Approximate Analytic Solutions of Forced Oscillations of Systems with Fractional Damping Affected by Quadratic Nonlinearity

Şenol Gümüş[1], Sümeyye Sınır[2], Bengi Yıldız[3]*
[1]*The Graduate School of Natural and Applied Sciences, Bilecik Şeyh Edebali University, Bilecik, Türkiye*
[2]*General Project Coordination Office, İzmir Kâtip Çelebi University, İzmir, Türkiye*
[3]*Department of Mathematics, Bilecik Şeyh Edebali University, Bilecik, Türkiye*
**Corresponding author: senolgumus@hotmail.com*

Abstract

In this study, the two frequency excitation oscillations of system having quadratic nonlinearities with fractional damping is considered via Riemann-Liouville fractional derivative. The system is assumed under the effect of external harmonic force. The approximated first-order uniform expansion of the dynamic response is derived for nonlinear forced oscillations of systems by employing a perturbation technique; that is, the method of multiple scales. The damping term is also assumed in a linear form. Perturbation is a well-known technique gives approximate analytical solutions [1-3]. It is frequently preferred for equations that cannot be found analytically solutions, such as equations involving nonlinearity and fractional derivatives.

In this research, perturbed solutions are examined for the system. The effects of the coefficient of a fractional damping term on the approximate solution are observed by graphs.

Keywords: fractional derivative, fractional damping, method of multiple time scales, perturbation method, stability

Introduction

Fractional calculus theory arose from the classical calculus, and it has been effectively used to the mathematical modeling of many scientific fields, such as mechanics, physics, biology, etc. The theory of fractional calculus and its applications are given in great depth in [4-7]. The equation of motion and boundary conditions for several different mechanical problems, such as oscillators, beams, plates, and shells, etc., can be described using the fractional derivative.

Like most types of fractional differential equations, equation systems and equations having fractional derivatives that have no analytical solutions are solved by a variety of methods and numerical techniques [8,9]. Approximate analytical approaches such as the perturbation method are employed in the solution of such fractional differential equations. Method of multiple time scales, which is widely used perturbation methods, is applied to seek analytical approximations to solve the system of equations [2,3,10]. This approximation is important as it simplified the complex dynamics of the system, rendering it more amenable to analysis.

This study investigates the forced oscillations of nonlinear systems having two degree of freedom with fractional damping characterized by quadratic nonlinearities. The multiple time scales method is applied to the equation of motion. The approximate analytical solutions of the equation is calculated and the effects of the coefficient of a fractional damping are observed from the solution, also are demonstrated in graphs.

Mathematical Model

For simplicity only the case of a single-frequency excitation is considered. Then, consider the modified model of [1] as:

$$\begin{aligned} \ddot{u}_1 + \omega_1^2 u_1 &= -2\mu_1 D^\alpha u_1 + u_1 u_2 + F_1 cos(\Omega t + \tau_1) \\ \ddot{u}_2 + \omega_2^2 u_2 &= -2\mu_2 D^\alpha u_2 + u_1^2 + F_2 cos(\Omega t + \tau_2) \end{aligned} \tag{1}$$

with ω_2 being larger than ω_1. Thus, there is no internal resonance. Here $\omega_n, (i = 1,2)$ are denotes the natural frequencies of the system (1) and Ω is the frequency of the excitation, D^α denotes the fractional operator with fractional order α.

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The Riemann-Liouville operator is one of the most useful and often used tool in the theory of fractional calculus [4]. Optimal accuracy, quick convergence and a better stability are the benefits of this operator. The Riemann-Liouville's definition introduced in the case 0<α<1 is (see [5])

$$D^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{-\infty}^{t}\frac{f(s)}{(t-s)^{\alpha}}ds, \tag{2}$$

where Γ denotes the Euler-Gamma function.

In the cases, mentioned in [11-13], it is possible to ignore the improper integral in the obtained fractional derivatives of the exponential function and the cosine function from definition in (2). Then, the fractional derivative of the exponential function and the cosine function can be used as, respectively

$$D^{\alpha}e^{i\lambda t}=(i\lambda)^{\alpha}e^{i\lambda t},$$

and

$$D^{\alpha}cos\lambda t=\lambda^{\alpha}cos\left(\lambda t+\frac{\pi\alpha}{2}\right).$$

Here, λ is constant coefficient and i denotes complex number $\sqrt{-1}$.

Solution Method

The Method of Multiple Time Scales is directly applied to the system (1). An approximate solution function is assumed in the following expansion:

$$\begin{aligned}u_1(t;\varepsilon)&=\varepsilon u_{11}(T_0,T_1,\dots)+\varepsilon^2u_{12}(T_0,T_1,\dots)+\cdots\\u_2(t;\varepsilon)&=\varepsilon u_{21}(T_0,T_1,\dots)+\varepsilon^2u_{22}(T_0,T_1,\dots)+\cdots\end{aligned} \tag{3}$$

where ε is a small, dimensionless parameter related to the amplitudes and the T_n are defined by

$$T_0=t,T_1=\varepsilon t,\dots$$

Here each variable represents a different time scale: T_0 is the fastest, T_1 is slower, and so on. Time derivatives are expressed in terms of fast and slow time scales as follows [14,15]:

$$\frac{d}{dt}=D_0+\varepsilon D_1+\cdots, \tag{4}$$

$$\frac{d^2}{dt^2}={D_0}^2+2\varepsilon D_0D_1+\cdots, \tag{5}$$

$$\left(\frac{d}{dt}\right)^{\alpha}=D^{\alpha}+\varepsilon\alpha D^{\alpha-1}D_1+\cdots \tag{6}$$

where D_n represents $\frac{\partial}{\partial T_n}$. The damping coefficients are ordered so the effects of the damping and the nonlinearity appear in the same perturbations. Thus, $\mu_n=\varepsilon\mu_n,(i=1,2)$. In this research one major category is considered, which is the primary resonance ($\Omega\approx\omega_2$).

The case of Ω near ω_2

To analyze primary resonances, we order the forcing term so that it appears in the same perturbation equation as the nonlinear terms and the damping. First we consider the case in which $\Omega\approx\omega_2$. Hence, we let $F_1=\varepsilon f_1$ and $F_2=\varepsilon^2f_2$.

Substituting u_1 and u_2 from (3) and time derivatives from (4)-(6) into Eq. (1), recalling that $\mu_n=\varepsilon\mu_n,(i=1,2)$ and equating each of the coefficients of ε and ε^2, one have following equations;

O(ε):

$$\begin{aligned}&{D_0}^2u_{11}+{\omega_1}^2u_{11}=f_1cos(\Omega T_0+\tau_1)\\&{D_0}^2u_{21}+{\omega_2}^2u_{21}=0\end{aligned} \tag{7}$$

O(ε^2):

$${D_0}^2u_{12}+{\omega_1}^2u_{12}=-2D_0D_1u_{11}-2\mu_1{D_0}^{\alpha}u_{11}+u_{11}u_{21} \tag{8}$$

$$D_0{}^2u_{22} + \omega_2{}^2u_{22} = -2D_0D_1u_{21} - 2\mu_2D_0{}^\alpha u_{21} + u_{11}^2 + f_2cos(\Omega T_0 + \tau_1)$$

The solutions of (7) can be expressed in the form

$$\begin{aligned} u_{11} &= A_1(T_1)e^{i\omega_1T_0} + \Lambda e^{i(\Omega T_0+\tau_1)} + c.c. \\ u_{21} &= A_2(T_1)e^{i\omega_2T_0} + c.c. \end{aligned} \tag{9}$$

where "$c.c.$" denotes complex conjugate part to the preceding terms. A_1 and A_2 are arbitrary functions and $\Lambda = \frac{f_1}{2(\omega_1{}^2-\Omega^2)}$. Substituting (9) into (8) yields

$$\begin{aligned} D_0{}^2u_{12} + \omega_1^2u_{12} = &-2i\omega_1A_1'e^{i\omega_1T_0} - 2\mu_1(i\omega_1)^\alpha A_1e^{i\omega_1T_0} \\ &-2\mu_1\Lambda(i\Omega)^\alpha e^{i(\Omega T_0+\tau_1)} + A_1A_2e^{i(\omega_1+\omega_2)T_0} \\ &+\Lambda A_2e^{i(\omega_2T_0+\Omega T_0+\tau_1)} + \Lambda\overline{A_2}e^{i(-\omega_2T_0+\Omega T_0+\tau_1)} \\ &+A_2\overline{A_1}e^{i(\omega_2-\omega_1)T_0} + c.c. \\ D_0{}^2u_{22} + \omega_2^2u_{22} = &-2i\omega_2A_2{}'e^{i\omega_2T_0} - 2\mu_2(i\omega_2)^\alpha A_1e^{i\omega_2T_0} \\ &+\frac{f_2}{2}e^{i(\Omega T_0+\tau_2)} + A_1{}^2e^{2i\omega_1T_0} + 2A_1\overline{A_1} \\ &+2\Lambda^2 + 2A_1\Lambda e^{i(\omega_1T_0+\Omega T_0+\tau_1)} + \Lambda^2e^{2i(\Omega T_0+\tau_1)} \\ &+2\overline{A_1}\Lambda e^{i(-\omega_1T_0+\Omega T_0+\tau_1)} + c.c. \end{aligned} \tag{10}$$

where

$$\Omega = \omega_2 + \varepsilon\sigma.$$

Here, σ is detuning parameter. None of the nonlinear terms produces a secular term and the solvability conditions are

$$A_1{}' + \mu_1A_1(i\omega_1)^{\alpha-1} = 0$$

$$2i\omega_2A_2{}' + 2\mu_2(i\omega_2)^\alpha A_2 = \frac{f_2}{2}e^{i(\sigma T_1+\tau_2)}$$

whose solutions are

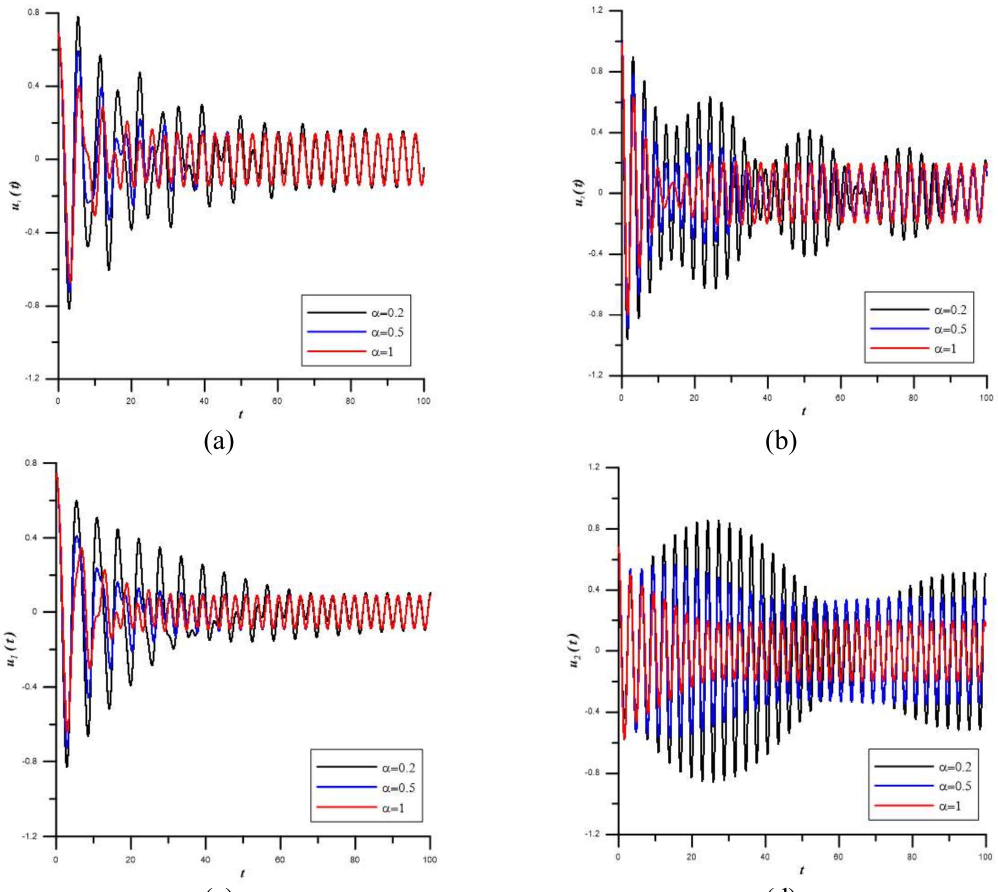
$$\begin{aligned} A_1 &= \frac{1}{2}a_1e^{-(i\omega_1)^{\alpha-1}\mu_1T_1+i\theta_1} \\ A_2 &= \frac{1}{2}a_2e^{-(i\omega_2)^{\alpha-1}\mu_2T_1+i\theta_2} + \frac{f_2}{4}(i\omega_2)^{-1}\frac{1}{i\sigma+(i\omega_2)^{\alpha-1}\mu_2}e^{i(\sigma T_1+\tau_2)} \end{aligned} \tag{11}$$

where a_n and θ_n are constants.

Substituting (11) into (9) and (3) and expressing the result in terms of the original variables, the following exact solution of the system (1) is obtained:

$$u_1 = \frac{1}{2}\varepsilon a_1e^{-(i\omega_1)^{\alpha-1}\mu_1\varepsilon t+i(\theta_1+\omega_1t)} + \frac{\varepsilon f_1}{2(\omega_1{}^2-\Omega^2)}e^{i(\Omega t+\tau_1)} + c.c.+O(\varepsilon^2)$$

$$\begin{aligned} u_2 = &\frac{\varepsilon f_2}{4}(i\omega_2)^{-1}\frac{1}{i\sigma+(i\omega_2)^{\alpha-1}\mu_2}e^{i(\sigma\varepsilon t+\tau_2+\omega_2t)} \\ &+\frac{1}{2}\varepsilon a_2e^{-(i\omega_2)^{\alpha-1}\mu_2\varepsilon t+i(\theta_2+\omega_2t)} + c.c.+O(\varepsilon^2). \end{aligned}$$



(a) (b)

(c) (d)

Figure 1. (a) - (b) The displacement u_1 and u_2 of the system (1) for $\sigma = -0.5$; (c) - (d) The displacement u_1 and u_2 of the system (1) for $\sigma = 0.5$. Parameters' values are: $\varepsilon = 1/3$, $a_1 = 5$, $a_2 = 5$, $\omega_1 = 1$, $\omega_2 = 2$, $\mu_1 = 0.4$, $\mu_2 = 0.4$, $\theta_1 = 0$, $\theta_2 = 0$, $\tau_1 = 0$, $\tau_2 = 0$, $f_1 = 2$, $f_2 = 3$.

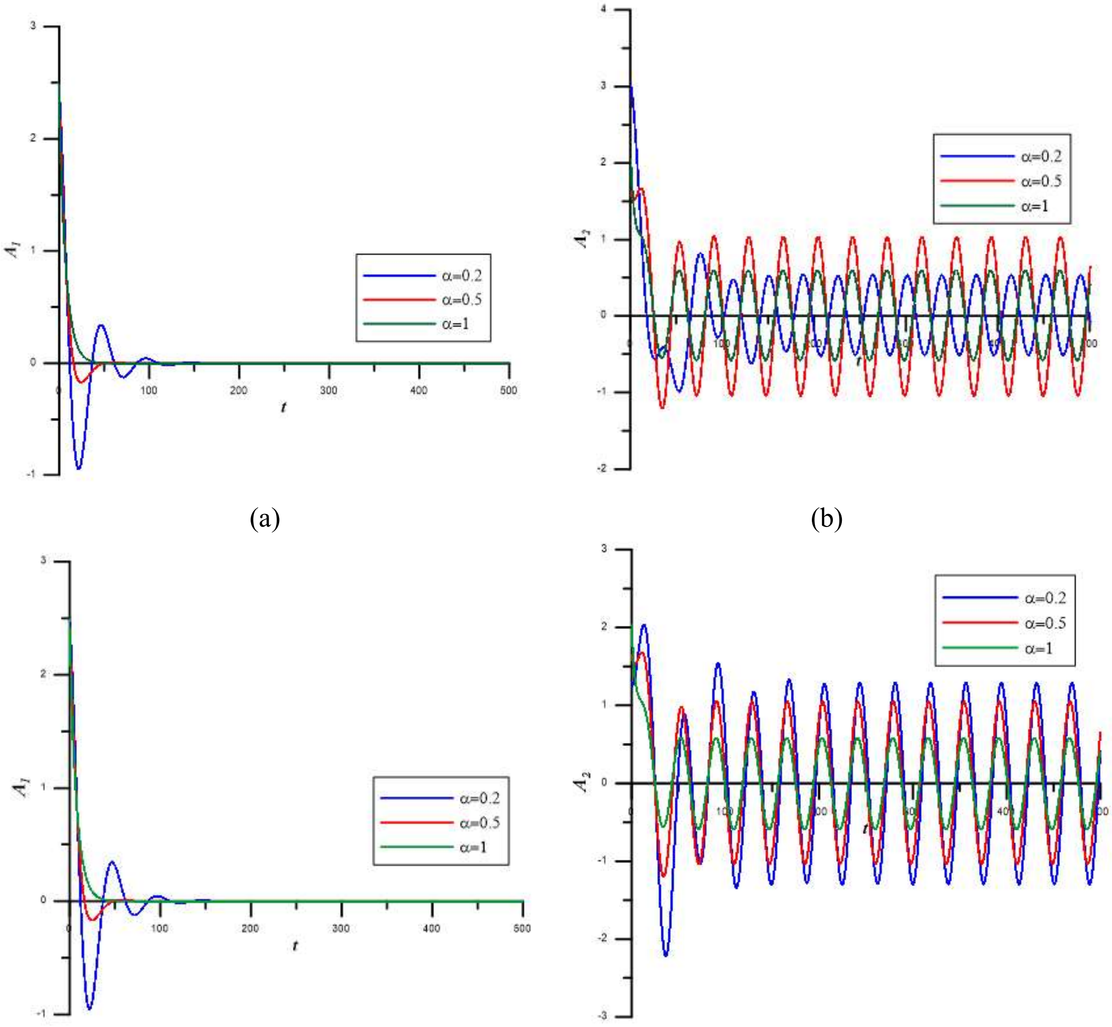
Figure 1 shows that by increasing α, the amplitude of vibration decreases. The velocity of the amplitude decrease is faster for larger values of the α. The amplitude of vibration decays until it reaches the limit cycle. Additionally, in the u_2 diagrams, there is a periodic oscillation that decreases after each cycle until the limit cycle.

As $t \to \infty, T_1 \to \infty$ and

$$A_1 \to 0, A_2 \to -\frac{f_2}{4}(i\omega_2)^{-1}\frac{1}{i\sigma + (i\omega_2)^{\alpha-1}\mu_2}e^{i(\sigma T_1+\tau_2)} \tag{12}$$

(See Figure 2)

z



(a) (b)

(c) (d)

Figure 2. (a) - (b) Variation of A_1 and A_2 over time for $\sigma = -0.5$; (c) - (d) Variation of A_1 and A_2 over time for $\sigma = 0.5$. Parameters' values are: $\varepsilon = 1/3$, $a_1 = 5$, $a_2 = 5$, $\omega_1 = 1$, $\omega_2 = 2$, $\mu_2 = 0.4$, $\theta_1 = 0$, $\theta_2 = 0$, $\tau_1 = 0$, $\tau_2 = 0$, $f_1 = 2$, $f_2 = 3$.

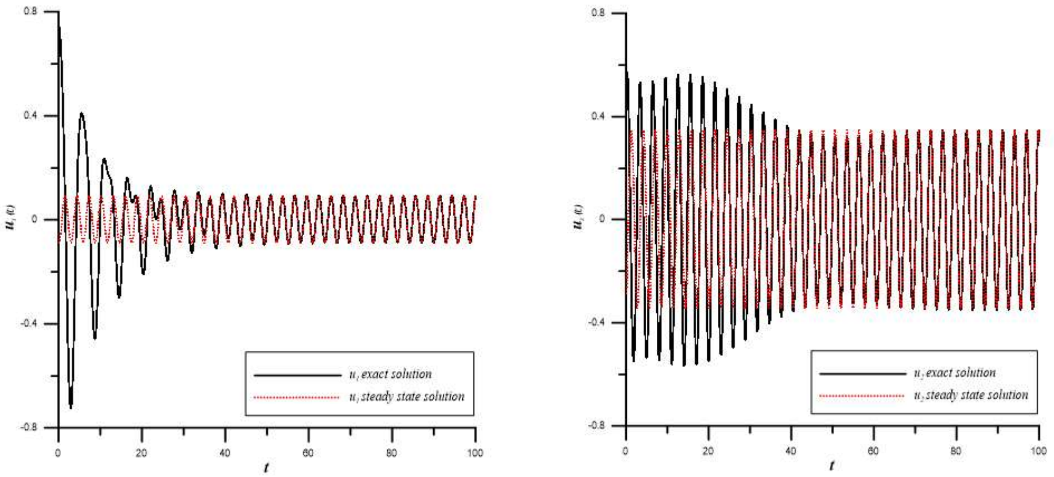
In Figure 2, it is observed that the value of A_1 approaches zero rapidly. With large values of α, it happens even faster. Since A_1 is independent of σ, obtained results for both negative and positive values of σ are the same. As stated in (12), A_2 converges to the $-\frac{f_2}{4}(i\omega_2)^{-1}\frac{1}{i\sigma+(i\omega_2)^{\alpha-1}\mu_2}e^{i(\sigma T_1+\tau_2)}$ over time. As α increases, convergence occurs faster and the amplitude decreases.

Substituting (12) into (9) and (3) the following steady-state response is obtained:

$$u_1 = \frac{\varepsilon f_1}{2({\omega_1}^2 - \Omega^2)} e^{i(\Omega t+\tau_1)} + c.c. + O(\varepsilon^2)$$

$$u_2 = \frac{\varepsilon f_2}{4}(i\omega_2)^{-1}\frac{1}{i\sigma + (i\omega_2)^{\alpha-1}\mu_2} e^{i(\sigma\varepsilon t+\tau_2+\omega_2 t)} + c.c. + O(\varepsilon^2)$$

Thus, when there is no internal resonance, the approximation is not influenced by the nonlinear terms; it is essentially the solution of the corresponding linear problem.



(a) (b)

Figure 3. Comparison of the exact solution and the steady-state solution for (a) u_1, (b) u_2. Parameters' values are: $\varepsilon = 1/3$, $a_1 = 5$, $a_2 = 5$, $\omega_1 = 1$, $\omega_2 = 2$, $\mu_2 = 0.4$, $\theta_1 = 0$, $\theta_2 = 0$, $\tau_1 = 0$, $\tau_2 = 0$, $f_1 = 2$, $f_2 = 3$, $\sigma = 0.5$, $\alpha = 0.5$.

In Figure 3, it is observed that after a certain time value, the amplitudes of the exact solution and the steady-state solution are equal to each other. Thus, the system continues to oscillate with the stable amplitude.

Conclusion

This study has been concerned with analytically approximating the solutions of fractional order differential equation systems. Two degree of freedom system with quadratic nonlinearity affected by the fractional linear damping is examined. The method of multiple time scales is performed to solve the system of equations by assigning the damping terms to the α-order. In the absence of internal resonance, exact and steady-state solutions are obtained. The effects of different α values on system behavior are given in the diagrams.

References

[1] Nayfeh AH, Mook DT. Nonlinear oscillations. New York: John Wiley & Sons; 2008.
[2] Sınır S, Yıldız B, Sınır BG. Approximate solutions of nonlinear pendulum with fractional damping. In 5th International Students Science Congress Proceedings Book, 2021; pages 295.
[3] Küzün D, Sınır S, Yıldız B. Euler-Bernoulli beam with fractional viscoelastic boundary conditions. In 18th International Scientific Research Congress Science and Engineering Full Text Book, 2023; pages 104-111.
[4] Podlubny I. Fractional differential equations, Mathematics in Science and Engineering, 1999; 198.
[5] Kilbas AA, Marichev OI, Samko SG. Fractional integrals and derivatives (Theory and Applications); 1993.
[6] Debnath L. Recent Applications of fractional calculus to science and engineering. International Journal of Mathematics and Mathematical Sciences, 2003; 2003(54):3413–3442.
[7] Ross B. Fractional calculus and its applications: Proceedings of the International Conference Held at the University of New Haven, June 1974, Volume 457. Springer; 2006.
[8] Javadi M, Noorian MA, Irani S. Stability analysis of pipes conveying fluid with fractional viscoelastic model. Meccanica, 2019; 54:399–410.
[9] Lewandowski R, Wielentejczyk P. Nonlinear vibration of viscoelastic beams described using fractional order derivatives. Journal of Sound and Vibration, 2017; 399:228–243.
[10] Dönmez Demir D, Bildik N, Sınır BG. Linear dynamical analysis of fractionally damped beams and rods. Journal of Engineering Mathematics, 2014; 85:131–147.
[11] Rossikhin YA, Shitikova MV. Analysis of nonlinear vibrations of a two-degree-of-freedom mechanical system with damping modelled by a fractional derivative. Journal of Engineering Mathematics, 2000; 37(4):343–362.
[12] Rossikhin YA, Shitikova MV. New approach for the analysis of damped vibrations of fractional oscillators. Shock and Vibration, 2009; 16(4):365–387.

[13] Rossikhin YA, Shitikova MV. Application of fractional derivatives for the analysis of nonlinear damped vibrations of suspension bridges. In Proceedings 1997 International Symposium on Nonlinear Theory and its Applications. NOLTA'97, pages 541–544; 1997.
[14] Rossikhin YA, Shitikova MV. Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results; 2010.
[15] Dönmez Demir D, Bildik N, Sınır BG. Application of fractional calculus in the dynamics of beams. Boundary Value Problems, 2012; 2012(1):1–13.