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Screw Theory in Lorentzian Space

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Abstract. In this paper we present various results about the six dimensional vectors obtained from the tangent operators of spatial motion, called as screws, in Lorentzian space. Each screw has an axis defined by six Plücker coordinates in Lorentzian space. The manipulation of screw coordinate transformations has been simplified by using Lorentz matrix multiplication and dual number algebra. Also, we showed that screw displacement is representation as the exponential of a dual angular velocity matrix by using the dual orthogonal matrices in Lorentzian space.

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1. Introduction

Kinematics is often referred to as the "geometry of motion" studies the geometric properties of the motion of points and is occasionally seen as a branch of mathematics. A set of points with the property that the distance between any two of them never varies is called a rigid body. The motion of a rigid body is represented as a continuous sequence of displacements $D(t) : F \rightarrow M$. The frame F is fixed but the position of the moving frame M varies with the parameter t [12].

Indeed, kinematics is used in astrophysics to describe the motion of celestial bodies and collections of such body. In mechanical engineering, robotics and biomechanics, it is used to describe the motion of systems composed of joined part such as an engine, a robot arm or the skeleton of human body.

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The tangent operator is used to describe the human body kinematic chain and robotics motion in Euclidean space [9]. At this point, it is natural to ask how important the tangent operator? Derivative of a motion gives velocity of a point according to the fixed frame F and moving frame M. The tangent operator is used to calculate the derivative of a motion and it computes the direction tangent to a motion. The elements of the tangent operators of spatial motion can be reassembled into six dimensional vectors known as screws [3].

Screw theory is without doubt an efficient mathematical tool for the study of spatial kinematics, which is the algebra and calculus of pairs of vectors, such as forces and moments and angular and linear velocity, that arise in the kinematics and dynamics of rigid bodies [20]. This theory provides a mathematical formulation for the geometry of lines which is central to rigid body dynamics, where lines form the screw axis of spatial movement and the lines of action of forces. The pair of vectors that form the Plücker coordinates of a line define a unit screw, and general screws are obtained by multiplication by a pair of real numbers and addition of vectors [1].

An important result of screw theory is that geometric calculations for points using vectors have parallel geometric calculations for lines obtained by replacing vectors with screws. This is termed the transfer principle (or E.Study's mapping) [11,12].

Dual numbers are useful for analytical treatment in kinematics and dynamics of spatial mechanisms [4]. Clifford (1873) introduced dual numbers in the form of $x + \varepsilon y$ with $\varepsilon^2 = 0$ to form biquaternions for studying the non-Euclidean geometry [2]. Eduard Study (1903) defined dual numbers as dual angles to specify the relation between two lines in the Euclidean space. Also, he devoted special attention to the representation of directed line by line unit vectors and defined the mapping which is called with his name; he proved that there exists a one-to-one correspondence between the points of the dual unit sphere $\mathbb{S}^2_D$ and the directed lines of Euclidean 3-space $\mathbb{R}^3$ (E.Study's mapping). Subsequently, in [19] by taking the 3- dimensional Lorentzian space L^3 instead of $\mathbb{R}^3$, Uğurlu and Çalışkan give a correspondence of E.Study mapping as follows: The line timelike and spacelike unit vectors of dual hyberbolic and Lorentzian unit sphere $\mathbb{H}^2_0$ and $\mathbb{S}^2_1$ in the dual Lorentzian space $\mathbb{D}^3_1$ is one-to-one correspondence between the directed timelike and spacelike lines of the L^3, respectively.

A new matrix multiplication, called as Lorentzian matrix multiplication, is defined in $\mathbb{R}^{m\times n} \times \mathbb{R}^{n\times p}$ where $\mathbb{R}^{m\times n}$ is set of matrices of m rows and n columns by using Lorentzian inner product in $\mathbb{R}^n$ in [5]. In recent years several authors used this multiplication to define coordinate transformations on n-dimensional Lorentzian space L^n, to obtain Cayley formula and Euler parameters of a Lorentzian orthogonal matrix in L^3 and to obtain rotations and screw motion in L^3 [5,8,14,15]. Also tangent operator is introduced to generalize the properites of the angular velocity matrix of a rotation to the cases of planar and spatial motion in L^3 by means of this multiplication in [3].

In this paper we present various results about the six dimensional vectors obtained from the tangent operators of spatial motion, called as screws, in Lorentzian space. Each screw has an axis defined by six Plücker coordinates in Lorentzian space. In Sect. 5, the manipulation of screw coordinate transformations has been simplified by using Lorentz matrix multiplication and dual number algebra. Also, in Sect. 6, we showed that screw displacement is representation as the exponential of a dual angular velocity matrix by using the dual orthogonal matrices in Lorentzian space. But, before this, we remind some concepts and give some notations. We refer mainly to O'Neill, Lopez and Ratcliffe [10,13,16].

2. Preliminary

We denote by L^3 the 3-dimensional Lorentzian space with coordinates x, y and z, endowed with the Lorentzian metric tensor

$$\langle .,. \rangle_L = -dx^2 + dy^2 + dz^2.$$

In Lorentzian space, the vectors are characterized by Lorentzian inner product. For a vector $v = (v_1, v_2, v_3) \in L^3$, the vector v is said to be a spacelike if $\langle v, v \rangle_L > 0$ or $v = 0$, timelike if $\langle v, v \rangle_L < 0$, lightlike (or null) if $\langle v, v \rangle_L = 0$. For $v \in L^3$, the norm of v is defined by $\|v\|_L = \sqrt{|\langle v, v \rangle_L|}$ and v is called a unit vector if $\|v\|_L = 1$.

We recall the notion of the Lorentzian cross-product (see e.g. [10]): For any $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in L^3$, Lorentzian cross product is defined by

$$a \times_L b = (a_3 b_2 - a_2 b_3,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)\,.$$

As analogue to the vector cross product in the Euclidean space, it has similar algebraic and geometric properties:

(i) $a \times_L b$ is perpendicular to a and b, i.e. $\langle a \times_L b, a \rangle_L = \langle a \times_L b, b \rangle_L = 0$;
(ii) $a \times_L b = -b \times_L a$;
(iii) $\langle a \times_L b, a \times_L b \rangle_L = - \langle a, a \rangle_L \langle b, b \rangle_L + \langle a, b \rangle_L^2$, for all $a, b \in L^3$.

Lorentzian matrix multiplication is defined in [5] as follows:

Let $\mathbb{R}^{m \times n}$ be the set of matrices of m rows and n columns. Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ and $B = [b_{jk}] \in \mathbb{R}^{n \times p}$. Lorentzian matrix multiplication denoted by "$\cdot_L$", and simply called L-multiplication, is defined as

$$A \cdot_L B = \left[-a_{i1} b_{1k} + \sum_{j=2}^{n} a_{ij} b_{jk} \right].$$

Note that $A \cdot_L B$ is an $m \times p$ matrix.

The $n \times n$ L-identity matrix corresponding to L- multiplication, denoted by I_n is defined by

$$I_n = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n}.$$

An $n \times n$ matrix A is called L-invertible if there exists an $n \times n$ matrix B such that $A \cdot_L B = B \cdot_L A = I_n$. Then B is called L-inverse of A and is denoted by A^{-1}.

A matrix A is called L-orthogonal matrix if $A^{-1} = A^T$. Observe that $\det A = \mp 1$ for L-orthogonal matrix A. If $\det A = -1$, then A is defines a rotation and if $\det A = 1$, then A is defines a reflection (see [5]).

A matrix A is called L-skew-symmetric matrix if $A^T = -A$. In L^3, it is known that every 3×3 skew-symmetric matrix B has only 3 independent elements, that is

$$B = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & -b_1 \\ b_2 & b_1 & 0 \end{bmatrix}.$$

These elements can be assembled into the vector $b = (b_1, b_2, b_3)$. Note that the matrix B satisfying the equality $B._L y = b \wedge_L y$ (see [14]).

To study the position of one body relative to another, we attach coordinate frames to each. One is chosen as the ground with coordinate frame F, and the other, the moving body, has the coordinate frame M. The coordinate transformation $D : F \to M$, is given by $X = A._L x + d$, where x is the coordinate vector point in F and X is the coordinate vector of the same point but measured in M. If the moving body is of dimension n, then A is an $n \times n$ matrix and d is an n-dimensional vector. This transformation must be rigid transformation then A is an $n \times n$ L-orthogonal matrix.

A displacement in L^n is defined by the matrix-vector pair $D = (A, d)$, where A is an $n \times n$ L-orthogonal matrix and d is an n-dimensional vector. Let us consider the function

$$f : L^n \longrightarrow L^n, \ f(x) = A \cdot_L x + d,$$

is not a linear transformation. Therefore, the displacement in L^n, may not be represented by $n \times n$ matrix transformation. This inconvenience is removed by embedding L^n in L^{n+1} as follows: if $(x_1, x_2, \ldots, x_n)$ be a vector in L^n then the corresponding $(n+1)$-dimensional vector in the Lorentzian hyperplane $H : x_{n+1} = 1$ is $(x_1, x_2, \ldots, x_n, 1)$. We show this vector as $(x, 1)$. This placement

$$X = A \cdot_L x + d,$$

in L^n, can be given as

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} A & d \\ 0 & 1 \end{bmatrix} \cdot_L \begin{bmatrix} x \\ 1 \end{bmatrix},$$

in L^{n+1}. The $(n+1) \times (n+1)$ matrix $T = [A, d]$ is called the homogeneous transform representing the displacement of $D = (A, d)$, and T is linear. The set of $(n+1) \times (n+1)$ homogeneous transforms is a matrix group denoted as $SE(1, n)$ [6].

The derivative and the tangent operator of a motion are computed in Lorentzian space in [3]. The tangent operator is introduced as a generalization of the angular velocity matrix of continuous rotation to all continuous displacements. Tangent operators are the elements of the Lie algebra of the set of rigid transformations [18]. The spherical and spatial motions that have constant tangent operators are shown to be pure rotations and screw displacements, respectively.

$O(1,2)$ is the set of 3×3 matrices which are nonzero of these determinant, $SO(3)$ is the set of 3×3 L-orthogonal matrix and $SE(1,3)$ and $SE(1,4)$ are the set of homogen maps. The continuous motion of a rigid body is the parameterized set of linear transformations $T(t) : \mathbb{R} \rightarrow O(1,2)$.

In L^3, planar motion is defined by $T(t) : \mathbb{R} \rightarrow SE(1,3)$, spherical motion by $T(t) : \mathbb{R} \rightarrow SO(3)$, and spatial motion by $T(t) : \mathbb{R} \rightarrow SE(1,4)$.

In general, each of the elements $t_{ij}(t)$ in $T(t) : \mathbb{R} \rightarrow O(1,2)$ is a continuous function of a real parameter. Thus, the derivative of this matrix function is the matrix of derivatives of every elements. $[T'(t_0)]$ defines the tangent direction of the motion at $[T(t_0)]$. The matrix function $T(t) : \mathbb{R} \rightarrow O(1,2)$ produces a continuous set of points

$$Y(t) = T(t) \cdot_L y,$$

called the trajectory of y. The direction of the tangent to the trajectory $Y(t)$ at $t = t_0$ is the derivative

$$Y'(t_0) = T'(t_0) \cdot_L y = \left[T' \cdot_L T^{-1}(t_0)\right] \cdot_L Y(t_0).$$

Hence, the matrix $\left[T' \cdot_L T^{-1}\right]$ calculates the derivative $Y'(t)$ by using the trajectory $Y(t)$. Since

$$[T'(t)] = \left[T' \cdot_L T^{-1}\right] \cdot_L [T(t)]$$

$\left[T' \cdot_L T^{-1}\right]$ also computes the derivative of $T(t)$. $\left[T' \cdot_L T^{-1}\right]$ matrix is called the $L-$Tangent operator on $O(1,2)$ [3].

If we shall identify the motion $[R(t)]$ that has a constant matrix $[S]$ as its tangent operator, $[S]$ calculate the derivative $[R'(t)]$ at every point $[R(t)]$. Then the matrix differential equation

$$[R'(t)] = [S] \cdot_L [R(t)].$$

If the initial condition is $[R(0)] = [I]$, then the solution becomes

$$[R(t)] = e^{t[S]}.$$

The tangent operators of $SO(3)$ is given by

$$\left[A' \cdot_L A^T\right] = [\Omega],$$

is skew symmetric. $[\Omega]$ is L-angular velocity matrix of the rotation $[A(t)]$. On spatial rotations $SO(3)$, for a given constant L-angular velocity matrix $[\Omega]$, we have one-parameter group of rotations from the matrix differential equation

$$[A'(t)] = [\Omega] \cdot_L [A(t)].$$

If the initial condition is $[A(0)] = [I]$, then the solution becomes

$$[A(t)] = e^{t[\Omega]}.$$

Also, tangent operators of $SE(1,3)$and $SE(1,4)$ is given by

$$\begin{aligned}\left[T' \cdot_L T^{-1}\right] &= \begin{bmatrix} A' & d' \\ 0 & 0 \end{bmatrix} \cdot_L \begin{bmatrix} A^T & -A^T \cdot_L d \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} A' \cdot_L A^T & -A' \cdot_L A^T \cdot_L d + d' \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix} \\ &= [\Omega, v],\end{aligned}$$

where $[\Omega] = \left[A' \cdot_L A^T\right]$ is the 2×2 and 3×3 L-angular velocity matrix of the moving body respectively and $v = -\Omega \cdot_L d + d'$ is its two and three dimensional linear velocity vector respectively. So $SE(1,3)$ has three dimension and $SE(1,4)$ has six.

The six-dimensional vector form $S = (w, v)$ of a tangent operator $[S] = [\Omega, v]$ of $SE(1,4)$ is called a screw [3], where w is the vector, corresponding to the L-skew symmetric matrix $[\Omega]$.

We now give a brief summary of the theory of dual numbers and line Lorentzian vectors. If x and y are real numbers and, the combination $z = x + \varepsilon y$ is called a dual number, where ε is dual unit, that is $\varepsilon \neq 0$, $\varepsilon^2 = 0$. Vectors and matrices with dual number elements are term line vectors and dual matrices, respectively.

Like a real number which can be considered as an angle, in differential geometry and motion analysis of spatial mechanisms, a dual number is also commonly referred as a dual angle between two lines in the space. The real part x of the dual angle is the projected angle between the lines, and the dual part y is the length along the common normal of the lines. The set of all dual numbers forms a commutative ring over the real number field and is denoted by $\mathbb{D}$. Then the set

$$\mathbb{D}^3 = \{A = (z_1, z_2, z_3) \mid z_i \in \mathbb{D}, 1 \leq i \leq 3\}$$

is a module over the ring $\mathbb{D}$ which is called a $\mathbb{D}$-module or dual space. The elements of $\mathbb{D}^3$ are called line vectors. Thus a line vector A can be written

$$A = a_1 + \varepsilon a_2$$

where a_1 and a_2 are real vectors at $\mathbb{R}^3$ [7]. The dual Lorentzian inner product of two line vectors $A = a_1 + \varepsilon a_2$, $B = b_1 + \varepsilon b_2$ is defined by

$$\langle A, B\rangle_L = \langle a_1, b_1\rangle_L + \varepsilon\left(\langle a_1, b_2\rangle_L + \langle a_2, b_1\rangle_L\right),$$

where $\langle a, b\rangle_L$ is the Lorentzian inner product of the vectors a and b in L^3.

Then a line vector $A = a_1 + \varepsilon a_2$ is said to be timelike if a_1 is timelike, spacelike if a_1 is spacelike or $a_1 = 0$, and lightlike (null) if a_1 is lightlike (null) and $a_1 \neq 0$. The set of all line Lorentzian vectors is called dual Lorentzian space and it is denoted by $\mathbb{D}^3_1$.

Let $A = a_1 + \varepsilon a_2 \in \mathbb{D}^3_1$. Then A is is said to be unit line timelike vector (resp., unit line spacelike vector) if the vectors a_1 and a_2 satisfy the following equations:

$$\langle a_1, a_1\rangle_L = -1 \text{ (resp. } \langle a_1, a_1\rangle_L = 1), \ \langle a_1, a_2\rangle_L = 0.$$

The set of all unit line timelike vectors (resp., all unit line spacelike vectors) is called the dual hyperbolic unit sphere (resp. dual Lorentzian unit sphere), and is denoted by $\widetilde{\mathbb{H}}_0^2$ (resp. $\widetilde{\mathbb{S}}_1^2$) [19].

Theorem: (E. Study's Mapping): The unit line timelike (resp. spacelike) vectors of the dual hyperbolic unit sphere $\widetilde{\mathbb{H}}_0^2$ (resp. $\widetilde{\mathbb{S}}_1^2$) are in one-to-one correspondence with the directed timelike (resp. spacelike) lines of L^3 [19,21].

The dual number $\widehat{\theta} = \theta + \varepsilon d$, the dual central angle (resp. the dual central hyperbolic angle) between line spacelike (resp. timelike) unit vectors. The dual angle (resp. dual hyperbolic angle) $\widehat{\theta} = \theta + \varepsilon d$ consists of the angle (resp. hyperbolic angle) between directed spacelike (resp. timelike) lines which are represented in L^3 of line spacelike (resp. timelike) unit vectors and the smallest distance d between two lines [19,21].

3. Screw Coordinate Transformations

As we mentioned before, the general tangent operator of $SE(1,4)$ is given by $[S] = [\Omega, v]$, where $[\Omega]$ is a 3×3 L-skew symmetric matrix and v is a three dimensional vector. Let w be the vector obtained from $[\Omega]$, then associated with $[S]$ is the six dimensional vector $S = (w, v)$, called a screw. We now determine how the coordinates of this screw change reference frames for the motion.

Let a motion be given as the parameterized displacement $T(t) : F \to M$, and the let position of fixed and moving frames F and M relative to the new coordinate frames, F' and M', be defined by the displacements $D : F' \to F$ and $D : M \to M'$; then the motion $T'(t) : F' \to M'$, is defined by $T'(t) = D \cdot_L T(t) \cdot_L D^{-1}$. The tangent operator $[S']$ of $T'(t)$ is determined from the tangent operator $[S]$ of the original motion $T(t)$ by the computation

$$\begin{aligned}[S'] &= \left[D \cdot_L T' \cdot_L D^{-1}\right] \cdot_L \left[D \cdot_L T^{-1} \cdot_L D^{-1}\right] \\ &= \left[D \cdot_L T' \cdot_L T^{-1} \cdot_L D^{-1}\right] \\ &= \left[D \cdot_L S \cdot_L D^{-1}\right].\end{aligned}$$

If the displacement is $[D] = [A, d]$, then this equation expands to yield

$$\begin{aligned}[S'] &= \begin{bmatrix} A & d \\ 0 & 1 \end{bmatrix} \cdot_L \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix} \cdot_L \begin{bmatrix} A^T & -A^T \cdot_L d \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} A \cdot_L \Omega \cdot_L A^T & -A \cdot_L \Omega \cdot_L A^T \cdot_L d + A \cdot_L v \\ 0 & 0 \end{bmatrix}\end{aligned}$$

The components of the L-skew symmetric matrix $[\Omega'] = \left[A \cdot_L \Omega \cdot_L A^T\right]$ can be assembled into the vector W. By using the relationship between skew-symmetric matrices and the Lorentzian cross product, we have

$$\begin{aligned}-\left[A \cdot_L \Omega \cdot_L A^T\right] \cdot_L d &= -\left[\Omega'\right] \cdot_L d \\ &= -W \times_L d \\ &= d \times_L W.\end{aligned}$$

Thus the screw $S = (w, v)$ becomes $S' = (W, V)$ given by

$$W = [A] \cdot_L w$$
$$V = d \times_L [A] \cdot_L w + [A] \cdot_L v.$$

This transformation can be written as the 6×6 matrix equation

$$\begin{bmatrix} W \\ V \end{bmatrix} = \begin{bmatrix} A & 0 \\ D \cdot_L A & A \end{bmatrix} \cdot_L \begin{bmatrix} w \\ v \end{bmatrix}, \tag{3.1}$$

where $[D]$ is L-skew symmetric matrix obtained from d and $[0]$ is the 3×3 zero matrix. This equation transforms a screw to account for a change of the fixed and moving frames defining a motion.

4. Standart Form for a Screw

Now, we want to obtain a special case of (3.1) the screw transformation for a pure translation $[T] = [I, d]$ of the coordinate frames F and M. In this case $S = (w, v)$ transforms to

$$S' = (w, d \times_L w + v).$$

It shows that translations of the origin of the coordinate frames only change the linear velocity component of the screw. We will use this fact to obtain a standart form for screws.

The linear velocity vector v of the screw, $S = (w, v)$ can be decomposed into components parallel and perpendicular to w as follows: The parallel component is kw, where k is given by

$$k = \frac{\langle v, w \rangle_L}{\langle w, w \rangle_L},$$

and the perpendicular component is $v - kw$, however we can determine a vector c such that this component is given by $c \times_L w$. That is, the vector c satisfied the equation

$$c \times_L w = v - kw.$$

The solution to this equation is obtained by computing cross product of both sides with w, the result is,

$$c = \frac{w \times_L v}{\langle w, w \rangle_L}. \tag{4.1}$$

Change the references frame by the translation $[T] = [I, -c]$, so that $S = (w, v)$ is transformed into the screw $S' = (w, -c \times_L w + v)$. By construction $c \times_L w$ equals the component of v perpendicular to w, therefore

$$-c \times_L w + v = kw.$$

So, the screw takes the form $S' = (w, kw)$ with its linear and angular velocity vectors aligned in the direction w, in new reference frame. In this frame we see explicity that the body moves in a screw motion at the instant defined by t. We called the axis of this screw motion in the original set of frames is the line $L = c + tw$, as the instantaneous screw axis of the motion in Lorentzian space. The point c, defined by (4.1) can be used to write a general screw, $S=(w,v)$, in the standart form

$$S=(w, c\times_L w+kw), \tag{4.2}$$

where v has been decomposed into components parallel and perpendicular to w. The line $L=c+tw$ is the axis of the screw S and the constant k is its pitch.

4.1. Plücker Coordinates of a Line

An important special case of a screw is obtained from (4.2) when the pitch $k=0$. So there is no velocity vector and the screw has the form

$$S=(w, c\times_L w).$$

The body has an angular velocity but no linear velocity relative to the line $L=c+tw$. In this case the screw can be identifying the line L and its components are called the Plücker coordinates of the line.

Another case in which screws degenerate to line screw occurs when the screw has the form $S=(0,v)$. This corresponds to a motion in which body has no angular velocity, only the linear velocity associated with a pure translation.

5. Screw and Dual Orthogonal Matrices

In this section, the manipulation of screw coordinate transformations has been simplified by using Lorentz matrix multiplication and dual number algebra. Let the screws $S=(w,v)$ and $S'=(W,V)$ be represented by the line vectors $\widehat{w}=w+\varepsilon v$ and $\widehat{W}=W+\varepsilon V$, then the screw transformation equation (3.1) can be written in the form

$$\widehat{W}=\left[\widehat{A}\right]\cdot_L \widehat{w}$$

where

$$\left[\widehat{A}\right]=[A]+\varepsilon\,[D]\cdot_L [A], \tag{5.1}$$

and $[D]$ is L-skew-symmetric matrix.

Since

$$\begin{aligned}\left[\widehat{A}\right]\cdot_L\left[\widehat{A}^T\right] &= ([A]+\varepsilon\,[D]\cdot_L[A])\cdot_L\left(\left[A^T\right]+\varepsilon\left[A^T\right]\cdot_L\left[D^T\right]\right)\\ &= I+\varepsilon\left([D]+\left[D^T\right]\right)+\varepsilon^2\,[D]\cdot_L\left[D^T\right]\\ &= I,\end{aligned}$$

the dual matrix $\left[\widehat{A}\right]$ has a property that $\left[\widehat{A}^{-1}\right]=\left[\widehat{A}^T\right]$ which means it is a 3×3 dual L-orthogonal matrix with dual number elements and defines a general transformation of screw coordinates associated with a change of reference frames.

Case 1: The L-dual orthogonal matrix associated with the screw displacement of hyperbolic angle θ about the spacelike z-axis, and a distance d along it, is easily obtained from (5.1) as

$$\left[\widehat{Z}\right] = \begin{bmatrix} -\left(\cosh\theta + \varepsilon d \sinh\theta\right) & \sinh\theta + \varepsilon d \cosh\theta & 0 \\ -\left(\sinh\theta + \varepsilon d \cosh\theta\right) & \cosh\theta + \varepsilon d \sinh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.2}$$

From the Taylor series expansion of the sine hyperbolic and cosine hyperbolic functions of a dual angle by the expression

$$\begin{cases} \cosh\widehat{\theta} & = \cosh\left(\theta + \varepsilon d\right) = \cosh\theta + \varepsilon d \sinh\theta, \\ \sinh\widehat{\theta} & = \sinh\left(\theta + \varepsilon d\right) = \sinh\theta + \varepsilon d \cosh\theta \end{cases} \tag{5.3}$$

The dual L-orthogonal matrix of the screw displacement (5.2) can now be written, using (5.3), as

$$\left[\widehat{Z}\right] = \begin{bmatrix} -\cosh\widehat{\theta} & \sinh\widehat{\theta} & 0 \\ -\sinh\widehat{\theta} & \cosh\widehat{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In the same way the L-dual orthogonal matrix associated with the screw displacement along the spacelike x-axis is given by

$$\left[\widehat{X}\right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cosh\widehat{\alpha} & \sinh\widehat{\alpha} \\ 0 & \sinh\widehat{\alpha} & \cosh\widehat{\alpha} \end{bmatrix},$$

where the dual hyperbolic angle $\widehat{\alpha} = \alpha + \varepsilon a$ is formed from the rotation α about the spacelike x-axis and translation a along it. The dual orthogonal transformation

$$\left[\widehat{T}\right] = \left[\widehat{Z}\right] \cdot_L \left[\widehat{X}\right],$$

which is often used to relate to coordinate frames in consecutive links of a manipulator, becomes

$$\left[\widehat{T}\right] = \begin{bmatrix} -\cosh\widehat{\theta} & \cosh\widehat{\alpha}\sinh\widehat{\theta} & \sinh\widehat{\theta}\sinh\widehat{\alpha} \\ -\sinh\widehat{\theta} & \cosh\widehat{\theta}\cosh\widehat{\alpha} & \cosh\widehat{\theta}\sinh\widehat{\alpha} \\ 0 & \sinh\widehat{\alpha} & \cosh\widehat{\alpha} \end{bmatrix}. \tag{5.4}$$

Case 2: The dual L-orthogonal matrix associated with the screw displacement of angle θ about the timelike z-axis, and a distance d along it, it is easily obtained from (5.1)

$$\left[\widehat{Z}\right] = \begin{bmatrix} -\left(\cos\theta - \varepsilon d \sin\theta\right) & \sin\theta + \varepsilon d \cos\theta & 0 \\ -\left(\sin\theta + \varepsilon d \cos\theta\right) & \cos\theta - \varepsilon d \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From the Taylor series expansion of the sine and cosine functions of a dual angle by the expression

$$\cos\widehat{\theta} = \cos\left(\theta + \varepsilon d\right) = \cos\theta - \varepsilon d \sin\theta, \tag{5.5}$$

$$\sin\widehat{\theta} = \sin\left(\theta + \varepsilon d\right) = \sin\theta + \varepsilon d \cos\theta.$$

The dual L-orthogonal matrix of the screw displacement (5.4) can now be written, using (5.5), as

$$\left[\widehat{Z}\right] = \begin{bmatrix} -\cos\widehat{\theta} & -\sin\widehat{\theta} & 0 \\ -\sin\widehat{\theta} & \cos\widehat{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In the same way the L-dual orthogonal matrix associated with the screw displacement along the timelike x-axis is given by

$$\left[\widehat{X}\right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\widehat{\alpha} & -\sin\widehat{\alpha} \\ 0 & \sin\widehat{\alpha} & \cos\widehat{\alpha} \end{bmatrix},$$

where the dual angle $\widehat{\alpha} = \alpha + \varepsilon a$ is formed from the rotation α about the timelike x-axis and translation a along it. The dual orthogonal transformation

$$\left[\widehat{T}\right] = \left[\widehat{Z}\right] \cdot_L \left[\widehat{X}\right],$$

which is often used to relate to coordinate frames in consecutive links of a manipulator, becomes

$$\left[\widehat{T}\right] = \begin{bmatrix} -\cos\widehat{\theta} & -\cos\widehat{\alpha}\sin\widehat{\theta} & \sin\widehat{\theta}\sin\widehat{\alpha} \\ -\sin\widehat{\theta} & \cos\widehat{\theta}\cos\widehat{\alpha} & -\cos\widehat{\theta}\sin\widehat{\alpha} \\ 0 & \sin\widehat{\alpha} & \cos\widehat{\alpha} \end{bmatrix}.$$

Example. Let the rotation

$$[A] = \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} & 0 \\ -\frac{3}{4} & \frac{5}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

be about z spacelike axis of the Lorentzian plane of Oxy coordinate system and translation vector be $d = (0, 0, 1)$. The L-skew symmetric matrix D corresponding to the vector d is

$$[D] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The dual L-orthogonal matrix associated with the screw displacement of hyberbolic angle $\theta = 0.69315$ about the z spacelike axis, and a distance d along it, is obtained

$$\begin{aligned}\left[\widehat{A}\right] &= [A] + \varepsilon\,[D] \cdot_L [A] \\ &= \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} & 0 \\ -\frac{3}{4} & \frac{5}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\quad +\varepsilon \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot_L \begin{bmatrix} -\frac{5}{4} & \frac{3}{4} & 0 \\ -\frac{3}{4} & \frac{5}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{5}{4} - \varepsilon\frac{3}{4} & \frac{3}{4} + \varepsilon\frac{5}{4} & 0 \\ -\frac{3}{4} - \varepsilon\frac{5}{4} & \frac{5}{4} + \varepsilon\frac{3}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}.\end{aligned}$$

Notice that $\left[\widehat{A}\right] \cdot_L \left[\widehat{A}\right]^T = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = I_2.$

6. Dual Matrix Exponential

Let $\left[\widehat{A}(t)\right]$ be a parameterized set of L-dual orthogonal matrices. The fact that the tangent operator $\left[\frac{d\widehat{A}}{dt}\right] \cdot_L \left[\widehat{A}^T\right] = \left[\widehat{\Omega}\right]$ is a L-skew symmetric is obtained from the computation

$$\begin{aligned}\frac{d\,[I]}{dt} &= \frac{d}{dt}\left[\widehat{A}\,(t) \cdot_L \widehat{A}^T\,(t)\right] \\ &= \left[\frac{d\widehat{A}(t)}{dt}\right] \cdot_L \left[\widehat{A}^T(t)\right] + \left[\widehat{A}(t)\right] \cdot_L \left[\frac{d\widehat{A}^T(t)}{dt}\right] \\ &= 0,\end{aligned}$$

which yields the result $\left[\widehat{\Omega}\right] + \left[\widehat{\Omega}^T\right] = 0$. We call the matrix $\left[\widehat{\Omega}\right]$ as the L-dual angular velocity matrix in Lorentzian space.

Let $\widehat{w}$ be a line vector obtained from $\left[\widehat{\Omega}\right]$ by the identifying the product $\left[\widehat{\Omega}\right] \cdot_L y$ with the cross product $\widehat{w} \times_L y$. This line vector has the form $\widehat{w} = w + \varepsilon w^0$. Its dual part can be decomposed into $w^0 = c \times_L w + kw$, where

$$c = \frac{w \times_L w^0}{\langle w, w\rangle_L} \text{ and } k = \frac{\langle w, w^0\rangle_L}{\langle w, w\rangle_L}.$$

Then we get,

$$\begin{aligned}\widehat{w} &= w + \varepsilon\,(c \times_L w + kw) \\ &= (1 + \varepsilon k)\,(w + \varepsilon c \times_L w)\,.\end{aligned}$$

The line vector $w + \varepsilon c \times_L w$ defines the instantaneous screw axis and k is the pitch of instantaneous screw motion defined by $\left[\widehat{A}(t)\right]$.

The motion $\left[\widehat{A}\right]$ with a constant dual angular velocity $\left[\widehat{\Omega}\right]$ is the solution to matrix differential equation

$$\left[\frac{d\widehat{A}}{dt}\right] \cdot_L \left[\widehat{A}^T\right] = \left[\widehat{\Omega}\right]$$

or

$$\left[\frac{d\widehat{A}}{dt}\right] = \left[\widehat{\Omega}\right] \cdot_L \left[\widehat{A}\right].$$

For the initial condition $\left[\widehat{A}(0)\right] = [I]$, the solution is,

$$\left[\widehat{A}(t)\right] = \exp(\widehat{\Omega}t). \tag{6.1}$$

Now, we show that $\left[\widehat{A}(t)\right]$ is a screw motion in Lorentzian space about the axis $w + \varepsilon c \times_L w$ obtained from $\left[\widehat{\Omega}\right]$.

First, consider the case $c \times_L w = 0$. This implies that the screw axis passes through the origin of fixed coordinate frame, in which case we have $\widehat{w} = (1 + \varepsilon k)\, w$, so the displacement $\left[\widehat{A}\right]$ has the special L-skew symmetric matrix $\left[\widehat{\Omega}\right] = (1 + \varepsilon k)\,[\Omega]$. Since $(1 + \varepsilon k)^n = 1 + n\varepsilon k$, we obtain from (6.1)

$$\begin{aligned} \exp\left((1 + \varepsilon k)\,\Omega t\right) &= [I + (1 + \varepsilon k)\,\Omega t \\ &\quad + \frac{1}{2}\left(1 + \varepsilon 2k\right)\Omega^2 t^2 + \cdots] \\ &= [I + \Omega t + \frac{1}{2}\Omega^2 t^2 + \cdots] \\ &\quad + \varepsilon k t \Omega [I + \Omega t + \frac{1}{2}\Omega^2 t^2 + \cdots]. \end{aligned} \tag{6.2}$$

Let $[A(t)]$ be the exponential of a L-skew symmetric matrix $[\Omega]$, then (6.2) defines the L-dual orthogonal matrix $\left[\widehat{A}(t)\right] = [A(t)] + \varepsilon k t\,[\Omega] \cdot_L [A(t)]$. This transformation defines a rotation of angle $\phi(t) = \omega t$ about w, where $\omega = \|w\|$, and a translation along it by the amount $d(t) = ktw$. Notice that ω is the angular velocity of this motion and $v = k\omega$ is its linear velocity. This is a screw motion of pitch k.

Remark 6.1. If w is spacelike (or timelike) vector, this motion consists of rotation of angular velocity $\omega = w$ about the line in the direction w through origin and the linear velocity kw along this line in timelike (or spacelike) plane, respectively.

If the screw axis does not pass through the origin, then $c \times_L w \neq 0$. In this case the motion $\left[\widehat{A}(t)\right]$ can be factored into

$$\left[\widehat{A}(t)\right] = \left[\widehat{D}\right] \cdot_L \exp\left((1 + \varepsilon k)\,\Omega t\right) \cdot_L \left[\widehat{D}\right]^{-1},$$

where $\left[\widehat{D}\right]$ is the translation $\left[\widehat{D}\right] = [I + \varepsilon C]$, and $[C]$ is the L-skew symmetric matrix obtained from c. The result is

$$\begin{aligned}
&\left[\widehat{D}\right] \cdot_L \exp\left(\left(1+\varepsilon k\right)\Omega t\right) \cdot_L \left[\widehat{D}\right]^{-1} \\
&\quad = \exp\left(\left[\widehat{D}\right] \cdot_L \left(1+\varepsilon k\right)[\Omega] \cdot_L \left[\widehat{D}\right]^{-1}\right) \\
&\quad = \exp\left(\left(1+\varepsilon k\right)\left[\Omega + \varepsilon\left(C \cdot_L \Omega - \Omega \cdot_L C\right)t\right]\right) \qquad (6.3)
\end{aligned}$$

Since $[C \cdot_L \Omega - \Omega \cdot_L C]$ is the L-skew symmetric matrix obtained from the vector $c \times_L w$, we see that $[\Omega + \varepsilon [C \cdot_L \Omega - \Omega \cdot_L C]]$ defines the screw axis $w + \varepsilon c \times_L w$. Expand (6.3) to obtain

$$\begin{aligned}
\left[\widehat{A}(t)\right] &= [A(t)] + \varepsilon\left[kt\Omega \cdot_L A(t) + C \cdot_L A(t) - A(t) \cdot_L C\right] \\
&= [A(t)] + \varepsilon\left[kt\Omega + C - A \cdot_L C \cdot_L A^T\right][A(t)].
\end{aligned}$$

The translation component of the motion is defined by the L-skew symmetric matrix $\left[kt\Omega + C - A \cdot_L C \cdot_L A^T\right]$, which takes the vector form

$$d(t) = ktw + [I - A(t)]\, c.$$

It can be see that, it is the displacement associated with a screw motion of pitch k about the axis $w + \varepsilon c \times_L w$ in Lorentzian space.

7. Conclusion

A lot of powerful tools are available which allow to describe the motion of rigid bodies in a geometrical and global way. These methods are related to the geometry of lines and screws and to the differential geometric concept of a Lie group. Screw theory describes the mathematical foundations, especially geometric, underlying the motions and force-transfers in robots and it has become an important tool in robot mechanics, mechanical design, computational geometry and multibody dynamics in Euclidean space.

Screw theory has become an important tool in robot mechanics, mechanical design, computational geometry and multibody dynamics. This is in part because of the relationship between screws and dual quaternions which have been used to interpolate rigid-body motions. Based on screw theory, an efficient approach has also been developed for the type synthesis of parallel mechanisms (parallel manipulators or parallel robots).

In this paper, screw theory is introduced in Lorentzian space and the advantages of screw theory in Euclidean space are transformed to Lorentzian space.

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