OPTİK FİBERLERDE IŞIK DARBELERİNİN YAYILIMINI MODELLEYEN DENKLEMİN TAM ÇÖZÜMLERİ

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ÖZET

Bu çalışmada, optik fiberlerde ışık darbelerinin yayılımını modelleyen ve dördüncü mertebeden doğrusal olmayan bir kısmi diferansiyel denklem ele alınmıştır. Söz konusu denklem, doğrusal olmayan terimler ve yüksek mertebeden dispersiyon etkilerini içermesi nedeniyle oldukça karmaşık bir yapıya sahiptir. Optik fiber sistemlerinde veri iletimi, soliton oluşumu ve solitonların dinamik davranışlarının anlaşılması için önemli bir matematiksel çerçeve sunmaktadır. Bu nedenle, denklemin tam çözümlerinin elde edilmesi hem teorik hem de uygulamalı bilimler açısından büyük önem taşımaktadır.

Denklemin tam çözümlerinin elde edilmesi için literatürde sistematik ve etkili yöntemler olarak bilinen üç farklı yöntem kullanılmıştır: Sardar alt denklem yöntemi, Riccati-Bernoulli yöntemi ve $\frac{G'}{kG'+G+r}$-genişleme yöntemi. Bu yöntemler, doğrusal olmayan kısmi diferansiyel denklemlerin çözümünde sıklıkla kullanılan güçlü tekniklerdir. Çalışma kapsamında, bu yöntemler ile yeni ve özgün tam çözümler türetilmiştir. Elde edilen çözümler, yalnızca matematiksel olarak önemli sonuçlar sunmakla kalmamış, aynı zamanda optik fiberlerde soliton davranışlarının modellenmesi, plazma fiziğinde elektromanyetik dalgaların yayılımının anlaşılması ve fiber optik teknolojilerinin geliştirilmesi gibi çeşitli bilimsel alanlarda uygulama potansiyeli göstermiştir.

Elde edilen analitik çözümlerin fiziksel anlamı, bazı özel çözümlere ait grafikler aracılığıyla görselleştirilmiştir. Bu grafikler, soliton davranışlarının zamana ve mekâna bağlı evrimini açıklamakta kritik bir rol oynamıştır. Çözümlerin pratik doğruluğunu ve bilimsel önemini vurgulayan bu görselleştirmeler, doğrusal olmayan fenomenlerin daha derinlemesine anlaşılmasına katkıda bulunmuştur.

Sonuç olarak, bu çalışma, zorlu yapıya sahip bir denklemin tam çözümlerini sunarak, doğrusal olmayan kısmi diferansiyel denklemler ve bunların çeşitli bilimsel ve mühendislik uygulamalarına yönelik literatüre önemli bir katkı sağlamaktadır.

Anahtar Kelimeler: Optik, Tam çözümler, Sardar-alt denklem yöntemi, Riccati-Bernouilli yöntemi, $\frac{G'}{kG'+G+r}$-genişleme yöntemi, lineer olmayan kısmi diferansiyel denklemler

EXACT SOLUTIONS OF THE EQUATION MODELING LIGHT PULSE PROPAGATION IN OPTICAL FIBERS

Abstract

In this study, a fourth-order nonlinear partial differential equation modeling the propagation of light pulses in optical fibers is addressed. The equation exhibits a highly complex structure due to its inclusion of nonlinear terms and higher-order dispersion effects. It provides a significant mathematical framework for understanding data transmission, soliton formation, and the dynamic behavior of solitons in optical fiber systems. Consequently, obtaining exact solutions to the equation holds great importance both in theoretical and applied sciences.

To derive exact solutions to the equation, three different methods widely recognized in the literature for their systematic and effective approach were employed: the Sardar sub-equation method, the Riccati-Bernoulli method, and the $\frac{G'}{kG'+G+r}$-expansion method. These techniques are powerful tools frequently used for solving nonlinear partial differential equations. Within the scope of this study, novel and original exact analytical solutions were obtained using these methods. The solutions not only provide valuable mathematical insights but also demonstrate applicability in various scientific fields, such as modeling soliton behavior in optical fibers, understanding the propagation of electromagnetic waves in plasma physics, and advancing fiber optic technologies.

The physical significance of the analytical solutions was visualized through graphs of specific solutions. These visualizations play a critical role in explaining the spatiotemporal evolution of soliton behaviors. By emphasizing the practical accuracy and scientific importance of the solutions, these graphs contribute to a deeper understanding of nonlinear phenomena.

In conclusion, this study offers exact solutions to a challenging equation and makes a substantial contribution to the literature on nonlinear partial differential equations and their diverse applications in scientific and engineering fields.

Keywords: Optics, Exact solutions, Sardar sub-equation method, Riccati-Bernoulli method, $\frac{G'}{kG'+G+r}$-expansion method, nonlinear partial differential equations.

1. INTRODUCTION

Differential equations are critical tools for understanding dynamic systems and mathematical modeling. These equations hold great significance with their exact solutions in various fields of science and engineering. For example, fundamental theories such as Newton's laws of motion, Maxwell's electromagnetic equations, and Schrödinger's equations can clearly explain physical phenomena under specific boundary conditions by providing exact solutions. These exact solutions not only offer theoretical insight but also enable the prediction of the future behavior of systems.

The fourth-order nonlinear Schrödinger equation, used to study the propagation of light pulses in optical fibers, also provides exact solutions under certain conditions. Such solutions specifically explain the emergence of stable and repeatable structures like solitons. Solitons are special light pulses where dispersion and nonlinear effects are balanced, allowing them to propagate through the fiber without distortion.

The fourth-order Schrödinger equation was developed to explain the relationship between dispersion and nonlinear effects. A more general form of this equation can be expressed as follows:

$$u_{xxxx}+\alpha u_{xx}+\mu|u|^2u+\nu|u|^4u+\kappa|u|^2u_{xx}+i\left(u_t+\beta u_{xxx}+\frac{\beta\chi}{2}|u|^2u_x\right)=0 \quad (1)$$

In this equation: u, represents the amplitude of the light pulse. utu_t is the derivative with respect to time, and u_x and u_{xx} are the first and second spatial derivatives, respectively. The parameters $\alpha,\mu,\nu,\kappa,\beta$ characterize the physical system and define dispersion, nonlinear effects, and higher-order effects.

In more complex cases, the width and behavior of solitons can change due to the effects of fourth-order dispersion and higher-order nonlinear terms. Exact solutions are powerful tools for analytically examining such effects and determining the properties of light pulses.

These solutions are used to study the stability of solitons in optical communication systems, model the propagation of femtosecond laser pulses, and design optical systems. The exact solutions of the fourth-order nonlinear Schrödinger equation play a critical role in understanding and controlling complex systems in science and engineering. This makes differential equations an indispensable tool in science and technology.

2. PRELIMINARIES AND APPLICATIONS

Exact solution methods enable the analytical solving of differential equations and provide a comprehensive understanding of the physical or mathematical systems these equations represent. The exact solutions of differential equations are indispensable for predicting the future behavior of a system, understanding boundary conditions, and analyzing the effects of parameters. Particularly, the methods used for solving nonlinear differential equations have a wide range of applications in physics, engineering, biology, and many other disciplines.

The Sardar sub-equation method, the Riccati-Bernoulli method, and the $\frac{G'}{kG'+G+r}$-expansion method are powerful analytical techniques used to obtain exact solutions of nonlinear differential equations. The common feature of these methods is assuming the solution in a specific functional form (e.g., hyperbolic, trigonometric, or rational functions), which transforms the differential equation into an algebraic form, making it solvable. Additionally, these methods are typically applied in scenarios where nonlinear effects and dispersion terms are balanced, revealing special structures such as soliton, periodic, or rational solutions.

In Equation (1), when the variables

$$u(x,t) = e^{i\xi}P(\eta), \quad \eta = x - ct, \quad \xi = kx - \omega t + \vartheta$$

(2)

are substituted, the real and imaginary parts of Equation (1) are obtained as follows,

Re: $(\alpha + 6k^2)P''(\eta) + P^{(4)}(\eta) + \kappa P''(\eta)P^2(\eta) + (-3k^4 - \alpha k^2 + \omega)P(\eta)$

$$+(k^2\kappa + \mu)P^3(\eta) + \upsilon P^5(\eta) = 0 \quad (3)$$

Im: $(-8k^3 - 6\beta k^2 + 4\alpha k - 2c)P'(\eta) + (8k + 2\beta)P^3(\eta) + (4k + \beta)\kappa P^2(\eta)P'(\eta) = 0$

(4)

From the imaginary part (4), the following equalities are obtained:

$\beta = -4k,$

$c = 2k\alpha + 8k^3.$

(5)

Additionally, when balancing is performed in the real part (3), $N = 1$ is obtained.

2.1. The Sardar Sub-Equation Method and Its Application to Equation (3)

Since $N = 1$ let, $P = P(\eta)$ and $g = G(\eta)$ with the assumption

$$P = f_0 + f_1 g$$

and the auxiliary equation

$$(g')^2 = g^4 + \gamma g^2 + \beta$$

Using these, the real part of Equation (3) is rewritten. The resulting equation is arranged as a polynomial in terms of the g function. By equating the coefficients of the polynomial to zero, the following algebraic system of equations is obtained:

$(g)^5:\ f_1^5\nu + 2\kappa f_1^3 + 24f_1 \ = 0,$

$(g)^4:\ 5f_0\ f_1^4\nu + 4f_0\ f_1^2\kappa = 0,$

$(g)^3:\ 10f_0^2 f_1^3\nu + f_1^3 k^2\kappa + f_1^3\kappa\gamma + 2f_0^2 f_1\ \kappa + f_1^3\mu + 12f_1 k^2 + 2\alpha f_1 + 20f_1\gamma = 0,$

$(g)^2:\ 10f_0^3 f_1^2\nu + 3f_0\ f_1^2 k^2\kappa + 2f_0\ f_1^2\mu = 0,$

$(g)^1:\ f_0^2 f_1(5f_0^2\nu + 3k^2\kappa + \kappa\gamma + 3\mu) + f_1(6k^2\gamma + \alpha\gamma + \gamma^2 + 12\beta + \omega - 3\kappa^4 - \alpha k^2) = 0,$

$(g)^0:\ f_0^5\nu + f_0^3 k^2\kappa - 3f_0\ k^4 - \alpha f_0\ k^2 + f_0^3\mu + f_0\ \omega = 0.$

When this algebraic system of equations is solved,

$$f_0 \ = 0,$$

$$f_1 \ = \pm\frac{\sqrt{\pm\nu\left(\pm\kappa + \sqrt{\kappa^2 - 24\nu}\right)}}{\nu},$$

$$\gamma = -\frac{\frac{\left(\pm\kappa + \sqrt{\kappa^2 - 24\nu}\right)}{\nu}(k^2\kappa \mp \mu) + 12k^2 + 2\alpha}{\mp\frac{\left(\pm\kappa + \sqrt{\kappa^2 - 24\nu}\right)}{\nu}\kappa + 20}$$

$$\omega = \pm\frac{\left(\begin{array}{c}\left(-\kappa + \sqrt{\kappa^2 - 24\nu}\right)\left(\begin{array}{c}-8k^4\kappa^3 - 2\alpha k^2\kappa^3 + 144k^4\kappa\nu + 40\alpha k^2\kappa\nu - 4k^2\kappa^2\mu + \alpha^2\kappa\nu \\ -\alpha\kappa^2\mu + 12\beta\kappa^3 + 48k^2\mu\nu + 8\alpha\mu\nu - 240\beta\kappa\nu + \kappa\mu^2\end{array}\right) \\ -96k^4\kappa^2 - 24\alpha k^2\kappa^2 + 1248k^4\nu + 416\alpha k^2\nu - 48k^2\kappa\mu \\ +18\alpha^2\nu - 12\alpha\kappa\mu + 144\beta\kappa^2 - 2400\beta\nu + 12\mu^2\end{array}\right)}{\mp\kappa\left(-\kappa + \sqrt{\kappa^2 - 24\nu}\right)(\kappa^2 - 20\nu) + 4\nu(3\kappa^2 - 50\nu)}$$

the coefficients are determined. When these coefficients are substituted into the appropriate places, the following solution classes are obtained:

Class 1:

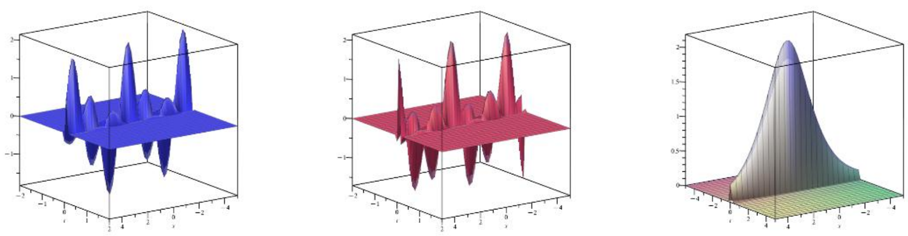
$$P_1 = \frac{2\sqrt{\nu\left(-\kappa + \sqrt{\kappa^2 - 24\nu}\right)\sqrt{\frac{pq\left((k^2\kappa + \mu)\left(-\kappa + \sqrt{\kappa^2 - 24\nu}\right) + 2(6k^2 + \alpha)\nu\right)}{\kappa\sqrt{\kappa^2 - 24\nu} - \kappa^2 + 20\nu}}}}{\nu\left(pe^{\sqrt{-\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)} + qe^{-\sqrt{-\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)}\right)}$$

as follows:

$$u(x,t) = P_1 e^{i\left[kx - \left(\frac{\left(\begin{array}{c}\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)\left(\begin{array}{c}-8k^4\kappa^3-2\alpha k^2\kappa^3+144k^4\kappa\nu+40\alpha k^2\kappa\nu-4k^2\kappa^2\mu+\alpha^2\kappa\nu \\ -\alpha\kappa^2\mu+12\beta\kappa^3+48k^2\mu\nu+8\alpha\mu\nu-240\beta\kappa\nu+\kappa\mu^2\end{array}\right) \\ -96k^4\kappa^2-24\alpha k^2\kappa^2+1248k^4\nu+416\alpha k^2\nu-48k^2\kappa\mu \\ +18\alpha^2\nu-12\alpha\kappa\mu+144\beta\kappa^2-2400\beta\nu+12\mu^2\end{array}\right)}{\mp\kappa\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)(\kappa^2-20\nu)+4\nu(3\kappa^2-50\nu)}\right)t+\theta\right]}$$

(6)

it is.



a. Real b. Imaginary c. Complex

Figure 1: Graphs corresponding to Equation (6) in Class 1.

Class 2:

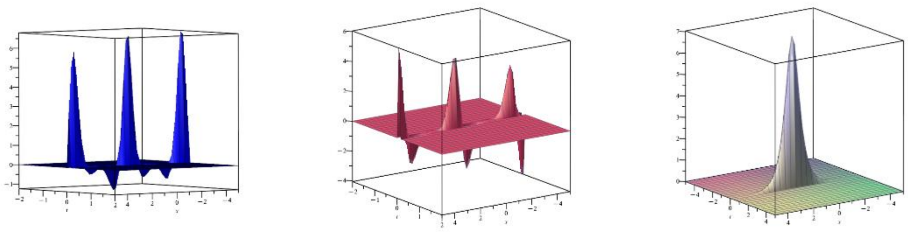
$$P_2 = \frac{2\sqrt{\nu\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)\sqrt{\frac{pq\left((k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu\right)}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}}}{\nu\left(pe^{\sqrt{\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)}+qe^{-\sqrt{\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)}\right)}$$

as follows:

$$u(x,t) = P_2 e^{i\left[kx-\left(\frac{\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)\left(\begin{array}{c}-8k^4\kappa^3-2\alpha k^2\kappa^3+144k^4\kappa\nu+40\alpha k^2\kappa\nu-4k^2\kappa^2\mu+\alpha^2\kappa\nu\\ -\alpha\kappa^2\mu+12\beta\kappa^3+48k^2\mu\nu+8\alpha\mu\nu-240\beta\kappa\nu+\kappa\mu^2\end{array}\right)-96k^4\kappa^2-24\alpha k^2\kappa^2+1248k^4\nu+416\alpha k^2\nu-48k^2\kappa\mu+18\alpha^2\nu-12\alpha\kappa\mu+144\beta\kappa^2-2400\beta\nu+12\mu^2}{\mp\kappa\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)(\kappa^2-20\nu)+4\nu(3\kappa^2-50\nu)}\right)t+\theta\right]}$$

(7)

it is.



a. Real b. Imaginary c. Complex

Figure 2: Graphs corresponding to Equation (7) in Class 2.

Class 3:

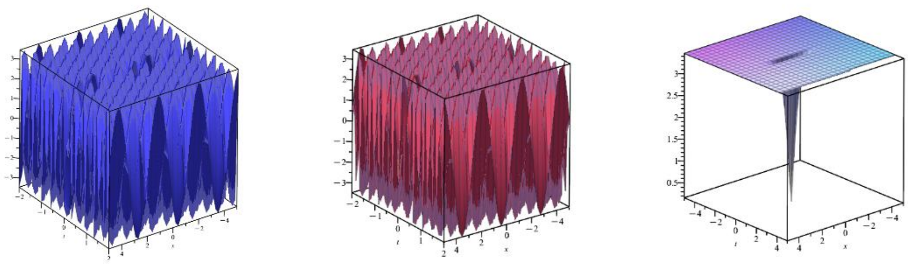
$$P_3 = \frac{\sqrt{\nu\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)}\left(pe^{\frac{\sqrt{2}\sqrt{\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)}{2}}-qe^{-\frac{\sqrt{2}\sqrt{\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)}{2}}\right)}{\nu\left(pe^{\frac{\sqrt{2}\sqrt{\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)}{2}}+qe^{-\frac{\sqrt{2}\sqrt{\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)}{2}}\right)}$$

as follows:

$$u(x,t)=P_3 e^{i\left[kx-\left(\frac{\left(\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)\left(\begin{array}{c}-8k^4\kappa^3-2\alpha k^2\kappa^3+144k^4\kappa\nu+40\alpha k^2\kappa\nu-4k^2\kappa^2\mu+\alpha^2\kappa\nu\\-\alpha\kappa^2\mu+12\beta\kappa^3+48k^2\mu\nu+8\alpha\mu\nu-240\beta\kappa\nu+\kappa\mu^2\end{array}\right)\right)-96k^4\kappa^2-24\alpha k^2\kappa^2+1248k^4\nu+416\alpha k^2\nu-48k^2\kappa\mu+18\alpha^2\nu-12\alpha\kappa\mu+144\beta\kappa^2-2400\beta\nu+12\mu^2}{\mp\kappa\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)\left(\kappa^2-20\nu\right)+4\nu\left(3\kappa^2-50\nu\right)}\right)t+\theta\right]}$$

(8)

it is.



a. Real b. Imaginary c. Complex

Figure 3: Graphs corresponding to Equation (8) in Class 3

<u>Class 4:</u>

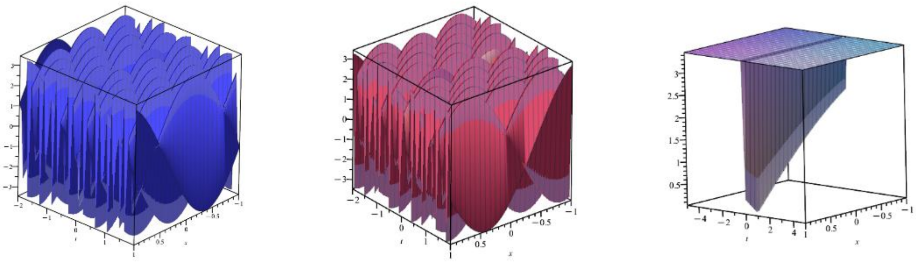
$$P_4=\frac{\sqrt{\nu\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)}\left(pe^{\frac{\sqrt{2}\sqrt{-\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)}{2}}-qe^{-\frac{\sqrt{2}\sqrt{-\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)}{2}}\right)}{\nu\left(pe^{\frac{\sqrt{2}\sqrt{-\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)}{2}}+qe^{-\frac{\sqrt{2}\sqrt{-\frac{(k^2\kappa+\mu)\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)+2(6k^2+\alpha)\nu}{\kappa\sqrt{\kappa^2-24\nu}-\kappa^2+20\nu}}(-8k^3t-2k\alpha t+x)}{2}}\right)}$$

as follows:

$$u(x,t)=P_4 e^{i\left[kx-\left(\frac{\left(\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)\left(\begin{array}{c}-8k^4\kappa^3-2\alpha k^2\kappa^3+144k^4\kappa\nu+40\alpha k^2\kappa\nu-4k^2\kappa^2\mu+\alpha^2\kappa\nu\\-\alpha\kappa^2\mu+12\beta\kappa^3+48k^2\mu\nu+8\alpha\mu\nu-240\beta\kappa\nu+\kappa\mu^2\end{array}\right)\right)-96k^4\kappa^2-24\alpha k^2\kappa^2+1248k^4\nu+416\alpha k^2\nu-48k^2\kappa\mu+18\alpha^2\nu-12\alpha\kappa\mu+144\beta\kappa^2-2400\beta\nu+12\mu^2}{\mp\kappa\left(-\kappa+\sqrt{\kappa^2-24\nu}\right)\left(\kappa^2-20\nu\right)+4\nu\left(3\kappa^2-50\nu\right)}\right)t+\theta\right]}$$

(9)

it is.



a. Real b. Imaginary c. Complex

Figure 4: Graphs corresponding to Equation (9) in Class 4.

2.2. $\frac{G'}{kG'+G+r}$-Expansion Method and Its Application to Equation (3)

$$F' = (\lambda - \gamma - 1)F^2 + \frac{1}{n}(2\gamma - \lambda)F - \frac{1}{n^2}\gamma$$

The auxiliary equation and the assumption

$$P = s_0 + s_1 F$$

are substituted into Equation (3). The resulting equation is then arranged as a polynomial in terms of the F function. By equating the coefficients of the polynomial to zero, an algebraic system of equations is obtained. Solving this system yields the following solution classes:

Class 1:

$$s_0 = 0,$$

$$s_1 = \frac{\sqrt{-2\mu(-9k^2n^2 + \alpha n^2 + 10\gamma^2 - 10\gamma)}(\gamma - 1)}{\mu n},$$

$$\lambda = 2\gamma,$$

$$\nu = \frac{\kappa^2}{25},$$

$$\kappa = \frac{15n^2\mu}{(-9k^2 + \alpha)n^2 + 10\gamma^2 - 10\gamma},$$

$$\omega=\frac{-16\gamma^4+32\gamma^3+2\left(-8+n^2(6k^2+\alpha)\right)\gamma^2-2(6k^2+\alpha)n^2\gamma+k^2n^4(3k^2+\alpha)}{n^4}$$

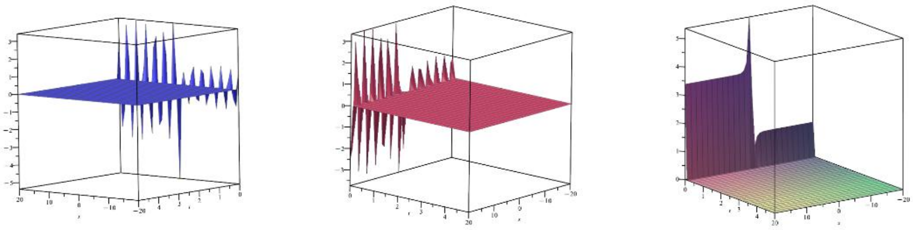
$$F=\frac{p_1\left(2\gamma+\sqrt{-2\gamma}\right)e^{\frac{-\left(2\gamma+\sqrt{-2\gamma}\right)\left(x-(8k^3+2k\alpha)t\right)}{2n}}+p_2\left(2\gamma-\sqrt{-2\gamma}\right)e^{\frac{\left(-2\gamma+\sqrt{-2\gamma}\right)\left(x-(8k^3+2k\alpha)t\right)}{2n}}}{n\left(p_1\left(2\gamma+\sqrt{-2\gamma}-2\right)e^{\frac{-\left(2\gamma+\sqrt{-2\gamma}\right)\left(x-(8k^3+2k\alpha)t\right)}{2n}}+p_2\left(2\gamma-\sqrt{-2\gamma}-2\right)e^{\frac{\left(-2\gamma+\sqrt{-2\gamma}\right)\left(x-(8k^3+2k\alpha)t\right)}{2n}}\right)}$$

as follows:

$$u(x,t)=s_1Fe^{i\left[kx-\left(\frac{-16\gamma^4+32\gamma^3+2\left(-8+n^2\left(6k^2+\alpha\right)\right)\gamma^2-2\left(2k^2+\alpha\right)n^2\gamma+\left(3k^2+\alpha\right)n^4k^2}{n^4}\right)t+\theta\right]}$$

(10)

it is.



a. Real b. Imaginary c. Complex

Figure 5: Graphs corresponding to Equation (10) in Class 1.

<u>Class 2:</u>

$$s_0=0,$$

$$s_1=\frac{\sqrt{-2\mu(-4k^2+\alpha)}(\gamma-1)}{\mu},$$

$$\lambda=2\gamma,$$

$$\nu=\frac{\kappa^2}{25},$$

$$\kappa=\frac{10\mu}{(-4k^2+\alpha)},$$

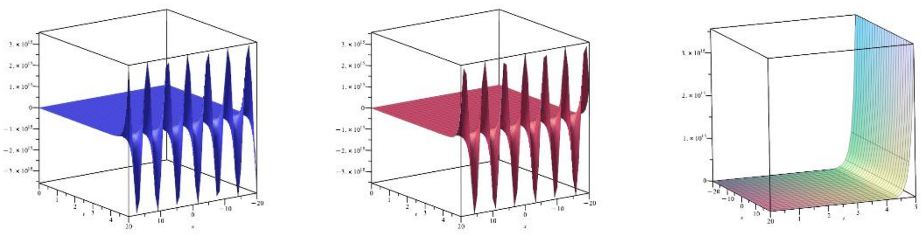
$$\omega = \frac{-16\gamma^4 + 32\gamma^3 + 2\left(-8 + n^2(6k^2 + \alpha)\right)\gamma^2 - 2(6k^2 + \alpha)n^2\gamma + k^2n^4(3k^2 + \alpha)}{n^4}$$

$$F = \frac{p_1(2\gamma + \sqrt{-2\gamma})e^{\frac{-(2\gamma+\sqrt{-2\gamma})(x-(8k^3+2k\alpha)t)}{2n}} + p_2(2\gamma - \sqrt{-2\gamma})e^{\frac{(-2\gamma+\sqrt{-2\gamma})(x-(8k^3+2k\alpha)t)}{2n}}}{n\left(p_1(2\gamma + \sqrt{-2\gamma} - 2)e^{\frac{-(2\gamma+\sqrt{-2\gamma})(x-(8k^3+2k\alpha)t)}{2n}} + p_2(2\gamma - \sqrt{-2\gamma} - 2)e^{\frac{(-2\gamma+\sqrt{-2\gamma})(x-(8k^3+2k\alpha)t)}{2n}}\right)}$$

as follows:

$$u(x,t) = s_1 F e^{i\left[kx - \left(\frac{-16\gamma^4+32\gamma^3+2\left(-8+n^2\left(6k^2+\alpha\right)\right)\gamma^2-2\left(2k^2+\alpha\right)n^2\gamma+\left(3k^2+\alpha\right)n^4k^2}{n^4}\right)t+\theta\right]} \quad (11)$$

it is.



a. Real b. Imaginary c. Complex

Figure 6: Graphs corresponding to Equation (11) in Class 2.

Class 3:

$$s_0 = 0,$$

$$s_1 = \frac{40}{\sqrt{-5\kappa}},$$

$$\gamma = 5,$$

$$\lambda = \gamma + 5,$$

$$\nu = \frac{\kappa^2}{25},$$

$$\mu = \frac{\kappa(-4k^2 + \alpha)}{10},$$

$$\omega = \frac{3k^4n^4 + \alpha k^2n^4 + 240k^2n^2 + 40\alpha n^2 - 6400}{n^4}$$

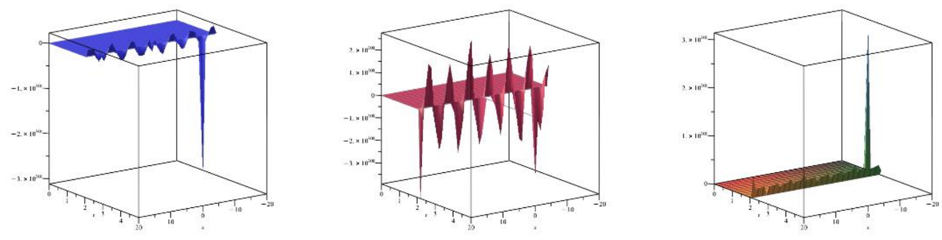
$$F = \frac{p_1(10 + i\sqrt{10})e^{\frac{-(10+i\sqrt{10})(x-(8k^3+2k\alpha)t)}{2n}} + p_2(10 - i\sqrt{10})e^{\frac{(-10+i\sqrt{10})(x-(8k^3+2k\alpha)t)}{2n}}}{n\left(p_1(8 + i\sqrt{10})e^{\frac{-(2\gamma+\sqrt{-2\gamma})(x-(8k^3+2k\alpha)t)}{2n}} + p_2(8 - i\sqrt{10})e^{\frac{(-10+i\sqrt{10})(x-(8k^3+2k\alpha)t)}{2n}}\right)}$$

as follows:

$$u(x,t) = s_1 F e^{i\left[kx - \left(\frac{3k^4n^4 + \alpha k^2n^4 + 240k^2n^2 + 40\alpha n^2 - 6400}{n^4}\right)t + \theta\right]}$$

(12)

it is.



a. Real b. Imaginary c. Complex

Figure 7: Graphs corresponding to Equation (12) in Class 3.

<u>Class 4:</u>

$$s_0 = 0,$$

$$s_1 = \frac{40}{\sqrt{-5\kappa}},$$

$$\gamma = \frac{31}{3} + \frac{\sqrt{-270k^2n^2 - 45\alpha n^2 + 18400}}{15},$$

$$\lambda = \gamma + 5,$$

$$\nu = \frac{\kappa^2}{25},$$

$$\mu = \frac{\kappa(-33k^2n^2 - 3\alpha n^2 + 2480 + +16\sqrt{-270k^2n^2 - 45\alpha n^2 + 18400})}{15n^2},$$

$$\omega = \frac{(1587200 - 2160(6k^2+\alpha)n^2)\sqrt{18400 - 45(6k^2+\alpha)n^2} + 217216000 + (17739k^4 + 5913\alpha k^2 + 324\alpha^2)n^4 - 565200(6k^2+\alpha)n^2}{2025n^4}$$

$$\Delta_1 = \left(\frac{46}{3} + \frac{\sqrt{-270k^2n^2-45\alpha n^2+18400}}{15} + \frac{\sqrt{-650-5\sqrt{-270k^2n^2-45\alpha n^2+18400}}}{5}\right)$$ ve

$$\Delta_2 = \left(\frac{46}{3} + \frac{\sqrt{-270k^2n^2-45\alpha n^2+18400}}{15} - \frac{\sqrt{-650-5\sqrt{-270k^2n^2-45\alpha n^2+18400}}}{5}\right)$$ olmak üzere

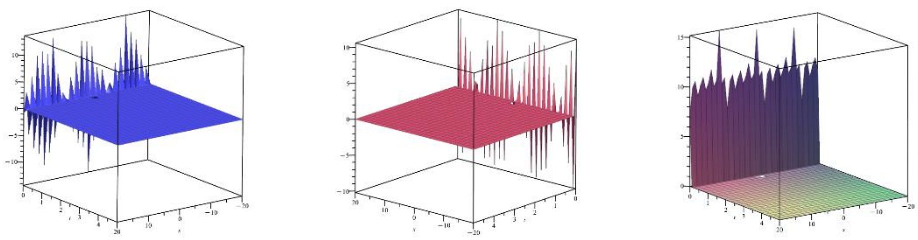
$$F = \frac{p_1\Delta_1\, e^{\frac{-\Delta_1(x-(8k^3+2k\alpha)t)}{2n}} + p_2\Delta_2\, e^{\frac{-\Delta_2(x-(8k^3+2k\alpha)t)}{2n}}}{n\left(p_1(\Delta_1-2)\, e^{\frac{-(\Delta_1-2)(x-(8k^3+2k\alpha)t)}{2n}} + p_2(\Delta_2-2)\, e^{\frac{-(\Delta_2-2)(x-(8k^3+2k\alpha)t)}{2n}}\right)}$$

as follows:

$$u(x,t) = s_1 F e^{i\left[kx - \left(\frac{3k^4n^4+\alpha k^2n^4+240k^2n^2+40\alpha n^2-6400}{n^4}\right)t+\theta\right]}$$

(13)

it is.



a. Real b. Imaginary c. Complex

Figure 8: Graphs corresponding to Equation (13) in Class 4.

2.3. The Riccati-Bernoulli Method and Its Application to Equation (3)

The auxiliary equation

$$P' = aP^{2-m} + bP + dP^{m}$$

and its derivatives are substituted into Equation (3). The resulting equation is arranged as a polynomial in terms of the P function. By equating the coefficients of the polynomial to zero, an algebraic system of equations is obtained. Solving this system yields the following solution classes:

Class 1:

$$m = 1,$$

$$\mu = -\kappa(a^2 + 2ab + 2ad + b^2 + 2bd + d^2 + k^2),$$

$$\nu = 0,$$

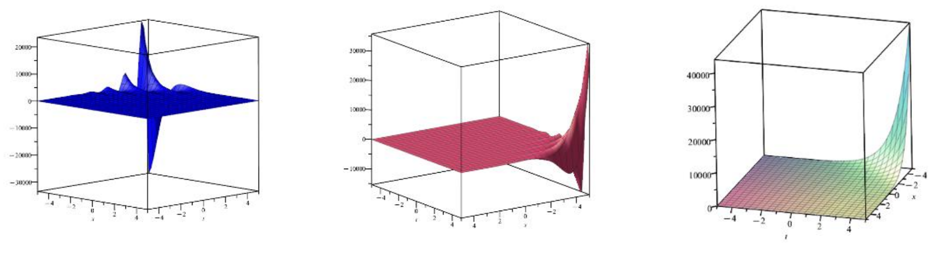
$$\omega = -\begin{pmatrix} a^4 + 4a^3b + 4a^3d + 6a^2b^2 + 12a^2bd + 6a^2d^2 + 6a^2k^2 + 4ab^3 + 12ab^2d \\ +12abd^2 + 12abk^2 + 4ad^3 + 12adk^2 + b^4 + 4b^3d + 6b^2d^2 + 6b^2k^2 + 4bd^3 \\ +12bdk^2 + d^4 + 6d^2k^2 - 3k^4 + a^2\alpha + 2ab\alpha + 2ad\alpha + b^2\alpha + 2bd\alpha + d^2\alpha - k^2\alpha \end{pmatrix}$$

$$P = Ce^{(a+b+d)(x-ct)}$$

as follows:

$$u(x,t) = Pe^{i\left[kx + \begin{pmatrix} a^4+4a^3b+4a^3d+6a^2b^2+12a^2bd+6a^2d^2+6a^2k^2+4ab^3+12ab^2d \\ +12abd^2+12abk^2+4ad^3+12adk^2+b^4+4b^3d+6b^2d^2+6b^2k^2+4bd^3 \\ +12bdk^2+d^4+6d^2k^2-3k^4+a^2\alpha+2ab\alpha+2ad\alpha+b^2\alpha+2bd\alpha+d^2\alpha-k^2\alpha \end{pmatrix} t + \theta\right]} \quad (14)$$

it is.



a. Real b. Imaginary c. Complex

Figure 9: Graphs corresponding to Equation (14) in Class 1.

Class 2:

$$b = d = \nu = 0,$$

$$m = 2,$$

$$\mu = -\kappa k^2,$$

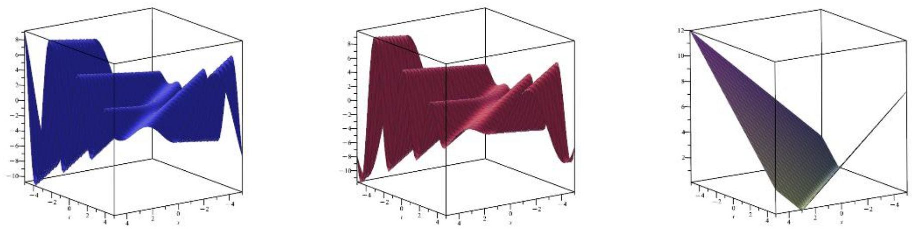
$$\omega = 3k^4 + \alpha k^2$$

$$P = \left(a(x - ct + C)\right)$$

as follows:

$$u(x,t) = Pe^{i\left[kx-(3k^4+\alpha k^2)t+\theta\right]} \quad (15)$$

it is.



a. Real b. Imaginary c. Complex

Figure 10: Graphs corresponding to Equation (15) in Class 2.

Class 3:

$$a = d = \nu = 0,$$

$$\mu = -\kappa(b^2 + k^2),$$

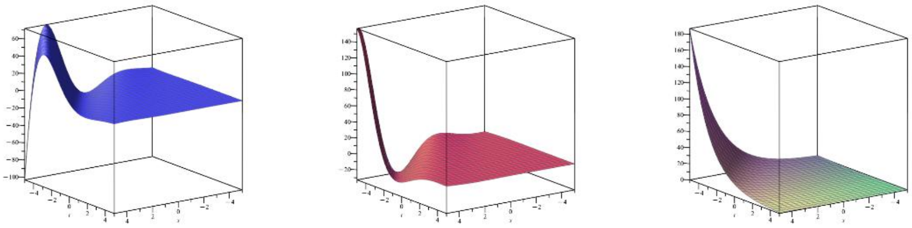
$$\omega = -b^4 - 6b^2k^2 + 3k^4 - \alpha b^2 + \alpha k^2$$

$$P = \left(Ce^{b(m-1)(x-ct)}\right)^{\frac{1}{m-1}}$$

as follows:

$u(x,t) = Pe^{i\left[kx-\left(-b^4-6b^2k^2+3k^4-\alpha b^2+\alpha k^2\right)t+\theta\right]}$ (16)

it is.



a. Real b. Imaginary c. Complex

Figure 11: Graphs corresponding to Equation (16) in Class 3.

Class 4:

$$d = \kappa = \mu = \nu = 0,$$

$$\alpha = -(b^2 + 6k^2),$$

$$m = 2,$$

$$\omega = -k^2(b^2 + 3k^2)$$

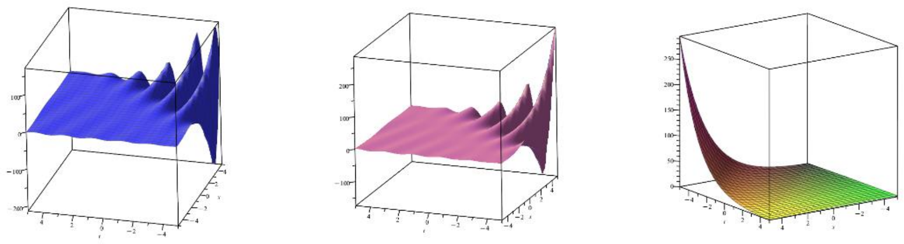
$$P = -\frac{a}{b} + Ce^{b(x-ct)}$$

as follows:

$u(x,t) = Pe^{i\left[kx+k^2(b^2+3k^2)t+\theta\right]}$

(17)

it is.



a. Real b. Imaginary c. Complex

Figure 12: Graphs corresponding to Equation (17) in Class 4.

<u>Class 5:</u>

$$d = \kappa = \mu = \nu = 0,$$

$$\alpha = -(b^2 + 6k^2),$$

$$\omega = -k^2(b^2 + 3k^2)$$

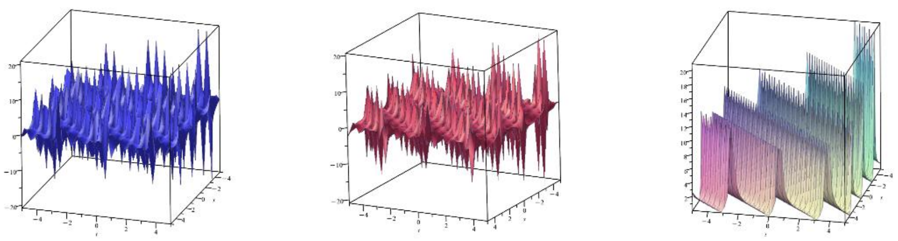
$$P = \frac{b}{2a}\left(-1 + \cot\left(\frac{b}{2}(x - ct + C)\right)\right)$$

as follows:

$$u(x,t) = Pe^{i\left[kx-\left(\frac{3k^4n^4+\alpha k^2n^4+240k^2n^2+40\alpha n^2-6400}{n^4}\right)t+\theta\right]}$$

(18)

it is.



a. Real b. Imaginary c. Complex

Figure 13: Graphs corresponding to Equation (18) in Class 5.

<u>Class 6:</u>

$$b = d = \nu = 0,$$

$$m = 2,$$

$$\mu = -\kappa k^2,$$

$$\omega = k^2(3k^2 + \alpha)$$

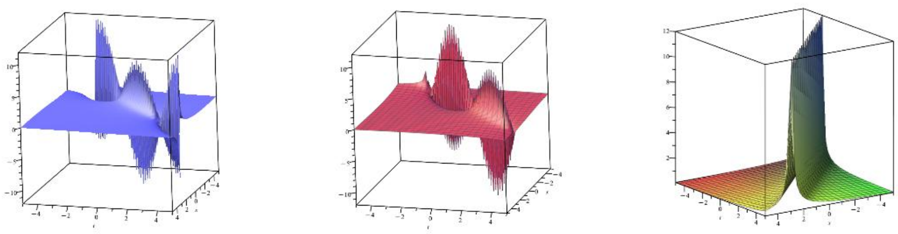
$$P = \frac{1}{a(x - ct + C)}$$

as follows:

$u(x,t) = Pe^{i[kx - k^2(3k^2+\alpha)t + \theta]}$

(19)

it is.



a. Real b. Imaginary c. Complex

Figure 14: Graphs corresponding to Equation (19) in Class 6.

<u>Class 7:</u>

$$b = d = \kappa = \nu = \mu = 0,$$

$$m = \frac{4}{3},$$

$$\alpha = -6k^2,$$

$$\omega = -3k^4$$

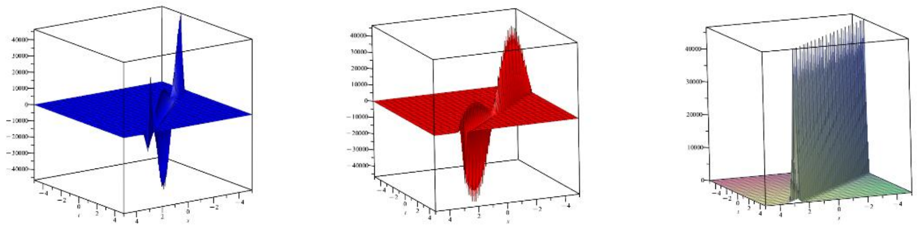
$$P = \frac{27}{a^3(x - ct + C)^3}$$

as follows:

$$u(x,t) = Pe^{i[kx+3k^4t+\theta]}$$

(20)

it is.



a. Real b. Imaginary c. Complex

Figure 14: Graphs corresponding to Equation (20) in Class 7.

<u>Class 8:</u>

$$b = d = \kappa = \nu = \mu = 0,$$

$$m = \frac{3}{2},$$

$$\alpha = -6k^2,$$

$$\omega = -3k^4$$

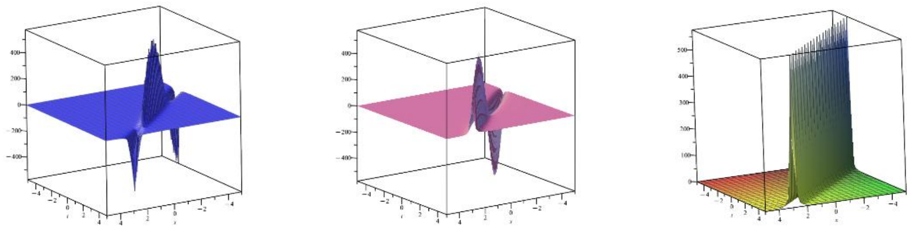
$$P = \frac{4}{a^2(x - ct + C)^2}$$

as follows:

$u(x,t) = Pe^{i[kx+3k^4t+\theta]}$

(21)

it is.



a. Real b. Imaginary c. Complex

Figure 15: Graphs corresponding to Equation (21) in Class 8.

3. RESULTS

In this study, the exact solutions of the equation modeling the propagation of light pulses in fourth-order nonlinear optical fibers were obtained using three different analytical methods: the Sardar sub-equation method, the $\frac{G'}{kG'+G+r}$-expansion method, and the Riccati-Bernoulli method. The solution classes provided by each method were examined, and the implications of these solutions in physical systems were discussed.

The results obtained have clearly demonstrated the effectiveness of these methods in solving nonlinear systems.

The Sardar Sub-Equation Method produced solutions in different classes within the solution space of the equation. The solutions obtained have made significant contributions to explaining the stability and behavior of optical solitons. Graphical analyses have revealed that the solutions exhibit distinct physical characteristics in real, imaginary, and complex domains. This highlights the precise modeling of the balance between dispersion and nonlinear effects.

The $\frac{G'}{kG'+G+r}$-Expansion Method has enabled the derivation of not only periodic and hyperbolic solutions but also new rational solutions. These solutions can be applied in practical scenarios, such as energy propagation and the control of light pulses in nonlinear optical fiber systems. Additionally, this method has facilitated a detailed examination of the effects of parameters on the solutions.

The Riccati-Bernoulli Method has yielded a broad range of solution classes, playing a significant role in understanding the effects of fourth-order dispersion on soliton width and stability. The analysis of these solution classes has provided a clearer understanding of the role of parameters in nonlinear systems and their influence on physical systems.

The solutions obtained offer valuable insights into the role of this nonlinear equation in various physical systems, such as optics and fluid dynamics.

In conclusion, this study has clearly demonstrated the effectiveness of the analytical methods used in solving nonlinear differential equations. The application of these methods to more complex physical systems in future studies could contribute to further advancements in modeling and solution techniques.

REFERENCES

Ablowitz, M. J., and Segur, H. (1981). *Solitons and the inverse scattering transform*. SIAM.

Bluman, G. W., and Kumei, S. (1989). *Symmetries and differential equations*. (Applied Mathematical Sciences, Vol. 81). Springer-Verlag.

Drazin, P. G., and Johnson, R. S. (1989). *Solitons: An introduction*. Cambridge University Press.

Giresunlu, İ. B., and Yıldız, B. (2023). Analytical Rational Solitons of the Modified Lakshmanan-Porsezian-Daniel Equation. *Universal Journal of Mathematics and Applications*, *6*(2), 53-64.

Ibragimov, N. H. (1995). *CRC handbook of Lie group analysis of differential equations* (Vols. 1-3). CRC Press.

Kopçasız, B., and Yaşar, E. (2023). The investigation of unique optical soliton solutions for dual-mode nonlinear Schrödinger's equation with new mechanisms. *Journal of Optics*, *52*(3), 1513-1527.

Kudryashov, N. A. (1990). Exact solutions of nonlinear differential equations. *Applied Mathematics Letters*, *3*(4), 75–78.

Kudryashov, N. A. (2019). First integrals and solutions of the traveling wave reduction for the Triki–Biswas equation. *Optik*, *185*, 275-281.

Kudryashov, N. A. (2019). General solution of traveling wave reduction for the Kundu–Mukherjee–Naskar model. *Optik*, *186*, 22-27.

Kudryashov, N. A. (2020). Periodic and solitary waves of the Biswas–Arshed equation. *Optik*, *200*, 163442.

Kudryashov, N. A. (2020). Traveling wave solutions of the generalized Gerdjikov–Ivanov equation. *Optik*, *219*, 165193.
Olver, P. J. (1993). *Applications of Lie groups to differential equations*. Springer.
Sardar, M. (2021). New exact solutions for nonlinear evolution equations via analytical methods. *Journal of Nonlinear Mathematical Physics, 28*(2), 150–165.
Sternberg, S. (2010). *Dynamical systems*. Dover Publications.
Strogatz, S. H. (2000). *Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering*. Perseus Books.
Strachan, I. A. B. (1992). Wave solutions of a (2+1)-dimensional generalization of the nonlinear Schrödinger equation. *Inverse Problems*, *8*(5), L21–L25.
Wang, M., Li, X., and Zhang, J. (2008). The G'/G-expansion method and travelling wave solutions of nonlinear evolution equations. *Physics Letters A*, *372*(4), 417–423.
Whitham, G. B. (1999). *Linear and nonlinear waves*. Wiley.
Zakharov, V. E., and Shabat, A. B. (1972). Exact theory of two-dimensional self-focusing. *Soviet Physics JETP, 34*(1), 62–69.
Zabusky, N. J., and Kruskal, M. D. (1965). Interaction of solitons in a collisionless plasma. *Physical Review Letters, 15*(6), 240–243.