\tau_G N$$

hold.

Proof. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc length parametrized curve with the Frenet apparatus $\{T, N, B\}$. Since $[T, N] \in Sp\{T, N, B\}$, we can write

$$[T, N] = \lambda_1 T + \lambda_2 N + \lambda_3 B. \tag{3.1}$$

If we multiply the two sides of the Eq. (3.1) with T, N and B, respectively

$$\langle [T, N], T\rangle = \lambda_1 = 0,$$
$$\langle [T, N], N\rangle = \lambda_2 = 0,$$
$$\langle [T, N], B\rangle = \lambda_3.$$

Thus we can write

$$[T, N] = \langle [T, N], B\rangle B$$

or using the Eq. (2.2) and the last equation, we get

$$[T,N] = 2\tau_G B.$$

On the other hand, using a similar method we can easily show that

$$[T,B] = -2\tau_G N.$$

Which complete the proof. □

Proposition 3.5. *Let $\alpha : I \subset \mathbb{R} \to G$ be a parametrized curve with arc length parameter s and $\{T,N,B\}$ denote the Frenet frame of the curve α. If the curve α is a slant helix in G, then the axis of α is*

$$X = \left\{ \frac{\varkappa H\left(1+H^2\right)}{H^{|}} T + N + \frac{\varkappa\left(1+H^2\right)}{H^{|}} B \right\} \cos\theta,$$

where $H = \frac{\tau-\tau_G}{\varkappa}$ is harmonic curvature function of the curve α and $\theta \neq \frac{\pi}{2}$ is a constant angle.

Proof. If the axis of slant helix α is X, then we can write

$$X = \lambda_1 T + \lambda_2 N + \lambda_3 B,$$

where $\lambda_1 = \langle T,X\rangle, \lambda_2 = \langle N,X\rangle$ and $\lambda_3 = \langle B,X\rangle$.

And we know from the Definition 3.1 that

$$\langle N(s),X\rangle = \cos\theta \quad \text{for all } s \in I, \tag{3.2}$$

where the left-invariant vector field $X \in \mathfrak{g}$ is unit length and θ is a constant angle between X and N which is the principal normal vector field of the curve α. By differentiating $\langle N(s),X\rangle = \cos\theta$, we get

$$\langle D_T N,X\rangle + \langle N,D_T X\rangle = 0$$

or using the Eq. (2.1) and the Frenet formulas

$$-\kappa\langle T,X\rangle + \tau\langle B,X\rangle - \frac{1}{2}\langle [T,N],X\rangle = 0$$

and with the help of the Proposition 3.4, we get

$$\langle T,X\rangle = H\langle B,X\rangle, \tag{3.3}$$

where $H = \frac{\tau-\tau_G}{\varkappa}$ is harmonic curvature function of the curve α.

Again differentiating the Eq. (3.3), we have

$$\langle D_T T,X\rangle + \langle T,D_T X\rangle = H^{|}\langle B,X\rangle + H\{\langle D_T B,X\rangle + \langle B,D_T X\rangle\},$$

then by using the Eq. (2.1) and the Proposition 3.4 we obtain

$$\langle B,X\rangle = \frac{\varkappa\left(1+H^2\right)}{H^{|}}\langle N,X\rangle. \tag{3.4}$$

Then if we write the Eq. (3.4) in the Eq. (3.3), we get

$$\langle T,X\rangle = \frac{\varkappa H}{H^{|}}\left(1+H^2\right)\langle N,X\rangle. \tag{3.5}$$

Consequently, using the Eqs. (3.2), (3.4) and (3.5) the axis of slant helix α is given by

$$X = \left\{ \frac{\varkappa H\left(1+H^2\right)}{H^{|}} T + N + \frac{\varkappa\left(1+H^2\right)}{H^{|}} B \right\} \cos\theta,$$

which completes the proof. □

Theorem 3.6. *Let $\alpha : I \subset \mathbb{R} \to G$ be a unit speed curve with the Frenet apparatus $(T,N,B,\varkappa,\tau)$. Then α is a slant helix if and only if*

$$\sigma_N = \frac{\varkappa(1+H^2)^{\frac{3}{2}}}{H^{|}} = \tan\theta$$

is a constant, where H is a harmonic curvature function of the curve α and $\theta \neq \frac{\pi}{2}$ is a constant.

Proof. If the axis of slant helix α is X, then using the Proposition 3.5 we have

$$X = \left\{ \frac{\varkappa H\left(1 + H^2\right)}{H^{|}} T + N + \frac{\varkappa\left(1 + H^2\right)}{H^{|}} B \right\} \cos\theta.$$

Since X is unit length vector field then we can easily see that

$$\frac{\varkappa(H^2 + 1)^{\frac{3}{2}}}{H^{|}} = \tan\theta$$

is a constant.

Conversely, if $\sigma_N(s)$ is constant then the result is obvious. This complete the proof. □

In the following remark, we note that three dimensional Lie groups admitting bi-invariant metrics are S^3, SO^3 and Abelian Lie groups using the same notation as in [6,15] as follows:

Remark 3.7. Let G be a Lie group with a bi-invariant metric $\langle , \rangle$. Then the following equalities can be given in different Lie groups:

(i) If G is abelian group then $\tau_G = 0$,
(ii) If G is SU^2 then $\tau_G = 1$,
(iii) If G is SO^3 then $\tau_G = \frac{1}{2}$
(see for details [6,15]).

Corollary 3.8. *Let α be a unit speed curve with the Frenet apparatus $\{T, N, B\}$ in the Abellian Lie group G. Then α is a slant helix if and only if*

$$\sigma_N = \frac{\left(\varkappa^2 + \tau^2\right)^{3/2}}{\varkappa^2 \left(\frac{\tau}{\varkappa}\right)^{|}}$$

is a constant function.

Proof. If G is Abellian Lie group then using the above Remark 3.7 and the Theorem 3.6 we have the result. □

So, the above Corollary shows that the study is a generalization of slant helices defined by Izumiya and Tkeuchi [7] in Euclidean 3-space. Moreover, with a similar proof, we have the following two corollaries.

Corollary 3.9. *Let α be unit speed curve with the Frenet apparatus $\{T, N, B\}$ in the Lie group SU^2. Then α is a slant helix if and only if*

$$\sigma_N = \frac{\left(\varkappa^2 + (\tau - 1)^2\right)^{3/2}}{\varkappa^2 \left(\frac{\tau - 1}{\varkappa}\right)^{|}}$$

is a constant function.

Corollary 3.10. *Let α be unit speed curve with the Frenet apparatus $\{T, N, B\}$ in the Lie group SO^3. Then α is a slant helix if and only if*

$$\sigma_N = \frac{\left(\varkappa^2 + \left(\tau - \frac{1}{2}\right)^2\right)^{3/2}}{\varkappa^2 \left(\frac{\tau - \frac{1}{2}}{\varkappa}\right)^{|}}$$

is a constant function.

4. Spherical images of slant helices in the three dimensional Lie group

In Euclidean geometry, the spherical indicatrix of a space curve is defined as follows: Let α be a unit speed regular curve in Euclidean 3 -space with Frenet vectors T, N and B. The unit tangent vectors along the curve α generate a curve β on the sphere of radius 1 about the origin. The curve β is called the spherical indicatrix of T or more commonly, β is called tangent indicatrix of the curve α. If $\alpha = \alpha(s)$ is a natural representation of α, then $\beta = T(s)$ will be a representation of β. Similarly one considers the principal normal indicatrix $\gamma = N(s)$ and binormal indicatrix $\delta = B(s)$. It is clear that, this definition is related with the spherical curve [2].

In this section, firstly we define spherical indicatrices of slant helices with the help of the studies [16,17] and then investigate the relation between slant helices and their spherical indicatrices in 3-dimensional Lie group. Morever, we give some theorems with their proofs and some results in Lie groups.

4.1. Tangent indicatrices of slant helices

Definition 4.1. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed regular curve. Its tangent indicatrix is the parametrized curve $\beta : I \subset \mathbb{R} \to S^2 \subset \mathfrak{g}$ defined by

$$\beta(s^*) = T(s) = \sum_{i=1}^{3} t_i X_i \quad \text{for all } s \in I,$$

where $\{X_1, X_2, X_3\}$ is an orthonormal basis of $\mathfrak{g}$ and s^* is the arc length parameter of β.

Theorem 4.2. *Let α be an arc-lengthed regular curve and β be the tangent indicatrix of the curve α. Then the curve α is a slant helix in three dimensional Lie group G if and only if the curve β is a general helix on S^2.*

Proof. We assume that the curve α is a slant helix in a three dimensional Lie group and β is the tangent indicatrix of the curve α. From the Definition 4.1 we get

$$\beta(s^*) = T(s),$$

then differentiating the last equation and using the Eq. (2.1), we have

$$\frac{d\beta}{ds^*}\frac{ds^*}{ds} = \dot{T} = D_T T - \frac{1}{2}[T, T],$$
$$\frac{d\beta}{ds^*}\frac{ds^*}{ds} = \varkappa N.$$

Then assuming that $\varkappa \rangle 0$, we obtain

$$\frac{ds^*}{ds} = \varkappa \tag{4.1}$$

and

$$T_\beta(s^*) = N(s). \tag{4.2}$$

If we differentiate the last equation and use Frenet formulas then we obtain

$$\varkappa_\beta N_\beta(s^*)\frac{ds^*}{ds} = \dot{N} = D_T N - \frac{1}{2}[T, N],$$
$$\varkappa_\beta N_\beta(s^*)\varkappa = -\kappa T + \tau B - \frac{1}{2}\langle [T, N], B\rangle B$$

or with the help of the Proposition 3.4, we get

$$\varkappa_\beta N_\beta(s^*) = -T + HB,$$

where $\varkappa_\beta$ is the curvature of β. Hence

$$\varkappa_\beta = \sqrt{1 + H^2}$$

and

$$N_\beta(s^*) = -\frac{1}{\sqrt{1 + H^2}}T + \frac{H}{\sqrt{1 + H^2}}B. \tag{4.3}$$

Then using the Eq. (4.2) and the Eq. (4.3), we have

$$B_\beta(s^*) = T_\beta(s^*) \times N_\beta(s^*) = \frac{H}{\sqrt{1 + H^2}}T + \frac{1}{\sqrt{1 + H^2}}B. \tag{4.4}$$

Using the differentiation of the last equation and the Proposition 3.4, this implies

$$(\tau_\beta - \tau_{G_\beta})N_\beta(s^*)\frac{ds^*}{ds} = -\frac{H'}{\left(1+H^2\right)^{3/2}}T + \frac{HH'}{\left(1+H^2\right)^{3/2}}B$$

or using the Eq. (4.1), we have

$$(\tau_\beta - \tau_{G_\beta})N_\beta(s^*) = -\frac{H'}{\varkappa\left(1+H^2\right)^{3/2}}T + \frac{HH'}{\varkappa\left(1+H^2\right)^{3/2}}B,$$

where $\tau_{G_\beta} = \frac{1}{2}\langle[T_\beta, N_\beta], B_\beta\rangle$. Thus we compute

$$\tau_\beta = \frac{H^|}{\varkappa\left(1+H^2\right)} + \tau_{G_\beta},$$

where τ_β is the torsion of β. The we can easily see that $\frac{\tau_\beta - \tau_{G_\beta}}{\varkappa_\beta} = \frac{H^|}{\varkappa\left(1+H^2\right)^{3/2}}$ is a constant function. In other words, using the Theorem 2.3 we can easily obtain that β is a general helix.

Conversely, we assume that β is a general helix then we can easily see that α is a slant helix. These complete the proof. □

Corollary 4.3. *Let α be an arc-lengthed regular curve with the Frenet vector fields $\{T, N, B\}$ in the Lie group G and β be the tangent indicatrix of the curve α. Then $\tau_{G_\beta} = \tau_G$ for the curves α and β.*

Proof. It is obvious using the Eqs. (4.2), (4.3) and (4.4). □

4.2. Normal indicatrices of slant helices

Definition 4.4. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed regular curve. Its normal indicatrix is the parametrized curve $\gamma : I \subset \mathbb{R} \to S^2 \subset \mathfrak{g}$ defined by

$$\gamma(s^*) = N(s) = \sum_{i=1}^{3} n_i X_i \quad \text{for all } s \in I,$$

where $\{X_1, X_2, X_3\}$ is an orthonormal basis of $\mathfrak{g}$ and s^* is the arc length parameter of γ.

Theorem 4.5. *Let α be an arc-lengthed slant helix in three dimensional Lie group G and γ be the normal indicatrix of the curve α. Then the curve γ is a plane curve on S^2.*

Proof. We assume that the curve α is a slant helix in a three dimensional Lie group and γ is the normal indicatrix of the curve α. From the Definition 4.4, we get

$$\gamma(s^*) = N(s). \tag{4.5}$$

Then differentiating the Eq. (4.5) and using the Eq. (2.1) we have

$$\begin{aligned}\frac{d\gamma}{ds^*}\frac{ds^*}{ds} &= \dot{N} = D_T N - \frac{1}{2}[T, N] \\ &= -\varkappa T + \tau B - \frac{1}{2}\langle[T, N], B\rangle B \\ &= -\varkappa T + (\tau - \tau_G)B, \\ \frac{d\gamma}{ds^*}\frac{ds^*}{ds} &= -\varkappa T + \varkappa H B.\end{aligned}$$

Then assuming that $\varkappa\rangle 0$, we obtain

$$\frac{ds^*}{ds} = \varkappa\sqrt{1+H^2} \tag{4.6}$$

and

$$\frac{d\gamma}{ds^*} = \frac{1}{\sqrt{1+H^2}}(-T + HB).$$

If we differentiate the last equation, then we obtain

$$\frac{d^2\gamma}{ds^{*2}}\frac{ds^*}{ds} = -\frac{HH^{|}}{\left(1+H^2\right)^{3/2}}(-T+HB) + \frac{1}{\sqrt{1+H^2}}\left(-\dot{T}+H^{|}B+H\dot{B}\right)$$
$$= -\frac{HH^{|}}{\left(1+H^2\right)^{3/2}}(-T+HB) + \frac{1}{\sqrt{1+H^2}}\left\{-\varkappa N+H^{|}B+H\left(-\tau N-\frac{1}{2}[T,B]\right)\right\}$$

and by using the Eq. (4.6) with together Proposition 3.4, we obtain

$$\frac{d^2\gamma}{ds^{*2}} = -\frac{H}{\sqrt{1+H^2}}\frac{H^{|}}{\varkappa\left(1+H^2\right)^{3/2}}(-T+HB) + \frac{1}{\varkappa\left(1+H^2\right)}\left\{(-\varkappa-H(\tau-\tau_G))N+H^{|}B\right\}$$
$$= -\frac{H}{\sqrt{1+H^2}}\frac{H^{|}}{\varkappa\left(1+H^2\right)^{3/2}}(-T+HB) + \frac{1}{\varkappa\left(1+H^2\right)}\left\{-\varkappa\left(1+H^2\right)N+H^{|}B\right\}.$$

Since α is a slant helix, $\sigma_N(s)$ is a constant function. So, we can obtain

$$\frac{d^2\gamma}{ds^{*2}} = \frac{1}{\sigma_N(s)}\frac{H}{\sqrt{1+H^2}}T - N + \frac{1}{\sigma_N(s)}\frac{1}{\sqrt{1+H^2}}B. \tag{4.7}$$

Hence

$$\varkappa_\gamma = \left\|\frac{d^2\gamma}{ds^{*2}}\right\| = \frac{1}{|\sigma_N|}\sqrt{1+\sigma_N^2},$$

where $\varkappa_\gamma$ is the curvature of γ. Then differentiating the Eq. (4.7) and using the Definition 3.3, we have

$$\frac{d^3\gamma}{ds^{*3}}\varkappa\sqrt{1+H^2} = -\frac{1}{\sigma_N}\left\{\frac{H^{|}}{\left(1+H^2\right)^{3/2}}(-T+HB) + \frac{H}{\sqrt{1+H^2}}\left(-\dot{T}+H^{|}B+H\dot{B}\right)\right\} - \dot{N} + \frac{1}{\sigma_N}\left(\frac{HH^{|}}{\sqrt{1+H^2}}B + \sqrt{1+H^2}\dot{B}\right)$$
$$= -\frac{1}{\sigma_N}\left\{\frac{H^{|}}{\left(1+H^2\right)^{3/2}}(-T+HB) + \frac{H}{\sqrt{1+H^2}}\left(-\varkappa\left(1+H^2\right)N+H^{|}B\right)\right\} - D_TN + \frac{1}{2}[T,N]$$
$$+ \frac{1}{\sigma_N}\left(\frac{HH^{|}}{\sqrt{1+H^2}}B + \sqrt{1+H^2}\left(D_TB - \frac{1}{2}[T,B]\right)\right)$$

then by using the Proposition 3.4, we obtain

$$\frac{d^3\gamma}{ds^{*3}} = \varkappa\frac{\sigma_N^2+1}{\sigma_N^2}T - \varkappa H\frac{\sigma_N^2+1}{\sigma_N^2}B. \tag{4.8}$$

Thus we compute

$$\tau_\gamma = \frac{\det\left(\gamma^{|},\gamma^{||},\gamma^{|||}\right)}{\|\gamma^{|}\times\gamma^{||}\|^2} = 0,$$

where τ_γ is the torsion of γ. Hence γ is a plane curve. This complete the proof. □

4.3. Binormal indicatrices of slant helices

Definition 4.6. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed regular curve. Its binormal indicatrix is the parametrized curve $\delta : I \subset \mathbb{R} \to S^2 \subset \mathfrak{g}$ defined by as

$$\delta(s^*) = B(s) = \sum_{i=1}^{3} b_i X_i \quad \text{for all } s \in I,$$

where $\{X_1, X_2, X_3\}$ is an orthonormal basis of $\mathfrak{g}$ and s^* is the arc length parameter of δ.

Theorem 4.7. *Let α be an arc-lengthed regular curve and δ be the binormal indicatrix of the curve α. Then the curve α is a slant helix in three dimensional Lie group G if and only if the curve δ is a general helix on S^2.*

Proof. We assume that α be a slant helix in a three dimensional Lie group and δ be the binormal indicatrix of the curve α. From the Definition 4.6, we get

$$\delta(s^*) = B(s), \tag{4.9}$$

then differentiating the Eq. (4.9) and using the Eq. (2.1), we have

$$\frac{d\delta}{ds^*}\frac{ds^*}{ds} = \dot{B} = D_T B - \frac{1}{2}[T, B],$$
$$\frac{d\delta}{ds^*}\frac{ds^*}{ds} = -\varkappa H N.$$

Then assuming that $\varepsilon = \left\{\begin{array}{ll} 1, & \text{if } \varkappa H\rangle 0 \\ -1, & \text{if } \varkappa H\langle 0 \end{array}\right\}$, we have

$$\frac{ds^*}{ds} = \varepsilon\varkappa H$$

and

$$T_\delta(s^*) = -\varepsilon N(s). \tag{4.10}$$

If we differentiate the last equation then we obtain

$$\varkappa_\delta N_\delta(s^*)\frac{ds^*}{ds} = -\varepsilon\dot{N} = -\varepsilon D_T N + \varepsilon\frac{1}{2}[T, N],$$
$$\varkappa_\delta N_\delta(s^*)\frac{ds^*}{ds} = \varepsilon\varkappa T - \varepsilon\tau B + \varepsilon\frac{1}{2}\langle[T, N], B\rangle B,$$
$$\varkappa_\delta N_\delta(s^*)\varepsilon\varkappa H = \varepsilon\varkappa T - \varepsilon(\tau - \tau_G)B,$$
$$\varkappa_\delta N_\delta(s^*) = \frac{1}{H}T - B,$$

where $\varkappa_\delta$ is the curvature of δ. Hence

$$\varkappa_\delta = \frac{1}{|H|}\sqrt{1 + H^2}$$

and assuming that $\varkappa\rangle 0$, we have

$$N_\delta(s^*) = \frac{\varepsilon}{\sqrt{1 + H^2}}T - \frac{\varepsilon H}{\sqrt{1 + H^2}}B. \tag{4.11}$$

Then using the Eq. (4.10) and the Eq. (4.11) we have

$$B_\delta(s^*) = T_\delta(s^*) \times N_\delta(s^*) = -\frac{H}{\sqrt{1 + H^2}}T + \frac{1}{\sqrt{1 + H^2}}B. \tag{4.12}$$

Using the differentiation of the last equation and the Proposition 3.4, this implies

$$(\tau_\delta - \tau_{G_\delta})N_\delta(s^*)\frac{ds^*}{ds} = \frac{H'}{\left(1 + H^2\right)^{3/2}}T + \frac{HH'}{\left(1 + H^2\right)^{3/2}}B$$

or using the equality $\frac{ds^*}{ds} = \varepsilon\varkappa H$, we have

$$(\tau_\delta - \tau_{G_\delta})N_\delta(s^*) = \frac{H'}{\varkappa\left(1 + H^2\right)^{3/2}}T + \frac{HH'}{\varkappa\left(1 + H^2\right)^{3/2}}B,$$

where $\tau_{G_\delta} = \frac{1}{2}\langle[T_\delta, N_\delta], B_\delta\rangle$. Thus we have

$$\tau_\delta = \frac{H^|}{\varkappa H\left(1 + H^2\right)} + \tau_{G_\delta},$$

where τ_δ is the torsion of δ and so $\frac{\tau_\delta - \tau_{G_\delta}}{\varkappa_\delta} = \frac{H^|}{\varkappa\left(1+H^2\right)^{3/2}}$ is a constant function, that is δ is a general helix. Conversely, we assume that δ is a general helix then we can see easily that α is a slant helix. These complete the proof. □

Corollary 4.8. *Let α be an arc-lengthed regular curve with the Frenet vector fields $\{T, N, B\}$ in the Lie group G and δ be the binormal indicatrix of the curve α. Then $\tau_{G_\delta} = \tau_G$ for the curves α and δ.*

Proof. It is obvious using the Eqs. (4.10), (4.11) and (4.12). □

4.4. Involutes of slant helices

Definition 4.9. Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed regular curve. Then the curve $x : I^* \subset \mathbb{R} \to G$ is called the involute of the curve α if the tangent vector field of the curve α is perpendicular to the tangent vector field of the curve x. That is,

$$\langle T(s), T_x(s^*)\rangle = 0,$$

where T and T_x are the tangent vector fields of the curves α and x, respectively. Moreover (x, α) is called the involute-evolute curve couple, which are given by (I, α) and (I^*, x) coordinate neighbourhoods, respectively. Then the distance between the curves x and α are given by

$$d_L(\alpha(s), x(s)) = |c - s|, \quad c = \text{constant } \forall s \in I,$$

[2]. We should remark that the parameter s generally is not an arc-length parameter of x. So, we define the arc-length parameter of the curve x by

$$s^* = \psi(s) = \int_0^s \left\|\frac{dx(s)}{ds}\right\| ds,$$

where $\psi : I \longrightarrow I^*$ is a smooth function and holds the following equality

$$\psi'(s) = (c - s)\varkappa \tag{4.13}$$

for $s \in I$.

Theorem 4.10. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed regular curve and x be an involute of α. Then α is a slant helix in a three dimensional Lie group if and only if x is a general helix.*

Proof. Let x be the involute of α, then we have

$$x(s) = \alpha(s) + (c - s)T(s), \quad c = \text{constant}.$$

Let us derive both side with respect to s

$$\frac{d\beta}{ds^*}\frac{ds^*}{ds} = (c - s)\dot{T}(s),$$
$$T_x(s^*)\frac{ds^*}{ds} = (c - s)\varkappa N,$$

where s and s^* are arc-parameters of α and x, respectively. Then we calculate as

$$\frac{ds^*}{ds} = \psi'(s) = (c - s)\varkappa.$$

and using this fact we can write

$$T_x(s^*) = N. \tag{4.14}$$

If we differentiate the last equation and use Frenet formulas, then we obtain

$$\varkappa_x N_x(s^*)\frac{ds^*}{ds} = \dot{N} = D_T N - \frac{1}{2}[T, N],$$
$$\varkappa_x N_x(s^*)\varkappa = -\kappa T + \tau B - \frac{1}{2}\langle [T, N], B\rangle B$$

or with the help of the Proposition 3.4, we get

$$\varkappa_x N_x(s^*) = -T + HB,$$

where $\varkappa_x$ is the curvature of x. Hence

$$\varkappa_x = \sqrt{1 + H^2}$$

and

$$N_x(s^*) = -\frac{1}{\sqrt{1+H^2}}T + \frac{H}{\sqrt{1+H^2}}B. \tag{4.15}$$

Then using the Eq. (4.14) and the Eq. (4.15), we have

$$B_x(s^*) = T_x(s^*) \times N_x(s^*) = \frac{H}{\sqrt{1+H^2}}T + \frac{1}{\sqrt{1+H^2}}B. \tag{4.16}$$

Using the differentiation of the last equation and the Proposition 3.4, this implies

$$(\tau_x - \tau_{G_x})N_x(s^*)\frac{ds^*}{ds} = -\frac{H'}{\left(1+H^2\right)^{3/2}}T + \frac{HH'}{\left(1+H^2\right)^{3/2}}B$$

or using the Eq. (4.13), we have

$$(\tau_x - \tau_{G_x})N_\beta(s^*) = -\frac{H'}{\varkappa\left(1+H^2\right)^{3/2}}T + \frac{HH'}{\varkappa\left(1+H^2\right)^{3/2}}B,$$

where $\tau_{G_x} = \frac{1}{2}\langle [T_x, N_x], B_x\rangle$. Thus we compute

$$\tau_x = \frac{H^{|}}{\varkappa\left(1+H^2\right)} + \tau_{G_x},$$

where τ_x is the torsion of x. The we can easily see that $\frac{\tau_x - \tau_{G_x}}{\varkappa_x} = \frac{H^{|}}{\varkappa\left(1+H^2\right)^{3/2}}$ is a constant function. In other words, using the Theorem 2.3, x is a general helix. Conversely, we assume that x is a general helix then we can easily see that α is a slant helix. These complete the proof. □

Corollary 4.11. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed regular curve and $\beta : I \subset \mathbb{R} \to S^2 \subset \mathfrak{g}$ be the tangent indicatrix of the curve α. If α is a slant helix, then β is one of the involutes of the curve α.*

Proof. It is obvious from the Theorem 4.2 and the Theorem 4.10. □

Corollary 4.12. *Let $\alpha : I \subset \mathbb{R} \to G$ be an arc-lengthed regular curve and $\delta : I \subset \mathbb{R} \to S^2 \subset \mathfrak{g}$ be the binormal indicatrix of the curve α. If α is a slant helix, then δ is one of the involutes of the curve α.*

Proof. It is obvious from the Theorem 4.7 and the Theorem 4.10. □

Corollary 4.13. *Let α be an arc-lengthed regular curve with the Frenet vector fields $\{T, N, B\}$ in the Lie group G and x be the involute of the curve α. Then $\tau_{G_x} = \tau_G$ for the curves α and x.*

Proof. It is obvious using the Eqs. (4.14)–(4.16). □

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