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On the Fundamental Forms of the B-scroll with Null Directrix and Cartan Frame in Minkowskian 3-Space

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Abstract

In this work we will study on Minkowskian $3-space$, with *null* directrix and Cartan frame.In this paper, we investigate the first fundamental form, the second fundamental form of B-scroll with Cartan framed null directrix in the Minkowskian 3-space. The third fundamental form of B-scroll with Cartan framed null directrix are examined in Minkowskian 3-space. Also we have that the ruled surface $B-scroll$ is minimal surface under the condition $-k_1 - u\dot{k}_2 + u^2k_2^3 = 0$.

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1 Introduction and Preliminaries

In a semi-Riemannian manifold, spacelike, timelike, and null or lightlike curves are three families of curves, according to their causal characters. In the case of null curves, many different situations appear compared with the cases of spacelike and timelike curves. In this work we will study the fundamental forms of the $B-scroll$ with *null* directrix and Cartan frame in the Minkowskian $3-space$, which is introduced by Graves in [4].

The set, whose elements are frame vectors and curvatures of a curve α, is called Frenet-Serret apparatus of the curves. Let Frenet vector fields be V_1, V_2, V_3 of the curve α and let the first and second curvatures of the curve α be k_1 and k_2, respectively. The quantities $\{V_1, V_2, V_3, k_1, k_2\}$ are collectively Frenet-Serret apparatus of the curves. It is well known that the null frame, also called Cartan Frame, of a null curve $\eta(t)$ parametrized by natural parametrization, is a frame field $\{V_1, V_2, V_3\}$, having properties

$$\langle V_1, V_1\rangle = \langle V_2, V_2\rangle = 0 \text{ and } \langle V_3, V_3\rangle = 1,$$
$$\langle V_1, V_3\rangle = \langle V_2, V_3\rangle = 0 \text{ and } \langle V_1, V_2\rangle = -1.$$

Here the Lorentzian inner product is defined by

$$\left\langle \vec{a}, \vec{b} \right\rangle = a_1 b_1 + a_2 b_2 - a_3 b_3, \quad \vec{a}, \vec{b} \in IR^3$$

[14] and

$$\begin{aligned} V_1 \times V_2 &= V_3, \\ V_1 \times V_3 &= V_1, \\ V_3 \times V_2 &= V_2, \\ \det(V_1, V_2, V_3) &= 1. \end{aligned}$$

Then, this curve is a Cartan framed null curve with constant k_1 and k_2 curvatures. And the matrix form of the derivatives of the null frame is

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & k_1 \\ 0 & 0 & k_2 \\ k_2 & k_1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

The infinitesimal displacement of the derivatives of null frame is given as

$$\begin{aligned} \dot{\eta}(t) &= V_1 \text{ and} \\ \dot{V}_1 &= k_1 V_3 \\ \dot{V}_2 &= k_2 V_3 \\ \dot{V}_3 &= k_2 V_1 + k_1 V_2 \end{aligned}$$

where V_1 is the tangent null vector field of the curve, V_2 is the pirincipal normal vector field and V_3 is the binormal vector field. The functions $k_1(t)$ and $k_2(t)$ are the curvature and torsion of the curve $\eta(t)$, recpectively.

A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. A ruled surface is one which can be generated by the motion of a straight line in Euclidean $3-space$ (see [1], [5]). Choosing a directrix on the surface, i.e. a smooth unit speed curve $\alpha(s)$ orthogonal to the straight lines, and then choosing $v(s)$ to be unit vectors along the curve in the direction of the lines, the velocity vector α_s and v satisfy $\langle \alpha', v \rangle = 0$. To illustrate the current situation, we bring here the famous example of L. K. Graves (see [4]), so called the $B-scroll$. The special ruled surfaces $B-scroll$ over null curves with null rulings in 3-dimensional Lorentzian space form has been introduced by L. K. Graves. The Gauss map of B-scrolls has been examined in [3]. In [6] there are applications of ruled surface.The properties of the $B-scroll$ are also examined in Euclidean $3-space$ and $n-space$ and in Lorentzian $3-space$ and $n-space$ with time-like directrix curve and null rulings (see [10], [11]). Also *involutive* $B-scroll$ (binormal scroll) of the curve α is defined as in the following definition and examined in [12]. Also ruled surface with Bishop Frame is studied in [13]. In [8], [9], the differential geometric elements of the Involute $\tilde{\text{D}}$-scroll is examined too.

Definition 1.1 *In Minkowskian $3-space$, let $\eta(t)$ be a null curve.*

$$\varphi(t,u) = \eta(t) + uV_2(t)$$

is the parametrization of the ruled surface which is called $B-scroll$ with the Cartan frame defined above [4]. The directrix of this $B-scroll$ is the null curve $\eta(t)$. The generating space of $B-scroll$ is spaned by normal subvector V_2. Here

$$Sp\{V_1, V_3\}$$

is the rectifian plane of $\eta(t)$.

2 Fundamental forms of the B-scroll with null directrix and Cartan frame

It is well known that the fundamental forms of a surface characterize the basic intrinsic properties of the surface and the way it is located in space in a neighbourhood of a given point; one usually singles out the so-called first, second and third fundamental forms.The first and the second fundamental forms define two important common scalar quantities which are invariant under a transformation of the coordinates on the surface [7].

2.1 The first fundamental form of the B-scroll with null directrix and Cartan frame in Minkowskian 3-space

In this section we will examine the first fundemantal form of the special ruled surface, already defined. The first fundamental form characterizes the interior geometry of the surface in a neighbourhood of a given point M. Suppose that the surface is given by the equation $\varphi = \varphi(t,u)$;where t and u are parametres of the surface; and

$$d\varphi \ = \varphi_t dt + \varphi_u du.$$

is the differential of the radius vector of φ along a chosen direction from a point M to an infinitesimally close point M'. The principal linear part of growth of the arc length $MM\,'$is expressed by the square of $d\varphi$,[7].The form I is the first fundamental form of the surface and

$$I = \langle d\varphi, d\varphi \rangle \, .$$

Theorem 2.1 *In Minkowskian* $3-space$ *the first fundamental form of the* $B-scroll$ *with null directrix and Cartan frame is denoted by* I *and it is given by*

$$I = u^2 k_2^2 dtdt - 2dtdu$$

Proof. In Minkowskian $3-space$, if $\varphi(t,u) = \eta(t) + uV_2(t)$ is the parametrization of the $B-scroll$ with space-like directrix and cartan frame , then φ_t and φ_u are the ortogonal tangent vectors at the point $\eta(t)$. That is

$$\chi(M) = Sp\left\{\varphi_t, \varphi_u\right\}$$

We know that the fundamental form I is

$$I = \ \langle d\varphi, d\varphi \rangle \, .$$

Here

$$d\varphi = \left(V_1 + uk_2V_3\right)\, dt + V_2\, du$$

If we suppose $uk_2 > 0$, then

$$I = u^2 k_2^2 dtdt - 2dtdu.$$

■

Corollary 2.1 *In Minkowskian* $3-space$, *if the second curvature* k_2 *of the null directrix curve* $\eta(t)$ *is constant but not equal to zero, then* $\dot{k}_2 = 0$. *Then we have*

$$I = u^2 k_2^2 dtdt - 2dtdu.$$

Corollary 2.2 *In Minkowskian* $3-space$, *if the second curvature* k_2 *of the null directrix curve* $\eta(t)$ *is equal to zero, that is* $\eta(t)$ *is a planar curve. Then we have*

$$I = -2dtdt.$$

2.2 The second fundamental form of the B-scroll with null directrix and Cartan frame in Minkowskian 3-space

In this section we will examine the second fundemantal form of the $B-scroll$, already defined.The second fundamental form characterizes the local structure of the surface in a neighbourhood of a regular point. Let N be the unit normal vector of the surface at the point M.The doubled principal linear part 2ρ, [7] of the deviation of the point M' on the surface from the tangent plane at the point M is given by,

$$II = \langle d\varphi, dN\rangle = 2\rho$$

Theorem 2.2 *In Minkowskian $3-space$, the unit normal vector field of the $B-scroll$ with null directrix and Cartan frame is denoted by N and it is given by*

$$N = uk_2V_2 + V_3.$$

Proof. In Minkowskian $3-space$, it is well known that

$$N = \frac{\varphi_t \times \varphi_u}{\|\varphi_t \times \varphi_u\|} = \frac{uk_2V_2 + V_3}{\|uk_2V_2 + V_3\|}.$$

Since

$$\|uk_2V_2 + V_3\|^2 = \langle uk_2V_2 + V_3, uk_2V_2 + V_3\rangle = 1,$$

we get the unit normal vector field

$$N = uk_2V_2 + V_3.$$

■

Theorem 2.3 *In Minkowskian $3-space$, the second fundamental form of the $B-scroll$ with null directrix and Cartan frame is denoted by II and it is given by*

$$II = \left(-k_1 - u\dot{k}_2 + u^2k_2^3\right) dtdt - 2k_2dtdu.$$

Proof. Where N is the unit normal vector of the surface at the point M. If we take $uk_2 > 0$, then

$$d\varphi = \varphi_t\, dt + \varphi_u\, du = (V_1 + uk_2V_3)\ dt + V_2du$$

Since we have

$$N = uk_2V_2 + V_3$$

we get

$$dN = N_t\, dt + N_u du = \left(k_2 V_1 + (k_1 + u\dot{k}_2)V_2 + uk_2^2 V_3\right) dt + (k_2 V_2)\, du.$$

Therefore

$$II = \langle d\varphi, dN\rangle = \left(-k_1 - u\dot{k}_2 + u^2 k_2^3\right) dtdt - 2k_2 dtdu.$$

This completes the proof of. ■

Corollary 2.3 *In Minkowskian $3-$ space, if the second curvature k_2 of the null directrix curve $\eta(t)$ is constant but not equal to zero, then $\dot{k}_2 = 0$. Hence we have that*

$$II = \left(-k_1 + u^2 k_2^3\right) dtdt - 2k_2 dtdu.$$

Corollary 2.4 *In Minkowskian $3-$ space, if the second curvature k_2 of the null directrix curve $\eta(t)$ is equal to zero, that is $\eta(t)$ is a planar curve, then*

$$II = -k_1 dtdt.$$

2.3 The third fundamental form of the B-scroll with null directrix and Cartan frame in Minkowskian 3-space

The third fundamental form of a surface is equal to the principal linear part of the growth of the angle between the vectors N and N' under displacement along the surface from the point M to M', [7]. The third fundamental form of the surface is the square of the differential of the unit normal vector N of the surface at the point M which is denoted by III and given by

$$III = \langle dN, dN\rangle\,.$$

Theorem 2.4 *In Minkowskian $3-$ space, the third fundamental Form of the $B-$ scroll with null directrix and cartan frame is denoted by III and*

$$III = \left(-2k_1 k_2 - 2uk_2\dot{k}_2 + u^2 k_2^4\right) dtdt - 2k_2^2 dtdu$$

Proof. *In Minkowskian $3-$space, if $\varphi(t,u) = \eta(t) + uV_2(t)$ is the parametrization of the $B-$ scroll with null directrix and Cartan frame, then φ_t and φ_u are the ortogonal tangent vectors at the point $\eta(t)$. That is $\chi(M) = Sp\left\{\varphi_t, \varphi_u\right\}$. We know that the third fundamental form is denoted by III and given by*

$$III = \langle dN, dN\rangle\,.$$

We have

$$III = \langle N_t, N_t\rangle\, dtdt + 2\left\langle N_t\,, N_u\right\rangle dtdu + \left\langle N_u\,, N_u\,\right\rangle dudu$$

and

$$III = \left\langle k_2V_1 + (k_1 + u\dot{k}_2)V_2 + uk_2^2V_3, k_2V_1 + (k_1 + u\dot{k}_2)V_2 + uk_2^2V_3 \right\rangle dtdt$$
$$+ 2\left\langle k_2V_1 + (k_1 + u\dot{k}_2)V_2 + uk_2^2V_3\, , k_2V_2 \right\rangle dtdu + \langle k_2V_2\, , k_2V_2 \rangle dudu$$
$$III = \left(-2k_1k_2 - 2uk_2\dot{k}_2 + u^2k_2^4\right) dtdt - 2k_2^2 dtdu.$$

■

Corollary 2.5 *In Minkowskian* $3-space$*, if the second curvature* k_2 *of the null directrix curve* $\eta(t)$ *is constant but not equal to zero, then* $\dot{k}_2 = 0$*. Hence we have*

$$III = \left(-2k_1k_2 + u^2k_2^4\right) dtdt - 2k_2^2 dtdu.$$

Corollary 2.6 *In Minkowskian* $3-space$*, if the second curvature* k_2 *of the null directrix curve* $\eta(t)$ *is equal to zero, that is* $\eta(t)$ *is a planar curve, then*

$$III = 0.$$

Theorem 2.5 *In Minkowskian* $3-space$*, the matrix corresponding to the shape operator* S *of* $B-scroll$ *is*

$$S = \begin{bmatrix} -k_1 - u\dot{k}_2 + u^2k_2^3 & -k_2 \\ -k_2 & 0 \end{bmatrix}.$$

Proof. In Minkowskian $3-space$, the normal vector of $B-scroll$ [4] is $N = uk_2V_2 + V_3$. The derivatives of N are

$$N_t = k_2V_1 + (k_1 + u\dot{k}_2)V_2 + uk_2^2V_3, \;\; N_u = k_2V_2.$$

$$\langle N_t, \varphi_t \rangle = \left\langle k_2V_1 + (k_1 + u\dot{k}_2)V_2 + uk_2^2V_3,\; V_1 + uk_2V_3 \right\rangle = -k_1 - u\dot{k}_2 + u^2k_2^3,$$
$$\langle N_t, \varphi_u \rangle = \left\langle k_2V_1 + (k_1 + u\dot{k}_2)V_2 + uk_2^2V_3, \;\; V_2 \right\rangle = -k_2$$

and

$$\langle N_u, \varphi_t \rangle = \langle k_2V_2, \quad V_1 + uk_2V_3 \rangle = -k_2,$$
$$\langle N_u, \varphi_u \rangle = \langle k_2V_2, V_2 \rangle = 0.$$

Hence

$$S = \begin{bmatrix} -k_1 - u\dot{k}_2 + u^2k_2^3 & -k_2 \\ -k_2 & 0 \end{bmatrix}$$

is the matrix corresponding to the shape operator. ■

Theorem 2.6 ***(Gaussian curvature and Mean curvature)*** *In the Minkowskian $3-space$, the Gaussian curvature and the mean curvature of $B-scroll$ are denoted by K and H, respectively. The values of K and H are*

$$K = k_2^2,$$
$$H = \frac{\left(-k_1 - u\dot{k}_2 + u^2 k_2^3\right)}{2}.$$

Proof. *It can be easily obtained by the following equations:*

$$K = -\det(S) = k_2^2,$$
$$H = \frac{Tr\,(S)}{2} = \frac{\left(-k_1 - u\dot{k}_2 + u^2 k_2^3\right)}{2}.$$

Theorem 2.7 *The ruled surface $B-scroll$ is developable if curve is planar planar.*

Proof. Gaussian curvature of the $B-scroll$ is $K = k_2^2 = 0$; which means that $k_2 = 0$. ■

Theorem 2.8 *The ruled surface $B-scroll$ is minimal surface under the condition*

$$u\frac{dk_2}{ds} - u^2 k_2^3\,(s) + k_1\,(s) = 0.$$

Proof. Minimal surfaces are classically defined as surfaces of zero mean curvature H of the ruled surface $B-scroll$. With the condition $H = 0$, the solution set of the differantial eqution $-k_1 - u\dot{k}_2 + u^2 k_2^3 = 0$ gives us the proof. Physically this means that for a given boundary a minimal surface can not be changed without increasing the area of the surface. ■

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