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Rotations and Screw Motion with Timelike Vector in 3-Dimensional Lorentzian Space

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Abstract. In the present paper we obtain the timelike Euler parameters of a Lorentzian orthogonal matrix in Lorentz space $L^3 = \mathbb{R}^{2,1}$ by using Lorentzian matrix multiplication. Then, by using the timelike Euler parameters of a given rotation in a split quaternion formulation, we produce split quaternion equation of a rotation motion in L^3. Moreover the components of a dual split quaternion are obtained by replacing the timelike L-Euler parameters with their split dual versions.

Keywords. Lorentzian space, Lorentzian matrix multiplication, rotation, split quaternions.

1. Introduction

Kinematics studies the geometrical properties at the motion of points. In classical mechanics a set of (mass) points with the property that the distances between any two of them never varies is called a rigid body. A given mechanical device is constructed by connecting rigid bodies together with joints that constrain their relative motion.

In 1775–1776, Leonhard Euler published a remarkable result stating that in three dimensions every rotation of a sphere about its center has an axis, and provided a geometric construction for finding it.

After recalling some basic concepts and notations in Section 2 we direct our attention in Sections 3 and 4 to the main goal of this study which consists in obtaining the timelike Euler parameters of a Lorentzian orthogonal matrix in Lorentz space L^3, and by expressing these timelike Euler parameters in a split quaternion formulation, to get the split quaternion equation of a rotation motion in L^3. Moreover the components of a dual split quaternion are obtained by replacing the timelike L-Euler parameters with their split dual versions.

Birkhäuser

2. Preliminary

For the vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, the Lorentzian inner product on $\mathbb{R}^3$ is given by

$$\langle x, y\rangle_L = -x_1y_1 + x_2y_2 + x_3y_3.$$

The vector space on $\mathbb{R}^3$ equipped with the Lorentzian inner product is called 3-dimensional Lorentzian space and denoted by L^3. For a vector $x \in L^3$ is called spacelike, lightlike or timelike if $\langle x, x\rangle_L > 0$, $\langle x, x\rangle_L = 0$ or $\langle x, x\rangle_L < 0$, respectively. Moreover, x is called a positive or negative vector if the first component x_1 of x is positive or negative, respectively. For $x \in L^3$, the norm of x is defined by $\|x\| = \sqrt{\langle x, x\rangle_L}$. The norm $\|x\|$ is either positive or zero or positive imaginary. If $\|x\|$ is positive imaginary, then the notation $|\|x\||$ used instead of $\|x\|$.

Two vectors x, y in L^3 are said to be orthogonal if $\langle x, y\rangle_L = 0$.

Let V be a vector subspace of L^3. Then V is said to be timelike iff V has a timelike vector, spacelike iff every non zero vector in V is spacelike or lightlike otherwise.

If x is a timelike vector in a Lorentz vector space, then the subspace $x^{\perp}$ is spacelike.

For the vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, the Lorentzian cross product is defined by

$$x \wedge_L y = (x_3y_2 - x_2y_3 \ , \ x_3y_1 - x_1y_3 \ , \ x_1y_2 - x_2y_1).$$

In Lorentzian space L^3, the angle φ between vectors x, y is defined as follows:

(1) x and y be spacelike vectors in L^3 that span a spacelike vector subspace. Then

$$\langle x, y\rangle_L = \|x\| \cdot \|y\| \cos\varphi,$$
$$\|x \wedge_L y\| = \|x\| \cdot \|y\| \sin\varphi.$$

(2) For x the spacelike vector x and positive timelike vector y in L^3,

$$|\langle x, y\rangle_L| = \|x\| \cdot |\|y\|| \sinh\varphi,$$
$$\|x \wedge_L y\| = \|x\| \cdot |\|y\|| \cosh\varphi$$

[5, 6].

Let L^n be the vector space $\mathbb{R}^n$ defined with the Lorentzian inner product

$$\langle x, y\rangle_L = -x_1y_1 + \sum_{i=2}^{n} x_iy_i \text{ for } x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n).$$

Let $\mathbb{R}_n^m$ be the set of matrices of m rows and n columns.

Let $A = [a_{ij}] \in \mathbb{R}_n^m$ and $B = [b_{jk}] \in \mathbb{R}_p^n$. Lorentzian matrix multiplication denoted by "$\cdot_L$", and simply say L-multiplication is defined as

$$A \cdot_L B = \left[-a_{i1}b_{1k} + \sum_{j=2}^{n} a_{ij}b_{jk} \right].$$

Note that $A \cdot_L B$ is an $m \times p$ matrix. $\mathbb{R}^m_n$ with L-multiplication by L^m_n.

An $n \times n$ L-identity matrix according to L-multiplication, denoted by I_n is defined by

$$I_n = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

An $n \times n$ matrix A is called L-invertible if there exists an $n \times n$ matrix B such that $A \cdot_L B = B \cdot_L A = I_n$. Then B is called L-inverse of A and is denoted by A^{-1}.

The transpose of a matrix $A = [a_{ij}] \in L^m_n$ is denoted by A^T and defined as $A^T = [a_{ji}] \in L^n_m$.

A matrix $A \in L^n_n$ is called L-orthogonal matrix if $A^{-1} = A^T$.

For every $A, B \in L^n_n$, $\det(A \cdot_L B) = -\det A \cdot \det B$ [3].

A split quaternion is defined by the base $\{1, i, j, k\}$ where i, j, k satisfy the equalities $i^2 = -1$, $j^2 = +1$, $k^2 = +1$, $i.j = -j.i = k$, $j.k = -k.j = -i$, $k.i = -i.k = j$. So, a split quaternion can be expressed as $q = d + ai + bj + ck$ where a, b, c, d are real scalars. If we take $S_q = d$ and $V_q = ai + bj + ck$ then the split quaternion $q = d + ai + bj + ck$ can be re-written as $q = S_q + V_q$. The split quaternion addition is defined as

$$q_1 + q_2 = S_{q_1} + S_{q_2} + V_{q_1} + V_{q_2}$$

for every split quaternions q_1, q_2. The scalar product of split quaternion is

$$\lambda q = \lambda S_q + \lambda V_q$$

where λ is real scalar. The conjugate $\overline{q}$ of split quaternion $q = S_q + V_q$ is defined as $\overline{q} = S_q - V_q$. The norm of split quaternion $q = d+ai+bj+ck$ denoted by $N(q)$ is $N(q) = \sqrt{\overline{q}q} = \sqrt{q\overline{q}}$. Observe that $N(q) = \sqrt{d^2 + a^2 - b^2 - c^2}$ [7].

Now, we recall some basic notions of L-Cayley formula and L-Rodrigues equation for rotation.

A rotation about the origin is defined by $A \cdot_L x = X$. Here, A is an 3×3 L-orthogonal matrix and $x \in L^3$. L-Cayley formula is defined

$$A = (I - B)^{-1} \cdot_L (I + B),$$

or its equivalent form

$$A = (I + B) \cdot_L (I - B)^{-1}$$

where B is a skew-symmetric matrix. In L^3, it is known that every skew-symmetric matrix B defines an L-orthogonal matrix by L-Cayley formula. A 3×3 skew-symmetric B has only 3 independent elements, that is

$$B = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & -b_1 \\ b_2 & b_1 & 0 \end{bmatrix}.$$

These elements can be assembled into the vector $b = (b_1, b_2, b_3)$. Note that the matrix B satisfying the equality $B \cdot_L y = b \wedge_L y$.

Given an L-orthogonal matrix A, we have from L-Cayley formula,

$$X - x = B \cdot_L (X + x).$$

By using the relationship between skew-symmetric matrices and the Lorentzian cross product, we can write this equation in the form

$$X - x = b\Lambda_L (X + x). \tag{2.1}$$

This equation called as L-Rodrigues equation for rotations, and called the vector b as L-Rodrigues vector. Furthermore, $A \cdot_L b = b$ [8].

Theorem 1. [8] *In L^3, the rotation matrix A transforms the timelike vectors to timelike vectors, the spacelike vectors to spacelike vectors and the null vectors to null vectors.*

In L-Rodrigues equation $X - x = b\Lambda_L (X + x)$, we consider b as timelike vector, and x is any vector in L^3. Let x^* and X^* be projections of x and X onto a plane perpendicular to b. Then there exists a $\lambda \in \mathbb{R}$ provided that $x = x^* + \lambda b$. Since x^* perpendicular b, $A \cdot_L b = b$ and $\langle x, b\rangle_L = \langle Ax, Ab\rangle_L$,

$$\begin{aligned} 0 &= \langle x^*, b\rangle_L \\ &= \langle x - \lambda b, b\rangle_L \\ &= \langle x, b\rangle_L - \lambda \langle b, b\rangle_L \\ &= \langle Ax, Ab\rangle_L - \lambda \langle b, b\rangle_L \\ &= \langle X, b\rangle_L - \lambda \langle b, b\rangle_L \\ &= \langle X - \lambda b, b\rangle_L . \end{aligned}$$

Thus, for the same $\lambda \in \mathbb{R}$ we can write $X = X^* + \lambda b$ and x^* and X^* are spacelike vectors in spacelike subspace. b perpendicular to $X^* - x^*$ and $X^* + x^*$. Also

$$\begin{aligned} A \cdot_L x^* &= A \cdot_L (x - \lambda b) \\ &= A \cdot_L x - \lambda A \cdot_L b \\ &= X - \lambda b \\ &= X^*. \end{aligned}$$

From L-Rodrigues equation, we get

$$X^* - x^* = b\Lambda_L (X^* + x^*).$$

Then

$$\|X^* - x^*\| = \|b\Lambda_L (X^* + x^*)\| = |\|b\|| \cdot \|X^* + x^*\| \cosh \phi_1.$$

Since $\langle b, X^* + x^*\rangle_L = 0$,

$$|\langle b, X^* + x^*\rangle_L| = |\|b\|| \cdot \|X^* + x^*\| \sinh \phi_1 = 0.$$

Thus, $\sinh \phi_1 = 0$, so $\phi_1 = 0$, and so $\cosh \phi_1 = 1$. Hence,

$$\|X^* - x^*\| = |\|b\|| \cdot \|X^* + x^*\|.$$

Moreover, by taking

$$\begin{aligned}\|X^* - x^*\| &= \sqrt{\langle X^*, X^*\rangle_L - 2\langle X^*, x^*\rangle_L + \langle x^*, x^*\rangle_L} \\ &= \sqrt{\langle X^*, X^*\rangle_L - 2\|X^*\|\|x^*\|\cos\phi + \langle x^*, x^*\rangle_L}\end{aligned}$$

and letting $\langle X^*, X^*\rangle_L = \langle x^*, x^*\rangle_L = k^2 \ (k > 0)$, we get

$$\begin{aligned}\|X^* - x^*\| &= \sqrt{k^2 - 2k^2\cos\phi + k^2} \\ &= 2k\sin\frac{\phi}{2}.\end{aligned}$$

Similarly,

$$\|X^* + x^*\| = 2k\cos\frac{\phi}{2}.$$

Since

$$\|X^* - x^*\| = |\|b\|| \cdot \|X^* + x^*\|,$$

then we obtain

$$\|b\| = \tan\frac{\phi}{2}.$$

Let the unit vector in the direction of b be $s = (s_x, s_y, s_z)$. Then the components of the vector b, or equivalently of the skew-symmetric matrix B are:

$$\begin{aligned}b_1 &= \left(\tan\frac{\phi}{2}\right)s_x, \\ b_2 &= \left(\tan\frac{\phi}{2}\right)s_y, \\ b_3 &= \left(\tan\frac{\phi}{2}\right)s_z.\end{aligned}$$

We call these constants as timelike Lorentzian Rodrigues parameters or in short timelike L-Rodrigues parameters.

3. Lorentzian Timelike Euler Parameters

L-Cayley formula for the L-orthogonal matrix A can be written in terms of the rotation angle ϕ and the unit timelike vector s determined from

$$B = \left(\tan\frac{\phi}{2}\right)S.$$

The result is:

$$A = \left(\left(\cos\frac{\phi}{2}\right)I - \left(\sin\frac{\phi}{2}\right)S\right)^{-1} \cdot_L \left(\left(\cos\frac{\phi}{2}\right)I + \left(\sin\frac{\phi}{2}\right)S\right). \quad (3.1)$$

If we let $C = \left(\cos \frac{\phi}{2}\right) I + \left(\sin \frac{\phi}{2}\right) S$, then we call the constants

$$\begin{aligned} c_0 &= \cos \frac{\phi}{2}, \\ c_1 &= \left(\sin \frac{\phi}{2}\right) s_x, \\ c_2 &= \left(\sin \frac{\phi}{2}\right) s_y, \\ c_2 &= \left(\sin \frac{\phi}{2}\right) s_z. \end{aligned}$$

in the matrix C, as the timelike Lorentzian Euler parameters or in short timelike L-Euler parameters of A.

In equation (3.1), by taking $X = \left(\cos \frac{\phi}{2}\right) I - \left(\sin \frac{\phi}{2}\right) S$ and $\frac{\phi}{2} = \theta$, we obtain

$$X = \begin{bmatrix} -\cos\theta & -s_z \sin\theta & s_y \sin\theta \\ s_z \sin\theta & \cos\theta & s_x \sin\theta \\ -s_y \sin\theta & -s_x \sin\theta & \cos\theta \end{bmatrix}.$$

If we let X^{-1} be the L-inverse of X, we get

$$\begin{aligned} X^{-1} &= \begin{bmatrix} -s_x^2 \frac{\sin^2\theta}{\cos\theta} - \cos\theta & -s_x s_y \frac{\sin^2\theta}{\cos\theta} + s_z \sin\theta & -s_x s_z \frac{\sin^2\theta}{\cos\theta} - s_y \sin\theta \\ -s_x s_y \frac{\sin^2\theta}{\cos\theta} - s_z \sin\theta & -s_y^2 \frac{\sin^2\theta}{\cos\theta} + \cos\theta & -s_y s_z \frac{\sin^2\theta}{\cos\theta} - s_x \sin\theta \\ -s_x s_z \frac{\sin^2\theta}{\cos\theta} + s_y \sin\theta & -s_y s_z \frac{\sin^2\theta}{\cos\theta} + s_x \sin\theta & -s_z^2 \frac{\sin^2\theta}{\cos\theta} + \cos\theta \end{bmatrix} \\ &= \cos\theta \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & s_z & -s_y \\ -s_z & 0 & -s_x \\ s_y & s_x & 0 \end{bmatrix} \\ &\quad + \frac{\sin^2\theta}{\cos\theta} \begin{bmatrix} -1 - s_y^2 - s_z^2 & -s_x s_y & -s_x s_z \\ -s_x s_y & 1 - s_x^2 + s_z^2 & -s_y s_z \\ -s_x s_z & -s_y s_z & 1 - s_x^2 + s_y^2 \end{bmatrix} \\ &= (\cos\theta) I + (\sin\theta) S + \frac{\sin^2\theta}{\cos\theta} \left(I + S^2\right). \end{aligned}$$

Thus

$$\begin{aligned} A &= X^{-1} \cdot_L C \\ &= \left((\cos\theta) I + (\sin\theta) S + \frac{\sin^2\theta}{\cos\theta} \left(I + S^2\right) \right) \cdot_L ((\cos\theta) I + (\sin\theta) S) \\ &= I + (\sin 2\theta) S + (1 - \cos 2\theta) S^2 + \frac{\sin^3\theta}{\cos\theta} \left(S^3 + S\right), \end{aligned}$$

Since the characteristic polynomial of the matrix S is $\lambda^3 + \lambda$, we have $S^3 + S = 0$. Hence,

$$A = I + (\sin\phi) S + (1 - \cos\phi) S^2. \tag{3.2}$$

4. Rotations in L^3 with Timelike Vectors

Let c_0, c_1, c_2, c_3 be L-Euler parameters for a rotation. Then rotations in L^3 can be identified by the split quaternion

$$q = c_0 + c_1 i + c_2 j + c_3 k.$$

For the rotation angle ϕ and timelike rotation axis $s = (s_x, s_y, s_z)$, the split quaternion $q = c_0 + c_1 i + c_2 j + c_3 k$ can be written as

$$q = \cos\frac{\phi}{2} + s_x\left(\sin\frac{\phi}{2}\right) i + s_y\left(\sin\frac{\phi}{2}\right) j + s_z\left(\sin\frac{\phi}{2}\right) k.$$

Here, since $N(q) = 1$, q is a unit split quaternion.

A vector $X = (x, y, z)$ in L^3 as a split quaternion is identified with the element

$$X = xi + yj + zk.$$

The rotation in L^3 is given by the split quaternion equation

$$X' = qX\overline{q}$$

where $\overline{q}$ is the conjugate of q and $\overline{q} = c_0 - c_1 i - c_2 j - c_3 k$.

Now, we consider the equation $X' = qX\overline{q}$ by using the matrix form of the split quaternion product. We choose to order the components of a split quaternion as $Z = Z_4 + Z_1 i + Z_2 j + Z_3 k$ so that each split quaternion can be identified with a four dimensional vector $Z = (Z_1, Z_2, Z_3, Z_4)$. For $W = W_4 + W_1 i + W_2 j + W_3 k$ the vector WZ which is product of two split quaternions W and Z is given by the matrix product

$$WZ = W^+ \cdot_L Z$$

or

$$WZ = Z^- \cdot_L W$$

where

$$W^+ = \begin{bmatrix} -W_4 & W_3 & -W_2 & W_1 \\ -W_3 & W_4 & -W_1 & W_2 \\ W_2 & W_1 & W_4 & W_3 \\ W_1 & W_2 & W_3 & W_4 \end{bmatrix}$$

$$Z^- = \begin{bmatrix} -Z_4 & -Z_3 & Z_2 & Z_1 \\ Z_3 & Z_4 & Z_1 & Z_2 \\ -Z_2 & -Z_1 & Z_4 & Z_3 \\ Z_1 & Z_2 & Z_3 & Z_4 \end{bmatrix}.$$

W^+ and Z^- are defined as follows: i, j, k being the basis vectors of split quaternions, each column of the matrix W^+ is the result of the product of W on the right with basis bivectors $-i$, j, $k, 1$ and each column of the matrix Z^- is the result of the product of Z on the left with basis bivectors $-i$, j, $k, 1$.

For $q=\left(s_x\sin\frac{\phi}{2}, s_y\sin\frac{\phi}{2}, s_z\sin\frac{\phi}{2}, \cos\frac{\phi}{2}\right)$ and $X=(x,y,z,0)$, the matrix form of the equation $X' = qX\overline{q}$ can be considered as

$$X' = q^{+}\cdot_L \overline{(q^{-})}\cdot_L X$$

where

$$q^{+}=\begin{bmatrix} -\cos\frac{\phi}{2} & s_z\sin\frac{\phi}{2} & -s_y\sin\frac{\phi}{2} & s_x\sin\frac{\phi}{2} \\ -s_z\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & -s_x\sin\frac{\phi}{2} & s_y\sin\frac{\phi}{2} \\ s_y\sin\frac{\phi}{2} & s_x\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & s_z\sin\frac{\phi}{2} \\ s_x\sin\frac{\phi}{2} & s_y\sin\frac{\phi}{2} & s_z\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}$$

and

$$\overline{(q^{-})}=\begin{bmatrix} -\cos\frac{\phi}{2} & s_z\sin\frac{\phi}{2} & -s_y\sin\frac{\phi}{2} & -s_x\sin\frac{\phi}{2} \\ -s_z\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & -s_x\sin\frac{\phi}{2} & -s_y\sin\frac{\phi}{2} \\ s_y\sin\frac{\phi}{2} & s_x\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & -s_z\sin\frac{\phi}{2} \\ -s_x\sin\frac{\phi}{2} & -s_y\sin\frac{\phi}{2} & -s_z\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}.$$

Then

$$\begin{aligned} & q^{+}\cdot_L \overline{(q^{-})} \\ &= \cos^2\frac{\phi}{2}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\phi\begin{bmatrix} 0 & s_z & -s_y \\ -s_z & 0 & -s_x \\ s_y & s_x & 0 \end{bmatrix} \\ &\quad + \sin^2\frac{\phi}{2}\begin{bmatrix} -s_x^2-s_y^2-s_z^2 & -2s_xs_y & -2s_xs_z \\ -2s_xs_y & -s_x^2-s_y^2+s_z^2 & -2s_ys_z \\ -2s_xs_z & -2s_ys_z & -s_x^2+s_y^2-s_z^2 \end{bmatrix} \\ &= \left(\cos^2\frac{\phi}{2}\right)I + (\sin\phi)\,S + \sin^2\frac{\phi}{2}\left(I+2S^2\right) \\ &= I + (\sin\phi)\,S + (1-\cos\phi)\,S^2. \end{aligned}$$

This is the rotational matrix for the rotation angle ϕ and the equation

$$A = I + (\sin\phi)\,S + (1-\cos\phi)\,S^2$$

defined by the rotation axis s and given by the equation (3.2). In this case,

$$q^{+}\cdot_L \overline{(q^{-})} = \begin{bmatrix} A & 0 \\ 0^T & 1 \end{bmatrix},$$

where 0 is the 3-dimensional zero vector.

4.1. Dual Split Quaternions

The displacement $T=(A,d)$ can be represented by a dual split quaternion $\widetilde{q}=q+\varepsilon q^0$. The real part, $q=q_0+q_1i+q_2j+q_3k$, is defined by the timelike L-Euler parameters of the rotation A, see ([8]). The dual part, q^0, is given by the formula:

$$q^0=\frac{1}{2}Dq,$$

where $D=d_1i+d_2j+d_3k$ is the split quaternion formed from the translation vector $d=(d_1,d_2,d_3)$.

Let d be written in terms of the L-screw parameters of the displacement, $d = ds + c - A._L c$([9]). Written each of these vectors in split quaternion form we obtain

$$D = dS + C - qC\overline{q},$$

and

$$q^0 = \frac{1}{2}\left(dS + C - qC\overline{q}\right)q$$
$$q^0 = \frac{1}{2}\left(dSq + Cq - qC\right).$$

The matrix form of the equation q^0 can be considered as

$$q^0 = \frac{1}{2}\left(q^- ._L \left(dS + C\right) - q^+ ._L C\right),$$

where

$$q^- ._L \left(dS + C\right) = \begin{bmatrix} -\cos\frac{\phi}{2} & -s_z\sin\frac{\phi}{2} & s_y\sin\frac{\phi}{2} & s_x\sin\frac{\phi}{2} \\ s_z\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & s_x\sin\frac{\phi}{2} & s_y\sin\frac{\phi}{2} \\ -s_y\sin\frac{\phi}{2} & -s_x\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & s_z\sin\frac{\phi}{2} \\ s_x\sin\frac{\phi}{2} & s_y\sin\frac{\phi}{2} & s_z\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}$$
$$._L \begin{bmatrix} \left(-d_1s_x + d_2s_y + d_3s_z\right)s_x + c_1 \\ \left(-d_1s_x + d_2s_y + d_3s_z\right)s_y + c_2 \\ \left(-d_1s_x + d_2s_y + d_3s_z\right)s_z + c_3 \\ 0 \end{bmatrix}$$

and

$$q^+ ._L C = \begin{bmatrix} -\cos\frac{\phi}{2} & s_z\sin\frac{\phi}{2} & -s_y\sin\frac{\phi}{2} & s_x\sin\frac{\phi}{2} \\ -s_z\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & -s_x\sin\frac{\phi}{2} & s_y\sin\frac{\phi}{2} \\ s_y\sin\frac{\phi}{2} & s_x\sin\frac{\phi}{2} & \cos\frac{\phi}{2} & s_z\sin\frac{\phi}{2} \\ s_x\sin\frac{\phi}{2} & s_y\sin\frac{\phi}{2} & s_z\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix} ._L \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} c_1\cos\frac{\phi}{2} + c_2s_z\sin\frac{\phi}{2} - c_3s_y\sin\frac{\phi}{2} \\ c_1s_z\sin\frac{\phi}{2} + c_2\cos\frac{\phi}{2} - c_3s_x\sin\frac{\phi}{2} \\ -c_1s_y\sin\frac{\phi}{2} + c_2s_x\sin\frac{\phi}{2} + c_3\cos\frac{\phi}{2} \\ -c_1s_x\sin\frac{\phi}{2} + c_2s_y\sin\frac{\phi}{2} + c_3s_z\sin\frac{\phi}{2} \end{bmatrix}$$

we have

$$q^0 = \frac{1}{2}\begin{bmatrix} -d_1s_x^2\cos\frac{\phi}{2} + d_2s_xs_y\cos\frac{\phi}{2} + d_3s_xs_z\cos\frac{\phi}{2} \\ -d_1s_xs_y\cos\frac{\phi}{2} + d_2s_y^2\cos\frac{\phi}{2} + d_3s_ys_z\cos\frac{\phi}{2} \\ -d_1s_xs_z\cos\frac{\phi}{2} + d_2s_ys_z\cos\frac{\phi}{2} + d_3s_z^2\cos\frac{\phi}{2} \\ 0 \end{bmatrix}$$
$$+ \frac{1}{2}\begin{bmatrix} -2c_2s_z\sin\frac{\phi}{2} + 2c_3s_y\sin\frac{\phi}{2} \\ -2c_1s_z\sin\frac{\phi}{2} + 2c_3s_x\sin\frac{\phi}{2} \\ 2c_1s_y\sin\frac{\phi}{2} - 2c_2s_x\sin\frac{\phi}{2} \\ \left(\sin\frac{\phi}{2}\right)\left(d_2s_y - d_1s_x + d_3s_z\right)\left(-s_x^2 + s_y^2 + s_z^2\right) \end{bmatrix}.$$

By letting $d = \langle d, s \rangle = -d_1 s_x + d_2 s_y + d_3 s_z$ and $s^* = \left(s_x^*, s_y^*, s_z^*\right) = c \wedge_L s$, we have

$$q^0 = \frac{1}{2}\begin{bmatrix} d\cos\frac{\phi}{2}s_x + 2\sin\frac{\phi}{2}s_x^* \\ d\cos\frac{\phi}{2}s_y + 2\sin\frac{\phi}{2}s_y^* \\ d\cos\frac{\phi}{2}s_z + 2\sin\frac{\phi}{2}s_z^* \\ -d\sin\frac{\phi}{2} \end{bmatrix},$$

and

$$\begin{aligned} q^0 = \; & -\tfrac{d}{2}\sin\tfrac{\phi}{2} + \left(\tfrac{d}{2}\cos\tfrac{\phi}{2}s_x + \sin\tfrac{\phi}{2}s_x^*\right)i \\ & + \left(\tfrac{d}{2}\cos\tfrac{\phi}{2}s_y + \sin\tfrac{\phi}{2}s_y^*\right)j + \left(\tfrac{d}{2}\cos\tfrac{\phi}{2}s_z + \sin\tfrac{\phi}{2}s_z^*\right)k. \end{aligned}$$

Let the dual vector $\widetilde{s} = s + \varepsilon s^*$ represent the timelike L-screw axis and $\widetilde{\phi} = \phi + \varepsilon d$ be the dual hyperbolic angle defining the rotation and translation along $\widetilde{s}$, then we obtain

$$\begin{aligned} \widetilde{q} &= \cos\frac{\phi}{2} + s_x\left(\sin\frac{\phi}{2}\right)i + s_y\left(\sin\frac{\phi}{2}\right)j + s_z\left(\sin\frac{\phi}{2}\right)k \\ &\quad + \varepsilon\begin{pmatrix} -\tfrac{d}{2}\sin\tfrac{\phi}{2} + \left(\tfrac{d}{2}\cos\tfrac{\phi}{2}s_x + \sin\tfrac{\phi}{2}s_x^*\right)i \\ + \left(\tfrac{d}{2}\cos\tfrac{\phi}{2}s_y + \sin\tfrac{\phi}{2}s_y^*\right)j + \left(\tfrac{d}{2}\cos\tfrac{\phi}{2}s_z + \sin\tfrac{\phi}{2}s_z^*\right)k \end{pmatrix} \\ &= \left(\cos\frac{\phi}{2} - \varepsilon\frac{d}{2}\sin\frac{\phi}{2}\right) + s_x\left(\sin\frac{\phi}{2} + \varepsilon\frac{d}{2}\cos\frac{\phi}{2}\right)i + \varepsilon\sin\frac{\phi}{2}s_x^* i \\ &\quad + s_y\left(\sin\frac{\phi}{2} + \varepsilon\frac{d}{2}\cos\frac{\phi}{2}\right)j + \varepsilon\sin\frac{\phi}{2}s_y^* j \\ &\quad + s_z\left(\sin\frac{\phi}{2} + \varepsilon\frac{d}{2}\cos\frac{\phi}{2}\right)k + \sin\frac{\phi}{2}s_z^* k. \end{aligned}$$

On the other hand, since Taylor polynomial of analytic dual variable function and letting $\widetilde{\phi} = \phi + \varepsilon d$ we obtain

$$\widetilde{q} = \cos\frac{\widetilde{\phi}}{2} + \widetilde{s}_x\left(\sin\frac{\widetilde{\phi}}{2}\right)i + \widetilde{s}_y\left(\sin\frac{\widetilde{\phi}}{2}\right)j + \widetilde{s}_z\left(\sin\frac{\widetilde{\phi}}{2}\right)k$$

Thus the components of a dual split quaternion are obtained by replacing the L-Euler parameters with their dual versions. We call this dual timelike L-Euler parameters Lorentzian spatial displacement.

Using the timelike L-Euler parameters, we can represent the dual L-orthogonal matrix,

$$\widetilde{A} = I + \left(\sin\widetilde{\phi}\right)\widetilde{S} + \left(1 - \cos\widetilde{\phi}\right)\widetilde{S}^2$$

by a dual version of $A = I + (\sin\phi)\,S + (1 - \cos\phi)\,S^2$.

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