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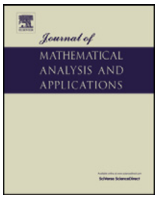

On the B-difference sequence space derived by generalized weighted mean and compact operators

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ABSTRACT

In the present paper, by using generalized weighted mean and difference matrix B, we introduce the paranormed sequence space $\ell(u, v, p; B)$ which consist of all sequences whose generalized weighted B-difference means are in the linear space $\ell(p)$ introduced by I.J. Maddox. Also, we give the basis of this space and compute its α-, β- and γ-duals. Further, we characterize the classes of matrix mappings from $\ell(u, v, p; B)$ to ℓ_∞, c and c_0. Finally, we apply the Hausdorff measure of noncompactness to characterize some classes of compact operators given by matrices on the space $\ell_p(u, v; B)$ $(1 \leqslant p < \infty)$.

1. Introduction

Let w be the space of real sequences. Any vector subspace of w is called as a sequence space. By ℓ_∞, c, c_0, ℓ_1 and ℓ_p $(1 < p < \infty)$, we denote the sequence spaces of all bounded, convergent, null sequences, all sequences which form absolutely and p-absolutely convergent series, respectively.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. We write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Then, we say that A defines a matrix mapping from X into Y and we denote it by writing $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$, the A-transform of x, is in Y; where

$$A_n(x) = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

By (X, Y), we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$ and we have $Ax \in Y$ for all $x \in X$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

The matrix domain X_A of an infinite matrix A in sequence space X is defined by

$$X_A = \{x = (x_k) \in w \colon Ax \in X\} \tag{1.2}$$

which is a sequence space.

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The linear spaces $\ell(p)$ and $\ell_\infty(p)$ were defined by Maddox [1] (see also Simons [2] and Nakano [3]) as follows:

$$\ell(p)=\left\{x=(x_k)\in w\colon \sum_k |x_k|^{p_k}<\infty\right\}$$

and

$$\ell_\infty(p)=\left\{x=(x_k)\in w\colon \sup_{k\in\mathbb{N}}|x_k|^{p_k}<\infty\right\},$$

which are complete spaces paranormed by

$$g_1(x)=\left(\sum_k |x_k|^{p_k}\right)^{1/M}<\infty \quad\text{and}\quad g_2(x)=\sup_{k\in\mathbb{N}}|x_k|^{p_k/M} \quad \text{iff } \inf p_k>0,$$

respectively, where (p_k) is a bounded sequence of strictly positive real numbers with $\sup p_k=H$ and $M=\max\{1,H\}$.

Recently, several authors have defined the new sequence spaces which are matrix domains and contained in $\ell(p)$ or ℓ_p. They introduced the sequence spaces $X_p=(\ell_p)_{C_1}$ in [4], $r^t(p)=(\ell(p))_{R^t}$ in [5], $e^r_p=(\ell_p)_{E^r}$ and $e^r(p)=(\ell(p))_{E^r}$ in [6–8], $Z(u,v,\ell_p)=(\ell_p)_{G(u,v)}$ and $\ell(u,v;p)=(\ell(p))_{G(u,v)}$ in [9,10], $a^r_p=(\ell_p)_{A^r}$ and $a^r(u,p)=(\ell(p))_{A^r_u}$ in [11,12], $bv_p=(\ell_p)_\Delta$ and $bv(u,p)=(\ell(p))_{A^u}$ in [13–15], $\overline{\ell(p)}=(\ell(p))_S$ in [16], $\ell^\lambda_p=(\ell_p)_\Lambda$ in [17], $\hat{\ell}_p=(\ell_p)_{B(r,s)}$ in [18] and etc., where C_1, R^t, E^r and $G(u,v)$ denote the Cesàro, Riesz, Euler and generalized weighted means, respectively, S is the summation matrix, Δ and $B(r,s)$ denote the band matrices defining the difference and generalized difference operators, respectively, and A^r, A^r_u, A^u and Λ are respectively defined in [11,12,15,17]. On the other hand, by using the generalized weighted mean $G(u,v)$ or generalized difference matrix $B(r,s)$, some new sequence spaces have been studied by several authors. For instance see [9,19,10,20,18,21,22]

In the present paper, we introduce the new sequence spaces derived by generalized weighted mean $G(u,v)$ and generalized difference matrix $B(r,s)$. This paper contains six sections as follows:

In Section 2, we give some notations and basic concepts including BK-space, paranormed space, generalized weighted mean and generalized difference matrix. The sequence spaces $\ell(u,v,p;B)$ and $\ell_p(u,v;B)$ have been defined in Section 3 and we prove that the spaces $\ell(u,v,p;B)$ and $\ell(p)$ are linearly isomorphic. Also, we construct the basis of the space $\ell(u,v,p;B)$. In Section 4, we determine the α-, β- and γ-duals of the space $\ell(u,v,p;B)$ which are used to find the necessary and sufficient conditions for matrix transformations. We characterize the matrix classes $(\ell(u,v,p;B),\ell_\infty)$, $(\ell(u,v,p;B),c)$ and $(\ell(u,v,p;B),c_0)$ in Section 5. Finally, we devote the last section of the paper to the characterizations of some classes of compact operators given by infinite matrices from $\ell_p(u,v;B)$ to c_0, c, ℓ_∞ and ℓ_1. Further, we give the necessary and sufficient conditions for $A\in(\ell_1(u,v;B),\ell_p)$ to be compact, where $1\leqslant p<\infty$.

2. Preliminaries and notation

A sequence space X with a linear topology is called a K-space provided each of the maps $p_n:X\to\mathbb{R}$ defined by $p_n(x)=x_n$ is continuous for all $n\in\mathbb{N}$; where $\mathbb{R}$ denotes the real field. A K-space X is called an FK-space provided X is a complete linear metric space. An FK-space whose topology is normable is called a BK-space. The space ℓ_p $(1\leqslant p<\infty)$ is a BK-space with $\|x\|_p=(\sum_{k=0}^\infty |x_k|^p)^{1/p}$ and c_0, c and ℓ_∞ are BK-spaces with $\|x\|_\infty=\sup_k|x_k|$.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g:X\to\mathbb{R}$ such that $g(\theta)=0$, $g(x)=g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n-\alpha|\to 0$ and $g(x_n-x)\to 0$ imply $g(\alpha_n x_n-\alpha x)\to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X. Assume here and after that (p_k) is a bounded sequence of strictly positive real numbers with $\sup p_k=H$, $M=\max\{1,H\}$ and $p_k^{-1}+(p'_k)^{-1}=1$ provided $1<\inf p_k\leqslant H<\infty$. Moreover, by $\mathcal{F}_r$ $(r\in\mathbb{N})$, we denote the subcollection of $\mathcal{F}$ consisting of all nonempty and finite subsets of $\mathbb{N}$ with elements that are greater than r, that is

$$\mathcal{F}_r=\{N\in\mathcal{F}\colon n>r \text{ for all } n\in N\}\quad (r\in\mathbb{N}).$$

By U, we denote for the set of all sequences $u=(u_n)$ such that $u_n\neq 0$ for all $n\in\mathbb{N}$. For $u\in U$, let $1/u=(1/u_n)$. Let $u,v\in U$ and define the matrix $G(u,v)=(g_{nk})$ by

$$g_{nk}=\begin{cases} u_n v_k & (0\leqslant k\leqslant n),\\ 0 & (k>n)\end{cases}$$

for all $k,n\in\mathbb{N}$, where u_n depends only on n and v_k only on k. The matrix $G(u,v)$, defined above, is called as generalized weighted mean or factorable matrix.

Altay and Başar [23] introduced the generalized difference matrix $B(r,s)=(b_{nk})$ by

$$b_{nk}=\begin{cases} r & (k=n),\\ s & (k=n-1),\\ 0 & (0\leqslant k<n-1) \text{ or } (k>n)\end{cases}$$

for all $k,n\in\mathbb{N}$, $r,s\in\mathbb{R}-\{0\}$. If we take $r=1$, $s=-1$ in this matrix $B(r,s)$ then we obtain the difference matrix Δ.

We assume throughout this paper that $u=(u_k)$, $v=(v_k)\in U$ and $r,s\in\mathbb{R}-\{0\}$. Also, we shall write for brevity that

$$R=R(u,v,B)=G(u,v).B(r,s)$$

and

$$\bar{\Delta}(j,k)=(-1)^{j-k}\left(\frac{s^{j-k}}{r^{j-k+1}v_k}+\frac{s^{j-k-1}}{r^{j-k}v_{k+1}}\right)\quad (j,k\in\mathbb{N}). \tag{2.1}$$

3. The paranormed sequence space $\ell(u,v,p;B)$

Now, we define the sequence space $\ell(u,v,p;B)$ by

$$\ell(u,v,p;B)=\{x=(x_k)\in w\colon\ y=(y_k)\in\ell(p)\},$$

where the sequence $y=(y_k)$, which will be frequently used, by the $R=R(u,v,B)$-transform of a sequence $x=(x_k)$, i.e.,

$$y_0=ru_0v_0\quad\text{and}\quad y_k=u_k\left(\sum_{j=0}^{k-1}(rv_j+sv_{j+1})x_j+rv_kx_k\right)\quad\text{for } k\geqslant 1. \tag{3.1}$$

If $p_k=p$ $(1\leqslant p<\infty)$ for every $k\in\mathbb{N}$, then we write $\ell_p(u,v;B)$ instead of $\ell(u,v,p;B)$. It is natural that the spaces $\ell(u,v,p;B)$ and $\ell_p(u,v;B)$ may also be defined with the notation of (1.2) that

$$\ell(u,v,p;B)=\big(\ell(p)\big)_R\quad\text{and}\quad \ell_p(u,v;B)=(\ell_p)_R.$$

This definition includes the following special cases:

i) If $r=1$ and $s=-1$, then $\ell(u,v,p;B)=\ell(u,v,p;\Delta)$.

ii) If $\lambda=(\lambda_k)$ is a strictly increasing sequence of positive reals tending to infinity, $v=(\lambda_k-\lambda_{k-1})$ and $u=(1/\lambda_n)$, then $\ell_p(u,v;B)=\ell_p^{\lambda}(B)$ and $\ell_p(u,v;B)=\ell_p^{\lambda}(\Delta)$ with $r=1$ and $s=-1$, where ℓ_p^{λ} is the sequence space defined by Mursaleen and Noman in [17] and $1\leqslant p<\infty$.

iii) If $v=(1+r^k)$, $u=(1/(n+1))$, $r=1$ and $s=-1$ then $\ell_p(u,v;B)=a_p^r(\Delta)$ (cf. [24]).

iv) If $v=(q_k)$ positive a sequence, $u=(1/Q_n)$ with $Q_n=\sum_{k=1}^{n}q_k$ $(n\in\mathbb{N})$, then $\ell(u,v,p;B)=r^q(p,B)$ (cf. [21]) and $\ell(u,v,p;B)=r^q(p,\Delta)$ with $r=1$ and $s=-1$ (cf. [25]).

v) If $v=e$, $u=(1/n)$, $r=1$ and $s=-1$, then $\ell_p(u,v;B)=X_p(\Delta)$ (cf. [26]).

We shall assume throughout the paper that the sequences $x=(x_k)$ and $y=(y_k)$ are connected by the relation (3.1).

Now, we may begin with the following theorem which is essential in the study.

Theorem 3.1. (a) *$\ell(u,v,p;B)$ is the complete linear metric space paranormed by h, defined by*

$$h(x)=\left(\sum_k\left|\sum_{j=0}^{k-1}u_k(rv_j+sv_{j+1})x_j+ru_kv_kx_k\right|^{p_k}\right)^{1/M}.$$

(b) *Let $1\leqslant p<\infty$. Then, $\ell_p(u,v;B)$ is the BK-space with the norm*

$$\|x\|_{\ell_p(u,v;B)}=\|y\|_{\ell_p}.$$

Proof. We prove only part (a) and omit the proof of part (b). The linearity of $\ell(u,v,p;B)$ with respect to the coordinate wise addition and scalar multiplication follows from the following inequalities which are satisfied for $z,x\in\ell(u,v,p;B)$ (see [27, p. 30]):

$$\begin{aligned}&\left(\sum_k\left|\sum_{j=0}^{k-1}u_k(rv_j+sv_{j+1})(z_j+x_j)+ru_kv_k(z_k+x_k)\right|^{p_k}\right)^{1/M}\\ &\quad\leqslant\left(\sum_k\left|\sum_{j=0}^{k-1}u_k(rv_j+sv_{j+1})z_j+ru_kv_kz_k\right|^{p_k}\right)^{1/M}+\left(\sum_k\left|\sum_{j=0}^{k-1}u_k(rv_j+sv_{j+1})x_j+ru_kv_kx_k\right|^{p_k}\right)^{1/M}\end{aligned} \tag{3.2}$$

and for any $\alpha \in \mathbb{R}$ (see [28])

$$|\alpha|^{p_k} \leqslant \max\{1, |\alpha|^M\}. \tag{3.3}$$

It is clear that $h(\theta) = 0$ and $h(x) = h(-x)$ for all $x \in \ell(u, v, p; B)$. Again the inequalities (3.2) and (3.3) yield the subadditivity of h and

$$h(\alpha x) \leqslant \max\{1, |\alpha|\} h(x).$$

Let (x^n) be any sequence of the points in $\ell(u, v, p; B)$ such that $h(x^n - x) \to 0$ and (α_n) also be any sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality

$$h(x^n) \leqslant h(x) + h(x^n - x)$$

holds by subadditivity of h, $\{h(x^n)\}$ is bounded and we thus have

$$\begin{aligned} h(\alpha_n x^n - \alpha x) &= \left(\sum_k \left| \sum_{j=0}^{k-1} u_k (r v_j + s v_{j+1}) (\alpha_n x_j^{(n)} - \alpha x_j) + r u_k v_k (\alpha_n x_k^{(n)} - \alpha x_k) \right|^{p_k} \right)^{1/M} \\ &\leqslant |\alpha_n - \alpha| h(x^n) + |\alpha| h(x^n - x) \end{aligned}$$

which tends to zero as $n \to \infty$. That is to say that the scalar multiplication is continuous. Hence, h is a paranorm on the space $\ell(u, v, p; B)$.

It remains to prove the completeness of the space $\ell(u, v, p; B)$. Let (x^i) be any Cauchy sequence in the space $\ell(u, v, p; B)$, where $x^i = (x_0^i, x_1^i, x_2^i, \ldots)$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$h(x^i - x^j) < \varepsilon \tag{3.4}$$

for all $i, j \geqslant n_0(\varepsilon)$. We obtain by using definition of h for each fixed $k \in \mathbb{N}$

$$\left| \{Rx^i\}_k - \{Rx^j\}_k \right| \leqslant \left(\sum_k \left| \{Rx^i\}_k - \{Rx^j\}_k \right|^{p_k} \right)^{1/M} < \varepsilon \tag{3.5}$$

for every $i, j \geqslant n_0(\varepsilon)$, which leads us to the fact that $\{(Rx^0)_k, (Rx^1)_k, (Rx^2)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(Rx^i)_k \to (Rx)_k$ as $i \to \infty$. Using these infinitely many limits $(Rx)_0$, $(Rx)_1$, $(Rx)_2$, $\ldots$ we define the sequence $\{(Rx)_0, (Rx)_1, (Rx)_2, \ldots\}$. We have from (3.5) for each $m \in \mathbb{N}$ and $i, j \geqslant n_0(\varepsilon)$ that

$$\sum_{k=0}^{m} \left| (Rx^i)_k - (Rx^j)_k \right|^{p_k} \leqslant h(x^i - x^j)^M < \varepsilon^M. \tag{3.6}$$

Take any $i \geqslant n_0(\varepsilon)$. Let us pass to limit first as $j \to \infty$ and next as $m \to \infty$ in (3.6) to obtain $h(x^i - x) \leqslant \varepsilon$. Finally, taking $\varepsilon = 1$ in (3.6) and letting $i \geqslant n_0(1)$ we have by Minkowski's inequality for each $m \in \mathbb{N}$ that

$$\left(\sum_{k=0}^{m} |(Rx)_k|^{p_k} \right)^{1/M} \leqslant h(x^i - x) + h(x^i) \leqslant 1 + h(x^i) \tag{3.7}$$

which implies that $x \in \ell(u, v, p; B)$. Since $h(x^i - x) \leqslant \varepsilon$ for all $i \geqslant n_0(\varepsilon)$ it follows that $x^i \to x$ as $i \to \infty$. Since (x^i) was an arbitrary Cauchy sequence, the space $\ell(u, v, p; B)$ is complete and this concludes the proof. □

Theorem 3.2. *The sequence space $\ell(u, v, p; B)$ is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \leqslant H < \infty$.*

Proof. To prove the theorem, we should show the existence of a linear bijection between the spaces $\ell(u, v, p; B)$ and $\ell(p)$ for $0 < p_k \leqslant H < \infty$. For this goal, consider the transformation T defined, with the notation of (3.1), from $\ell(u, v, p; B)$ to $\ell(p)$ by $x \to y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y \in \ell(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k-1} \frac{1}{u_j} \bar{\Delta}(k, j) y_j + \frac{1}{r u_k v_k} y_k \quad (k \in \mathbb{N}).$$

Then,

$$h(x) = \left(\sum_k \left| \sum_{j=0}^{k-1} u_k (r v_j + s v_{j+1}) x_j + r u_k v_k x_k \right|^{p_k} \right)^{1/M} = \left(\sum_k |y_k|^{p_k} \right)^{1/M} = g_1(y) < \infty.$$

Thus, we deduce that $x \in \ell(u, v, p; B)$ and consequently T is surjective and is paranorm preserving. Hence, T is linear bijection and this says us that the spaces $\ell(u, v, p; B)$ and $\ell(p)$ are linearly isomorphic. This completes the proof. □

If a sequence space λ paranormed by h_1 contains a sequence (b_k) with the property that for every $x \in \lambda$, there is a unique of scalars (α_k) such that

$$\lim_{n\to\infty} h_1\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) = 0,$$

then (b_k) is called a *Schauder* basis for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_k) and written as $x = \sum \alpha_k b_k$.

Now, we may give the sequence of the points of the space $\ell(u, v, p; B)$ which forms Schauder basis for that space. Because of the isomorphism T between the sequence spaces $\ell(u, v, p; B)$ and $\ell(p)$ is onto, the inverse image of the basis of the space $\ell(p)$ is the basis of the space $\ell(u, v, p; B)$. Therefore, we have:

Theorem 3.3. *Let $\alpha_k = \{Rx\}_k$ and $0 < p_k \leqslant H < \infty$ for all $k \in \mathbb{N}$. Define the sequence $c^{(k)} = \{c_n^{(k)}\}_{n\in\mathbb{N}}$ for every fixed $k \in \mathbb{N}$ by*

$$c_n^{(k)} = \begin{cases} \frac{\bar{\Delta}(n,k)}{u_k} & (0 \leqslant n \leqslant k-1), \\ \frac{1}{r u_k v_k} & (n = k), \\ 0 & (n > k). \end{cases} \tag{3.8}$$

Then, the sequence $\{c^{(k)}\}_{k\in\mathbb{N}}$ is a basis for the sequence $\ell(u, v, p; B)$ and any x in $\ell(u, v, p; B)$ has a unique representation of the form

$$x = \sum_k \alpha_k c^{(k)}.$$

4. The α-, β- and γ-duals of the space $\ell(u, v, p; B)$

For arbitrary sequence spaces X and Y, the set $S(X, Y)$ defined by

$$S(X, Y) = \left\{z = (z_k) \in w \colon xz = (x_k z_k) \in Y \text{ for all } x \in X\right\} \tag{4.1}$$

is called the multiplier space of X and Y. With the notation (4.1), the α-, β-, γ-duals of a sequence space X, which are respectively denoted by X^α, X^β and X^γ are, defined by

$$X^\alpha = S(X, \ell_1), \qquad X^\beta = S(X, cs) \quad \text{and} \quad X^\gamma = S(X, bs),$$

where cs and bs are the sets of all sequences which form convergent and bounded series, respectively.

Now, we give the following lemmas which are needed in proving our theorems.

Lemma 4.1. *(See [29, Theorem 5.1.0 with $q_n = 1$].)* (i) *Let $1 < p_k \leqslant H < \infty$ for every k. Then $A \in (\ell(p), \ell_1)$ if and only if there exists an integer $K > 1$ such that*

$$\sup_{N\in\mathcal{F}} \sum_k \left|\sum_{n\in N} a_{nk} K^{-1}\right|^{p_k'} < \infty. \tag{4.2}$$

(ii) *Let $0 < p_k \leqslant 1$ for every k. Then $A \in (\ell(p), \ell_1)$ if and only if*

$$\sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left|\sum_{n\in N} a_{nk}\right|^{p_k} < \infty. \tag{4.3}$$

Lemma 4.2. *(See [30, Theorem 1(i) and (ii)].)* (i) *Let $1 < p_k \leqslant H < \infty$ for every k. Then $A \in (\ell(p), \ell_\infty)$ if and only if there exists an integer $K > 1$ such that*

$$\sup_{n\in\mathbb{N}} \sum_k \left|a_{nk} K^{-1}\right|^{p_k'} < \infty. \tag{4.4}$$

(ii) *Let $0 < p_k \leqslant 1$ for every k. Then $A \in (\ell(p), \ell_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{4.5}$$

Lemma 4.3. *(See [30, Corollary for Theorem 1].) Let $0 < p_k \leqslant H < \infty$ for every k. Then $A \in (\ell(p), c)$ if and only if there exists an integer $K > 1$ such that* (4.4) *and* (4.5) *hold,*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \quad (k \in \mathbb{N}) \tag{4.6}$$

also holds.

Theorem 4.4. *Let $N_k^* = N \cap \{n \in \mathbb{N}\colon n \geqslant k\}$ for $N \in \mathcal{F}$ and define the sets $d_1(p)$ and $d_2(p)$ as follows:*

$$d_1(p) = \left\{a = (a_n) \in w\colon \sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left|\sum_{n\in N_k^*} c_{nk}\right|^{p_k} < \infty\right\}$$

and

$$d_2(p) = \bigcup_{K>1} \left\{a = (a_n) \in w\colon \sup_{N\in\mathcal{F}} \sum_k \left|\sum_{n\in N_k^*} c_{nk} K^{-1}\right|^{p_k'} < \infty\right\},$$

where the matrix $C = (c_{nk})$ is defined by

$$c_{nk} = \begin{cases} \frac{\bar{\Delta}(n,k)}{u_k} a_n & (0 \leqslant k \leqslant n-1), \\ \frac{a_n}{r u_n v_n} & (k = n), \\ 0 & (k > n). \end{cases}$$

(i) *Let $0 < p_k \leqslant 1$ for all $k \in \mathbb{N}$. Then, $\{\ell(u, v, p; B)\}^\alpha = d_1(p)$.*
(ii) *Let $1 < p_k \leqslant H < \infty$ for all $k \in \mathbb{N}$. Then, $\{\ell(u, v, p; B)\}^\alpha = d_2(p)$.*

Proof. Since the case (i) may be proved by analogy, we give the proof only for the case (ii). Let $a = (a_n) \in w$. We immediately derive with (3.1) that

$$a_n x_n = \sum_{k=0}^{n-1} \frac{\bar{\Delta}(n,k)}{u_k} y_k a_n + \frac{1}{r u_n v_n} y_n a_n = C_n(y) \quad (n \in \mathbb{N}). \tag{4.7}$$

Thus, we observe by (4.7) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in \ell(u, v, p; B)$ if and only if $Cy \in \ell_1$ whenever $y = (y_k) \in \ell(p)$. This means that the sequence $a = (a_n)$ is in the α-dual of the space $\ell(u, v, p; B)$ if and only if $C \in (\ell(p), \ell_1)$. We therefore obtain by Lemma 4.1(i) with C instead of A that $\{\ell(u, v, p; B)\}^\alpha = d_2(p)$. This concludes the proof. □

Theorem 4.5. *Define the sets $d_3(p)$, $d_4(p)$ and $d_5(p)$ as follows:*

$$d_3(p) = \left\{a = (a_n) \in w\colon \sup_{n,k\in\mathbb{N}} \left|\tilde{a}_k(n)\right|^{p_k} < \infty \text{ and } \left(\frac{a_k}{r u_k v_k}\right) \in \ell_\infty(p)\right\},$$

$$d_4(p) = \left\{a = (a_n) \in w\colon \sum_{j=k+1}^{\infty} \bar{\Delta}(j,k) a_j \text{ exists for all } k \in \mathbb{N}\right\}$$

and

$$d_5(p) = \bigcup_{K>1} \left\{a = (a_n) \in w\colon \sup_{n\in\mathbb{N}} \sum_{k=0}^{n-1} \left|\tilde{a}_k(n) K^{-1}\right|^{p_k'} < \infty \text{ and } \left(\frac{a_k}{r u_k v_k} K^{-1}\right) \in \ell_\infty\big(p'\big)\right\},$$

where $\tilde{a}_k(n) = \frac{1}{r u_k v_k} a_k + \frac{1}{u_k} \sum_{j=k+1}^n \bar{\Delta}(j,k) a_j$ $(k < n)$.
(i) *Let $0 < p_k \leqslant 1$ for all $k \in \mathbb{N}$. Then, $\{\ell(u, v, p; B)\}^\beta = d_3(p) \cap d_4(p)$.*
(ii) *Let $1 < p_k \leqslant H < \infty$ for all $k \in \mathbb{N}$. Then, $\{\ell(u, v, p; B)\}^\beta = d_4(p) \cap d_5(p)$.*

Proof. (i) Consider the equation

$$\begin{aligned} \sum_{k=0}^n a_k x_k &= \sum_{k=0}^n a_k \left(\sum_{j=0}^{k-1} \frac{1}{u_j} \bar{\Delta}(k,j) y_j + \frac{1}{r u_k v_k} y_k\right) \\ &= \sum_{k=0}^{n-1} \left(\frac{1}{r u_k v_k} a_k + \frac{1}{u_k} \sum_{j=k+1}^n \bar{\Delta}(j,k) a_j\right) y_k + \frac{a_n}{r u_n v_n} y_n \end{aligned}$$

$$=\sum_{k=0}^{n-1}\tilde{a}_k(n)y_k+\frac{a_n}{ru_nv_n}y_n=D_n(y)\quad(n\in\mathbb{N}),\tag{4.8}$$

where the matrix $D=(d_{nk})$ is defined by

$$d_{nk}=\begin{cases}\tilde{a}_k(n) & (k<n),\\ \frac{a_n}{ru_nv_n} & (k=n),\\ 0 & (k>n)\end{cases}\quad(n,k\in\mathbb{N}).$$

Then, we deduce by (4.8) that $ax=(a_kx_k)\in cs$ whenever $x=(x_k)\in\ell(u,v,p;B)$ if and only if $Dy\in c$ whenever $y=(y_k)\in\ell(p)$. This means that $a=(a_k)\in\{\ell(u,v,p;B)\}^{\beta}$ if and only if $D\in(\ell(p),c)$. Therefore, by using Lemma 4.3 with (4.8), we have that $\{\ell(u,v,p;B)\}^{\beta}=d_3(p)\cap d_4(p)$.

Since proof of the part (ii) may also be obtained in the similar way for the part (i), we leave the detail to reader. □

Theorem 4.6. (i) *Let $0<p_k\leqslant 1$ for all $k\in\mathbb{N}$. Then $\{\ell(u,v,p;B)\}^{\gamma}=d_3(p)$.*

(ii) *Let $1<p_k\leqslant H<\infty$ for all $k\in\mathbb{N}$. Then, $\{\ell(u,v,p;B)\}^{\gamma}=d_5(p)$.*

Proof. We see Lemma 4.2 with (4.8) that $ax=(a_kx_k)\in bs$ whenever $x=(x_k)\in\ell(u,v,p;B)$ if and only if $Dy\in\ell_\infty$ whenever $y=(y_k)\in\ell(p)$. Therefore, we respectively obtain from (4.4) and (4.5) that $\{\ell(u,v,p;B)\}^{\gamma}=d_3(p)$ for $0<p_k\leqslant 1$, $\{\ell(u,v,p;B)\}^{\gamma}=d_5(p)$ for $p_k>1$ and this completes the proof. □

5. Some matrix mappings on the space $\ell(u,v,p;B)$

For an infinite matrix $A=(a_{nk})$, we shall write for brevity that

$$\tilde{a}_{nk}(m)=\frac{1}{ru_kv_k}a_{nk}+\frac{1}{u_k}\sum_{j=k+1}^{m}\bar{\Delta}(j,k)a_{nj}\quad(k<m)\tag{5.1}$$

and

$$\tilde{a}_{nk}=\frac{1}{ru_kv_k}a_{nk}+\frac{1}{u_k}\sum_{j=k+1}^{\infty}\bar{\Delta}(j,k)a_{nj}\tag{5.2}$$

for all $n,k,m\in\mathbb{N}$ provided the convergence of the series.

Now, we give the characterization of the classes $(\ell(u,v,p;B),\ell_\infty)$, $(\ell(u,v,p;B),c)$ and $(\ell(u,v,p;B),c_0)$.

Theorem 5.1. (i) *Let $1<p_k\leqslant H<\infty$ for every $k\in\mathbb{N}$. Then, $A\in(\ell(u,v,p;B),\ell_\infty)$ if and only if there exists an integer $K>1$ such that*

$$C(K)=\sup_{n\in\mathbb{N}}\sum_k\left|\tilde{a}_{nk}K^{-1}\right|^{p_k'}<\infty,\tag{5.3}$$

$$\{a_{nk}\}_{k\in\mathbb{N}}\in d_3(p)\cap d_4(p)\quad(n\in\mathbb{N}).\tag{5.4}$$

(ii) *Let $0<p_k\leqslant 1$ for every $k\in\mathbb{N}$. Then, $A\in(\ell(u,v,p;B),\ell_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}}|\tilde{a}_{nk}|^{p_k}<\infty,\tag{5.5}$$

$$\{a_{nk}\}_{k\in\mathbb{N}}\in d_4(p)\cap d_5(p)\quad(n\in\mathbb{N}).\tag{5.6}$$

Proof. We consider only the case $1<p_k\leqslant H<\infty$ and leave the case $0<p_k\leqslant 1$ to the reader because it may be proved in a similar way.

Assume that conditions (5.3) and (5.4) are satisfied and take any $x=(x_k)\in\ell(u,v,p;B)$. Then, we have by Theorem 4.5(ii) that $\{a_{nk}\}_{k\in\mathbb{N}}\in\{\ell(u,v,p;B)\}^{\beta}$ for every fixed $n\in\mathbb{N}$ and this implies the existence of the A-transform of x, i.e., Ax exists. Let us now consider the following equality derived by using the relation (3.1) from the mth partial sum of the series $\sum_k a_{nk}x_k$:

$$\sum_{k=0}^{m}a_{nk}x_k=\sum_{k=0}^{m-1}\tilde{a}_{nk}(m)y_k+\frac{1}{ru_mv_m}a_{nm}y_m\quad(n,m\in\mathbb{N}).\tag{5.7}$$

Taking into account the hypothesis we derive from (5.7) as $m\to\infty$ that

$$\sum_k a_{nk}x_k = \sum_k \tilde{a}_{nk}y_k \quad (n \in \mathbb{N}). \tag{5.8}$$

Now, by combining (5.8) and the inequality which holds for any $K > 0$ and complex numbers a, b

$$|ab| \leqslant K\left\{\left|aK^{-1}\right|^{p'} + |b|^p\right\}, \tag{5.9}$$

where $p > 1$ and $p^{-1} + p'^{-1} = 1$ (see [30]), one can easily see that

$$\sup_{n\in\mathbb{N}}\left|\sum_k a_{nk}x_k\right| \leqslant \sup_{n\in\mathbb{N}} \sum_k |\tilde{a}_{nk}||y_k| \leqslant K\left[C(K) + g_1^K(y)\right] < \infty.$$

Conversely suppose that $A \in (\ell(u,v,p;B), \ell_\infty)$ and $1 < p_k \leqslant H < \infty$ for every $k \in \mathbb{N}$. Then Ax exists for every $x \in \ell(u,v,p;B)$ and this implies that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\ell(u,v,p;B)\}^\beta$ for each $n \in \mathbb{N}$. Now, the necessity of (5.4) is immediately obtained from Theorem 4.5(ii). Besides, we have from (5.8) that the matrix $\tilde{A} = (\tilde{a}_{nk})$ is in the class $(\ell(p), \ell_\infty)$. Then $\tilde{A}$ satisfies the condition (4.4) which is equivalent to (5.3). This completes the proof. □

Theorem 5.2. *Let* $0 < p_k \leqslant H < \infty$ *for every* $k \in \mathbb{N}$. *Then,* $A \in (\ell(u,v,p;B), c)$ *if and only if* (5.3)–(5.6) *hold, and there is a sequence* $(\tilde{\alpha}_k)$ *of the scalars such that*

$$\lim_{n\to\infty} \tilde{a}_{nk} = \tilde{\alpha}_k \quad (k \in \mathbb{N}). \tag{5.10}$$

Proof. Let $A \in (\ell(u,v,p;B), c)$ and $0 < p_k \leqslant H < \infty$ for every $k \in \mathbb{N}$. Then, since the inclusion $c \subset \ell_\infty$, the necessities of (5.3)–(5.6) are trivial by Theorem 5.1(i) and (ii).

To prove the necessity of (5.10), consider the sequence $c^{(k)}$ defined by (3.8), which is in the space $\ell(u,v,p;B)$ for every fixed $k \in \mathbb{N}$. Because the A-transform of every $x \in \ell(u,v,p;B)$ exists and is in c by the hypothesis, $Ac^{(k)} = \{\tilde{a}_{nk}\}_{n\in\mathbb{N}}$ is also in c for every fixed $k \in \mathbb{N}$, which shows the necessity of (5.10).

Conversely, suppose that the conditions (5.3)–(5.6) and (5.10) hold and take any $x \in \ell(u,v,p;B)$. Then, we have by Theorem 4.5 that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\ell(u,v,p;B)\}^\beta$ for every fixed $n \in \mathbb{N}$ and this implies the existence of the A-transform of x, i.e., Ax exists. Also, it is clear that the associated sequence is in the space $\ell(p)$. Further, it follows by combining Lemma 4.3 with the conditions (5.3) and (5.10) that the matrix $\tilde{A}$ is in the class $(\ell(p), c)$. Thus, we see that $y \in \ell(p)$ and $\tilde{A}y \in c$. Consequently, we obtain from (5.10) that $Ax \in c$ whenever $x \in \ell(u,v,p;B)$ and this is what we wished to prove. □

If the sequence space c is replaced by the space c_0 then Theorem 5.2 is reduced to

Corollary 5.3. *Let* $0 < p_k \leqslant H < \infty$ *for every* $k \in \mathbb{N}$. *Then,* $A \in (\ell(u,v,p;B), c_0)$ *if and only if* (5.3)–(5.6) *hold and* (5.10) *also holds with* $\tilde{\alpha}_k = 0$ *for all* $k \in \mathbb{N}$.

6. The Hausdorff measure of noncompactness and compact operators on the space $\ell_p(u,v;B)$ $(1 \leqslant p < \infty)$

In this section, we characterize classes of compact operators given by infinite matrices from $\ell_p(u,v;B)$ to c_0, c, ℓ_∞ and ℓ_1. Also we give the necessary and sufficient conditions for $A \in (\ell_1(u,v;B), \ell_p)$ to be compact, where $1 \leqslant p < \infty$.

The Hausdorff measure of noncompactness was defined by Goldenštein, Gohberg and Markus in 1957, later studied by Goldenštein and Markus in 1968. It is quite natural to find conditions for a matrix map between BK-spaces to define a compact operator since a matrix transformation between BK-spaces are continuous. This can be achieved by applying the Hausdorff measure of noncompactness. In past, several authors characterized classes of compact operators given by infinite matrices on the some sequence spaces by using this method. For example see [31–40]. Recently, Malkowsky and Rakočević [41], Djolović and Malkowsky [42] and Mursaleen and Noman [43] established some identities or estimates for the operator norms and Hausdorff measures of noncompactness of linear operators given by infinite matrices that map an arbitrary BK-space or the matrix domains of triangles in arbitrary BK-spaces.

Let X be a normed space. Then, we write S_X for the unit sphere in X, that is, $S_X = \{x \in X\colon \|x\| = 1\}$. If X and Y be Banach spaces then $B(X,Y)$ is the set of all continuous linear operators $L : X \to Y$; $B(X,Y)$ is a Banach space with the operator norm defined by $\|L\| = \sup\{\|L(x)\|\colon \|x\| \leqslant 1\}$ for all $L \in B(X,Y)$.

If $(X, \|.\|)$ is a normed sequence space then we write $\|a\|_X^* = \sup_{x\in S_X} |\sum_{k=0}^\infty a_k x_k|$ for $a \in w$ provided the expression on the right-hand side exists and is finite which is the case whenever X is a BK-space and $a \in X^\beta$ [44, Theorem 7.2.9, p. 107].

Throughout, let $1 \leqslant p < \infty$ and q denote the conjugate of p, that is, $q = p/(p-1)$ for $1 < p < \infty$ or $q = \infty$ for $p = 1$. Also, we write ϕ for the set of all finite sequences that terminate in zeros.

The following results are fundamental for our investigation.

Lemma 6.1. *(See [44, Theorem 4.2.8].) Let* X *and* Y *be* BK*-spaces. Then we have* $(X,Y) \subset B(X,Y)$, *that is, every* $A \in (X,Y)$ *defines a linear operator* $L_A \in B(X,Y)$, *where* $L_A(x) = Ax$ *for all* $x \in X$.

Lemma 6.2. *(See [32, Lemma 5.2].) Let $X \supset \phi$ be BK-space and Y be any of the spaces c_0, c or ℓ_∞. If $A \in (X, Y)$, then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

Lemma 6.3. *(See [45, Theorem 1.29(b)].) Let $1 \leqslant p < \infty$. Then, we have $\ell_p^\beta = \ell_q$ and $\|a\|_{\ell_p}^* = \|a\|_{\ell_q}$ for all $a = (a_k) \in \ell_q$.*

Lemma 6.4. *If $a = (a_k) \in (\ell_p(u, v; B))^\beta$, then $\tilde{a} = (\tilde{a}_k) \in \ell_q$ and the equality*

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \tilde{a}_k y_k \tag{6.1}$$

holds for every $x = (x_k) \in \ell_p(u, v; B)$, where

$$\tilde{a}_k = \frac{a_k}{r u_k v_k} + \frac{1}{u_k} \sum_{j=k+1}^{\infty} \bar{\Delta}(j, k) a_j \quad (k \in \mathbb{N}).$$

Proof. This is immediate by [41, Theorem 3.2]. □

On the other hand, let $1 \leqslant p < \infty$. Then, it can easily be shown that the inclusion $\ell_p(u, v; B) \supset \phi$ holds if and only if $u = (u_k) \in \ell_p$. So, we shall assume that $u = (u_k) \in \ell_p$ whenever we study the space $\ell_p(u, v; B)$.

Lemma 6.5. *Let $1 \leqslant p < \infty$ and $\tilde{a} = (\tilde{a}_k)$ be defined as in Lemma 6.4. Then, we have*

$$\|a\|_{\ell_p(u,v;B)}^* = \|\tilde{a}\|_{\ell_q} = \begin{cases} (\sum_{k=0}^{\infty} |\tilde{a}_k|^q)^{1/q} & (1 < p < \infty), \\ \sup_k |\tilde{a}_k| & (p = 1) \end{cases}$$

for all $a = (a_k) \in (\ell_p(u, v; B))^\beta$.

Proof. Let $a = (a_k) \in (\ell_p(u, v; B))^\beta$. Then, we have from Lemma 6.4 that $\tilde{a} = (\tilde{a}_k) \in \ell_q$ and the equality (6.1) holds for all sequences $x = (x_k) \in \ell_p(u, v; B)$ and $y = (y_k) \in \ell_p$ which are connected by the relation (3.1). Also, we can write by Theorem 3.1(ii) that $x \in S_{\ell_p(u,v;B)}$ if and only if $y \in S_{\ell_p}$. Thus, we have from (6.1) that

$$\|a\|_{\ell_p(u,v;B)}^* = \sup_{x \in S_{\ell_p(u,v;B)}} \left| \sum_{k=0}^{\infty} a_k x_k \right| = \sup_{y \in S_{\ell_p}} \left| \sum_{k=0}^{\infty} \tilde{a}_k y_k \right| = \|\tilde{a}\|_{\ell_p}^*. \tag{6.2}$$

Further, since $\tilde{a} \in \ell_q$, we get by Lemma 6.3 and (6.2) that

$$\|a\|_{\ell_p(u,v;B)}^* = \|\tilde{a}\|_{\ell_p}^* = \|\tilde{a}\|_{\ell_q} < \infty$$

which concludes the proof. □

Lemma 6.6. *Let X be a sequence space, $A = (a_{nk})$ an infinite matrix and $1 \leqslant p < \infty$. If $A \in (\ell_p(u, v; B), X)$, then $\tilde{A} \in (\ell_p, X)$ such that $Ax = \tilde{A}y$ for all $x \in \ell_p(u, v; B)$ and $y \in \ell_p$, where the sequences x and y are connected by the relation (3.1) and $\tilde{A} = (\tilde{a}_{nk})$ is defined as (5.2).*

Proof. Let $x \in \ell_p(u, v; B)$ and $A \in (\ell_p(u, v; B), X)$. Then, $A_n \in (\ell_p(u, v; B))^\beta$ for all $n \in \mathbb{N}$, where A_n is the sequence in the nth row of A. Thus, it follows by Lemma 6.4 that $\tilde{A}_n \in \ell_p^\beta = \ell_q$ for all $n \in \mathbb{N}$ and the equality $Ax = \tilde{A}y$ holds. Hence, $\tilde{A}y \in X$. Since every $y \in \ell_p$ is the associated sequence of $x \in \ell_p(u, v; B)$, we obtain that $\tilde{A} \in (\ell_p, X)$. This completes the proof. □

Lemma 6.7. *(See [36, Lemma 3.1].) Let $1 \leqslant p < \infty$ and q be the conjugate of p. IfA $\in (\ell_p, c)$, then the followings hold:*

$$\alpha_k = \lim_{n\to\infty} a_{nk} \quad \text{exists for every } k \in \mathbb{N},$$

$$\alpha = (\alpha_k) \in \ell_q,$$

$$\sup_n \|A_n - \alpha\|_{\ell_q} < \infty,$$

$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k \quad \text{for all } x = (x_k) \in \ell_p.$$

We recall that if X and Y are Banach spaces and L is a linear operator from X to Y, then L is said to be compact if its domain is all of X and for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. We denote the class of such operators by $K(X,Y)$.

If (X,d) is a metric space, we write M_X for the class of all bounded subsets of X. By $B(x,r)=\{y\in X\colon\ d(x,y)<r\}$, we denote the open ball of radius $r>0$ with centre in x. Then the Hausdorff measure of noncompactness of the set $Q\in M_X$, denoted by $\chi(Q)$, is given by

$$\chi(Q)=\inf\left\{\varepsilon>0\colon\ Q\subset\bigcup_{i=0}^{n}B(x_i,r_i),\ x_i\in X,\ r_i<\varepsilon\ (i=0,1,\ldots,n),\ n\in\mathbb{N}\right\}.$$

The function $\chi : M_X\to[0,\infty)$ is called the Hausdorff measure of noncompactness.

The basic properties of the Hausdorff measure of noncompactness can be found in [45],for example if Q, Q_1 and Q_2 are bounded subsets of a metric space (X,d), then

$$\chi(Q)=0\quad\text{if and only if}\quad Q\text{ is totally bounded},$$
$$Q_1\subset Q_2\quad\text{implies }\chi(Q_1)\leqslant\chi(Q_2).$$

Further if X is a normed space, then the function χ has some additional properties connected with the linear structure, e.g.

$$\chi(Q_1+Q_2)\leqslant\chi(Q_1)+\chi(Q_2),$$
$$\chi(\alpha Q)=|\alpha|\chi(Q)\quad\text{for all }\alpha\in\mathbb{C},$$

where $\mathbb{C}$ is the complex field.

Lemma 6.8. *(See [36, Theorem 1.6].) Let X be a Banach space with a Schauder basis $(b_k)_{k=0}^{\infty}$, $Q\in M_X$ and $P_n=X\to X$ $(n\in\mathbb{N})$ be the projector onto the linear span of $\{b_0,b_1,\ldots,b_n\}$. Then, we have*

$$\frac{1}{a}.\limsup_{n\to\infty}\Bigl(\sup_{x\in Q}\bigl\|(I-P_n)(x)\bigr\|\Bigr)\leqslant\chi(Q)\leqslant\limsup_{n\to\infty}\Bigl(\sup_{x\in Q}\bigl\|(I-P_n)(x)\bigr\|\Bigr),$$

where $a=\limsup_{n\to\infty}\|I-P_n\|$.

In particular, the following result shows how to compute the Hausdorff measure of noncompactness in the spaces ℓ_p $(1\leqslant p<\infty)$ and c_0.

Lemma 6.9. *(See [32, Lemma 5.5].) Let Q be a bounded subsets of the normed space X, where X is ℓ_p for $1\leqslant p<\infty$ or c_0. If $P_n : X\to X$ is the operator defined by $P_n(x)=x^{[n]}=(x_0,x_1,x_2,\ldots,x_n,0,0,\ldots)$ for all $x=(x_k)\in X$, then we have*

$$\chi(Q)=\lim_{n\to\infty}\Bigl(\sup_{x\in Q}\bigl\|(I-P_n)(x)\bigr\|\Bigr).$$

The next lemma is related to the Hausdorff measure of noncompactness of a bounded linear operator.

Lemma 6.10. *(See [45, Theorem 2.25, Corollary 2.26].) Let X and Y be Banach spaces and $L\in B(X,Y)$. Then we have*

$$\|L\|_{\chi}=\chi\bigl(L(S_X)\bigr)\tag{6.3}$$

and

$$L\in K(X,Y)\quad\text{if and only if}\quad\|L\|_{\chi}=0.\tag{6.4}$$

Lemma 6.11. *(See [43, Theorem 3.7 and 3.11].) Let $X\supset\phi$ be a BK-space. Then, we have*

(a) *If $A\in(X,c_0)$, then*

$$\|L_A\|_{\chi}=\limsup_{n\to\infty}\|A_n\|_X^{*}$$

and

$$L_A\text{ is compact if and only if }\lim_{n\to\infty}\|A_n\|_X^{*}=0.$$

(b) *If $A \in (X, \ell_\infty)$, then*

$$0 \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \|A_n\|_X^*$$

and

$$L_A \text{ is compact if } \lim_{n\to\infty} \|A_n\|_X^* = 0.$$

(c) *If $A \in (X, \ell_1)$, then*

$$\lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left\|\sum_{n\in N} A_n\right\|_X^*\right) \leqslant \|L_A\|_\chi \leqslant 4.\lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left\|\sum_{n\in N} A_n\right\|_X^*\right)$$

and

$$L_A \text{ is compact if and only if } \lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left\|\sum_{n\in N} A_n\right\|_X^*\right) = 0.$$

This lemma gives necessary and sufficient conditions for a matrix transformation from a BK-space $X$to c_0, ℓ_1 and ℓ_∞ to be compact (only sufficient condition for ℓ_∞). Thus, we have;

Theorem 6.12. *Let $1 < p < \infty$ and $q = p/(p-1)$. Then we have*
(a) *If $A \in (\ell_p(u,v;B), c_0)$, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}|^q\right)^{1/q} \tag{6.5}$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}|^q\right)^{1/q} = 0. \tag{6.6}$$

(b) *If $A \in (\ell_p(u,v;B), \ell_\infty)$, then*

$$0 \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}|^q\right)^{1/q} \tag{6.7}$$

and

$$L_A \text{ is compact if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}|^q\right)^{1/q} = 0. \tag{6.8}$$

Proof. (a) Let $A \in (\ell_p(u,v;B), c_0)$. Since $A_n \in (\ell_p(u,v;B))^\beta$ for all $n \in \mathbb{N}$, we have from Lemma 6.5 that

$$\|A_n\|_{\ell_p(u,v;B)}^* = \|\tilde{A}_n\|_{\ell_q} = \left(\sum_{k=0}^{\infty} |\tilde{a}_{nk}|^q\right)^{1/q} \tag{6.9}$$

for all $n \in \mathbb{N}$. Hence, we get (6.5) and (6.6) from (6.9) and Lemma 6.11(a).

Part (b) can be proved similarly by using Lemma 6.11(b) instead of Lemma 6.11(a). □

Theorem 6.13. *Let $1 \leqslant p < \infty$. If $A \in (\ell_p(u,v;B), c)$, then*

$$\frac{1}{2}.\lim_{r\to\infty}\left(\sup_{n\geqslant r}\|\tilde{A}_n - \tilde{\alpha}\|_{\ell_q}\right) \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty}\left(\sup_{n\geqslant r}\|\tilde{A}_n - \tilde{\alpha}\|_{\ell_q}\right), \tag{6.10}$$

where $\tilde{\alpha} = (\tilde{\alpha}_k)$ is defined as (5.10).

Proof. Let $A \in (\ell_p(u,v;B),c)$. Then we have by Lemma 6.6 that $\tilde{A} \in (\ell_p,c)$. Thus, the sequence $(\sup_{n\geqslant r}\|\tilde{A}_n - \tilde{\alpha}\|_{\ell_q})_{r=0}^{\infty}$ of non-negative reals is nonincreasing and bounded by Lemma 6.7. Hence, the limit in (6.10) exists.

Now, we write $S = S_{\ell_p(u,v;B)}$, for short. Then, we obtain by (6.3) and Lemma 6.1 that

$$\|L_A\|_\chi = \chi(AS) \tag{6.11}$$

and $AS \in M_c$, where M_c is the class of all bounded subsets of c. Further, we know that every $z = (z_n) \in c$ has a unique representation $z = \bar{z}e + \sum_{n=0}^{\infty}(z_n - \bar{z})e^{(n)}$, where $\bar{z} = \lim_{n\to\infty} z_n$. Thus, we define the projectors $P_r : c \to c$ $(r \in \mathbb{N})$ by $P_0(z) = \bar{z}e$ and $P_r(z) = \bar{z}e + \sum_{n=0}^{r-1}(z_n - \bar{z})e^{(n)}$ for $r \geqslant 1$. Then, we have for every $r \in \mathbb{N}$ that $(I - P_r)(z) = \sum_{n=r}^{\infty}(z_n - \bar{z})e^{(n)}$ and hence

$$\big\|(I - P_r)(z)\big\|_{\ell_\infty} = \sup_{n\geqslant r}|z_n - \bar{z}| \tag{6.12}$$

for all $z \in c$ and every $r \in \mathbb{N}$. Further, it can easily be shown that $\|I - P_r\| = 2$ for all $r \in \mathbb{N}$. Therefore, from (6.11) we obtain by applying Lemma 6.8 that

$$\frac{1}{2}.\mu(A) \leqslant \|L_A\|_\chi \leqslant \mu(A), \tag{6.13}$$

where

$$\mu(A) = \limsup_{r\to\infty}\Big(\sup_{x\in S}\big\|(I - P_r)(Ax)\big\|_{\ell_\infty}\Big).$$

Now, for every given $x \in \ell_p(u,v;B)$, let $y \in \ell_p$ be associated sequence defined by (3.1). Since $A \in (\ell_p(u,v;B),c)$, we have by Lemma 6.6 that $\tilde{A} \in (\ell_p,c)$ and $Ax = \tilde{A}y$. Further it follows from Lemma 6.7 that the limits $\alpha_k = \lim_{n\to\infty} a_{nk}$ exist for all k, $\tilde{\alpha} = (\tilde{\alpha}_k) \in \ell_q$ and

$$\lim_{n\to\infty}\tilde{A}_n(y) = \sum_{k=0}^{\infty}\tilde{\alpha}_k x_k.$$

Thus, we derive from (6.12) that

$$\big\|(I - P_r)(Ax)\big\|_{\ell_\infty} = \big\|(I - P_r)(\tilde{A}y)\big\|_{\ell_\infty} = \sup_{n\geqslant r}\left|\tilde{A}_n(y) - \sum_{k=0}^{\infty}\tilde{\alpha}_k y_k\right| = \sup_{n\geqslant r}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk} - \tilde{\alpha}_k)y_k\right|$$

for all $r \in \mathbb{N}$. Moreover, since $x \in S = S_{\ell_p(u,v;B)}$ if and only if $y \in S_{\ell_p}$, we obtain by the definition of the norm $\|.\|_X^*$ and Lemma 6.3 that

$$\sup_{x\in S}\big\|(I - P_r)(Ax)\big\|_{\ell_\infty} = \sup_{n\geqslant r}\left(\sup_{y\in S_{\ell_p}}\left|\sum_{k=0}^{\infty}(\tilde{a}_{nk} - \tilde{\alpha}_k)y_k\right|\right) = \sup_{n\geqslant r}\|\tilde{A}_n - \tilde{\alpha}\|_{\ell_p}^* = \sup_{n\geqslant r}\|\tilde{A}_n - \tilde{\alpha}\|_{\ell_q}$$

for all $r \in \mathbb{N}$. Hence, we get (6.10) from (6.13), since the limit in (6.10) exists. The concludes the proof. □

Now, let us recall that the *upper limit* (limit superior) of a bounded real sequence $x = (x_n)$ can be defined by

$$\limsup_{n\to\infty} x_n = \lim_{r\to\infty}\Big(\sup_{n\geqslant r} x_n\Big). \tag{6.14}$$

Further, if $x_n \geqslant 0$ for all n, then

$$\limsup_{n\to\infty} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} x_n = 0.$$

By using the above notation,we have the following result:

Corollary 6.14. *Let $1 < p < \infty$ and $q = p/(p-1)$. If $A \in (\ell_p(u,v;B),c)$, then*

$$\frac{1}{2}.\limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk} - \tilde{\alpha}_k|^q\right)^{1/q} \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty}\left(\sum_{\infty}^{k=0}|\tilde{a}_{nk} - \tilde{\alpha}_k|^q\right)^{1/q}$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk} - \tilde{\alpha}_k|^q\right)^{1/q} = 0,$$

where $\tilde{\alpha} = (\tilde{\alpha}_k)$ is defined as (5.10).

Proof. This result follows from Theorem 6.13 by using (6.14) and (6.4). □

Theorem 6.15. *Let* $1 \leqslant p < \infty$. *If* $A \in (\ell_1(u, v; B), \ell_p)$, *then*

$$\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r}^{\infty}|\tilde{a}_{nk}|^p\right)^{1/p}\right). \tag{6.15}$$

Proof. Let $S = S_{\ell_1(u,v;B)}$. Then, we have by Lemma 6.1 that $L_A(S) = AS \in \ell_p$. Thus, from (6.3) and Lemma 6.9 we can write that

$$\|L_A\|_\chi = \chi(AS) = \lim_{r\to\infty}\left(\sup_{x\in S}\big\|(I-P_r)(Ax)\big\|_{\ell_p}\right), \tag{6.16}$$

where $P_r : \ell_p \to \ell_p$ $(r \in \mathbb{N})$ is the operator defined by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in \ell_p$.

Now, let $x = (x_k) \in \ell_1(u, v; B)$. Since $A \in (\ell_1(u, v; B), \ell_p)$, we obtain from Lemma 6.6 that $\tilde{A} \in (\ell_1, \ell_p)$ and $Ax = \tilde{A}y$, where $y = (y_k) \in \ell_1$ is the associated sequence defined by (3.1). Therefore, we have that

$$\begin{aligned}\big\|(I-P_r)(Ax)\big\|_{\ell_p} &= \big\|(I-P_r)(\tilde{A}y)\big\|_{\ell_p} = \left(\sum_{n=r+1}^{\infty}\big|\tilde{A}_n(y)\big|^p\right)^{1/p} = \left(\sum_{n=r+1}^{\infty}\left|\sum_{k=0}^{\infty}\tilde{a}_{nk}y_k\right|^p\right)^{1/p}\\ &\leqslant \sum_{k=0}^{\infty}\left(\sum_{n=r+1}^{\infty}|\tilde{a}_{nk}y_k|^p\right)^{1/p} \leqslant \|y\|_{\ell_1}\left(\sup_k\left(\sum_{n=r+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{1/p}\right) = \|x\|_{r_1^q(B^m)}\left(\sup_k\left(\sum_{n=r+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{1/p}\right)\end{aligned}$$

for every $n \in \mathbb{N}$. This yields that

$$\sup_{x\in S}\big\|(I-P_r)(Ax)\big\|_{\ell_p} \leqslant \sup_k\left(\sum_{n=r+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{1/p}$$

for every $n \in \mathbb{N}$. Hence, from (6.16) we have that

$$\|L_A\|_\chi \leqslant \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{1/p}\right). \tag{6.17}$$

Conversely, let $c^{(k)} \in \ell_1(u, v; B)$ such that $R(c^{(k)}) = e^{(k)}$ $(k \in \mathbb{N})$, where $c^{(k)} = \{c_n^{(k)}\}_{n\in\mathbb{N}}$ is defined by (3.8) for each $k \in \mathbb{N}$. Then, we have by Lemma 6.6 that $Ac^{(k)} = \tilde{A}e^{(k)} = (\tilde{a}_{nk})_{n=0}^{\infty}$ for every $k \in \mathbb{N}$. Now, let $E = \{c^{(k)}\colon\ k \in \mathbb{N}\}$. Then, $E \subset S$ and hence $AE \subset AS$ which implies that

$$\chi(AE) \leqslant \chi(AS) = \|L_A\|_\chi. \tag{6.18}$$

Moreover, we can write from Lemma 6.8 and (6.18) that

$$\chi(AE) = \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{1/p}\right) = \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{1/p}\right) \leqslant \|L_A\|_\chi.$$

Thus, we get (6.15) from (6.17) and (6.18). □

Corollary 6.16. *Let* $1 \leqslant p < \infty$. *If* $A \in (\ell_1(u, v; B), \ell_p)$, *then*

$$L_A \text{ is compact if and only if } \lim_{r\to\infty}\left(\sup_k\left(\sum_{n=r+1}^{\infty}|\tilde{a}_{nk}|^p\right)^{1/p}\right) = 0.$$

Proof. This is an immediate consequence of Theorem 6.15 and (6.4). □

Theorem 6.17. *Let* $1 < p < \infty$ *and* $q = p/(p-1)$. *If* $A \in (\ell_p(u, v; B), \ell_1)$, *then*

$$\lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N}\tilde{a}_{nk}\right|^q\right)^{1/q}\right) \leqslant \|L_A\|_\chi \leqslant 4.\lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N}\tilde{a}_{nk}\right|^q\right)^{1/q}\right) \tag{6.19}$$

and

$$L_A \text{ is compact if and only if } \lim_{r\to\infty}\left(\sup_{N\in\mathcal{F}_r}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N}\tilde{a}_{nk}\right|^q\right)^{1/q}\right)=0. \tag{6.20}$$

Proof. Let $A \in (\ell_p(u,v;B),\ell_1)$. Since $A_n \in \{\ell_p(u,v;B)\}^\beta$ for all $n \in \mathbb{N}$, we derive from Lemma 6.5 that

$$\left\|\sum_{n\in N} A_n\right\|^*_{r_P^q(B)} = \left\|\sum_{n\in N}\tilde{A}_n\right\|_{\ell_q}. \tag{6.21}$$

Thus, we get (6.19) and (6.20) from Lemma 6.11(c) and (6.21). □

Remark 6.18. The conclusions of Theorem 6.12, Corollary 6.14 and Theorem 6.17 still hold for $\ell_1(u,v;B)$ instead of $\ell_p(u,v;B)$ with $q=1$ and on replacing the summation over k by the supremum over k.

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References

[1] I.J. Maddox, Space of strongly summable sequences, Quart. J. Math. Oxford 18 (2) (1967) 345–355.
[2] S. Simons, The sequence spaces $\ell(p_v)$ and $m(p_v)$, Proc. Lond. Math. Soc. 15 (3) (1965) 422–436.
[3] H. Nakano, Modulared sequence spaces, Proc. Japan Acad. 27 (1951) 508–512.
[4] P.-N. Ng, P.-Y. Lee, Cesàro sequence spaces of non-absolute type, Comment. Math. (Prace Mat.) 20 (2) (1978) 429–433.
[5] B. Altay, F. Başar, On the paranormed Riesz sequence spaces of non-absolute type, Southeast Asian Bull. Math. 26 (2002) 701–715.
[6] B. Altay, F. Başar, M. Mursaleen, On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I, Inform. Sci. 176 (10) (2006) 1450–1462.
[7] M. Mursaleen, F. Başar, B. Altay, On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ II, Nonlinear Anal. 65 (3) (2006) 707–717.
[8] E.E. Kara, M. Öztürk, M. Başarır, Some topological and geometric properties of generalized Euler sequence spaces, Math. Slovaca 60 (3) (2010) 385–398.
[9] E. Malkowsky, E. Savaş, Matrix transformations between sequence spaces of generalized weighted mean, Appl. Math. Comput. 147 (2004) 333–345.
[10] B. Altay, F. Başar, Generalization of the sequence space $\ell(p)$ derived by weighted mean, J. Math. Anal. Appl. 330 (2007) 174–185.
[11] C. Aydın, F. Başar, Some new sequence spaces which include the spaces ℓ_p and ℓ_∞, Demonstratio Math. 38 (3) (2005) 641–656.
[12] C. Aydın, F. Başar, Some generalizations of the sequence spaces a_p^r, Iran. J. Sci. Technol. Trans. A Sci. 30 (A2) (2006) 175–190.
[13] E. Malkowsky, V. Rakočević, S. Živković, Matrix transformations between the sequence space bv^p and certain BK spaces, Bull. Cl. Sci. Math. Nat. Sci. Math. 27 (2002) 33–46.
[14] F. Başar, B. Altay, On the space of sequences of p-bounded variation and related matrix mappings, Ukrainian Math. J. 55 (2003) 136–147.
[15] B. Altay, F. Başar, M. Mursaleen, Some generalizations of the space bv_p of p-bounded variation sequences, Nonlinear Anal. 68 (2008) 273–287.
[16] B. Choudhary, S.K. Mishra, On Köthe–Toeplitz duals of certain sequence spaces and their matrix transformations, Indian J. Pure Appl. Math. 24 (1993) 291–301.
[17] M. Mursaleen, A.K. Noman, On some new sequence spaces of non-absolute type related to the spaces ℓ_p and ℓ_∞ I, Filomat 25 (2) (2011) 33–51.
[18] M. Kirişçi, F. Başar, Some new sequence spaces derived by the domain of generalized difference matrix, Comput. Math. Appl. 60 (2010) 1299–1399.
[19] B. Altay, F. Başar, Some paranormed sequence spaces of non-absolute type derived by weighted mean, J. Math. Anal. Appl. 319 (2006) 494–508.
[20] H. Polat, V. Karakaya, N. Şimşek, Difference sequence spaces derived by generalized weighted mean, Appl. Math. Lett. 24 (5) (2011) 608–614.
[21] M. Başarır, On the generalized Riesz B-difference sequence spaces, Filomat 24 (4) (2010) 35–52.
[22] M. Kirişçi, F. Başar, Almost convergence and generalized difference matrix, Comput. Math. Appl. 61 (3) (2010) 602–611.
[23] B. Altay, F. Başar, On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c, Int. J. Math. Math. Sci. 18 (2008) 3005–3013.
[24] S. Demiriz, C. Çakan, Some topological and geometrical properties of a new difference sequence space, Abstr. Appl. Anal. (2011), doi:10.1155/2011/213878, 14 pp.
[25] M. Başarır, M. Öztürk, On the Riesz difference sequence space, Rend. Circ. Mat. Palermo 57 (2008) 377–389.
[26] M. Başarır, Paranormed Cesàro difference sequence space and related matrix transformations, Doğa Tr. J. Math. 15 (1991) 14–19.
[27] I.J. Maddox, Elements of Functional Analysis, second ed., Cambridge Univ. Press, Cambridge, UK, 1998.
[28] I.J. Maddox, Paranormed sequence spaces generated by infinite matrices, Proc. Cambridge Philos. Soc. 64 (1968) 335–340.
[29] K.-G. Grosse-Erdmann, Matrix transformations between the sequence spaces of Maddox, J. Math. Anal. Appl. 180 (1993) 223–238.
[30] C.G. Lascarides, I.J. Maddox, Matrix transformations between some classes of sequences, Proc. Cambridge Philos. Soc. 68 (1970) 99–104.
[31] E. Malkowsky, V. Rakočević, S. Živković, Matrix transformations between the sequence spaces $w_0^p(\Lambda)$, $v_0^p(\Lambda)$, $c_0^p(\Lambda)$ $(1<p<\infty)$ and certain BK spaces, Appl. Math. Comput. 147 (2) (2004) 377–396.
[32] I. Djolović, E. Malkowsky, Matrix transformations and compact operators on some new m^{th} order difference sequence spaces, Appl. Math. Comput. 198 (2008) 700–714.
[33] B. de Malafosse, E. Malkowsky, On the measure of noncompactness of linear operators in spaces of strongly α-summable and bounded sequences, Period. Math. Hungar. 55 (2) (2007) 129–148.
[34] E. Malkowsky, V. Rakočević, The measure of noncompactness of linear operators between certain sequence spaces, Acta Sci. Math. (Szeged) 64 (1998) 151–171.
[35] V. Rakočević, Measures of noncompactness and some applications, Filomat 12 (1998) 87–120.
[36] M. Mursaleen, A.K. Noman, Applications of the Hausdorff measure of noncompactness in some sequence spaces of weighted means, Comput. Math. Appl. 60 (5) (2010) 245–1258.
[37] E.E. Kara, M. Başarır, On compact operators and some Euler $B^{(m)}$ difference sequence spaces, J. Math. Anal. Appl. 379 (2011) 499–511.

[38] F. Başar, E. Malkowsky, The characterization of compact operators on spaces of strongly summable and bounded sequences, Appl. Math. Comput. 217 (2011) 5199–5207.
[39] M. Başarır, E.E. Kara, On compact operators on the Riesz B^m-difference sequence space, Iran. J. Sci. Technol. A A4 (2011) 279–285.
[40] M. Başarır, E.E. Kara, On some difference sequence spaces of weighted means and compact operators, Ann. Funct. Anal. 2 (2) (2011) 116–131.
[41] E. Malkowsky, V. Rakočević, On matrix domains of triangles, Appl. Math. Comput. 189 (2) (2007) 1146–1163.
[42] I. Djolović, E. Malkowsky, A note on compact operators on matrix domains, J. Math. Anal. Appl. 340 (1) (2008) 291–303.
[43] M. Mursaleen, A. K Noman, Compactness by the Hausdorff measure of noncompactness, Nonlinear Anal. 73 (8) (2010) 2541–2557.
[44] A. Wilansky, Summability Through Functional Analysis, North-Holland Math. Stud., vol. 85, Elsevier Science Publishers, Amsterdam, New York, Oxford, 1984.
[45] E. Malkowsky, V. Rakočević, An introduction into the theory of sequence spaces and measure of noncompactness, Zb. Rad. (Beogr.) 9 (17) (2000) 143–234.