

Conoid Surfaces with Given Mean or Gaussian Curvature in Euclidean 3-Space

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Abstract

In this paper, we construct a right conoid surface in the Euclidean 3-space $\mathbb{R}^3$ with a given mean curvature or a given Gaussian curvature satisfying certain conditions. We also provide examples of right conoid surfaces corresponding to given mean and Gaussian curvatures satisfying these conditions.

Keywords: Conoid surfaces, Mean curvature, Gaussian curvature

1. Introduction

Conoid surfaces constitute an important subclass of ruled surfaces, characterized by the property that all their rulings intersect a fixed straight line, referred to as the axis, while remaining parallel to a prescribed plane. This distinctive geometric feature endows conoids with a rich structural behavior, making them an object of considerable interest in differential geometry. In particular, their intrinsic and extrinsic properties have been extensively investigated in relation to curvature analysis, developable surfaces, and special surface families.

Moreover, due to their inherent geometric simplicity combined with structural versatility, conoid surfaces arise naturally in various practical contexts. They play a significant role in architectural design, where aesthetically appealing and structurally efficient forms are required, as well as in mechanical engineering and computer-aided geometric design (CAGD), where precise surface modeling is essential. Consequently, the study of conoid surfaces provides a valuable link between theoretical developments and real-world applications.

Furthermore, right conoids in three-dimensional Euclidean space have been the subject of extensive investigation in the literature. In particular, the works of Berger and Gostiaux [1], Do Carmo [2], Gray [3], and Kreyszig [4] provide comprehensive treatments of their geometric properties. These studies examine several notable examples of right conoids, including the helicoid, Whitney umbrella, Wallis's conical edge, Plücker's conoid, and the hyperbolic paraboloid, highlighting their fundamental role in both classical and modern differential geometry. Güler and Kişi consider a family of conoid surfaces and determine the necessary conditions for their minimality in $\mathbb{R}^3$ and $\mathbb{R}_1^3$ [5, 6].

The construction of surfaces with prescribed geometric quantities has long been a central topic in differential geometry due to its theoretical significance and broad range of applications. In particular, the problem of determining surfaces with prescribed mean curvature or prescribed Gaussian curvature has attracted considerable attention, as these curvatures characterize both the intrinsic and extrinsic geometry of a surface. Such problems naturally lead to nonlinear partial differential equations, whose solutions provide valuable insight into geometric analysis, minimal surface theory, and mathematical physics. Consequently, the study of surfaces with prescribed mean or Gaussian curvature continues to be an active area of research, inspiring the development of new analytical techniques and the discovery of novel classes of surfaces. For more details on the problem of determining surfaces with prescribed mean curvature or prescribed Gaussian curvature, see [7–15].

In this paper, we study right conoid surfaces in three-dimensional Euclidean space $\mathbb{R}^3$. We construct a right conoid surface with a given mean curvature satisfying certain conditions and a given Gaussian curvature satisfying certain conditions. Finally, we provide examples to illustrate our results.

2. Preliminaries

A ruled surface in three-dimensional Euclidean space $\mathbb{R}^3$, given by

$$r(u, v) = \alpha(v) + u\beta(v) = (0, 0, h(v)) + u(\cos f(v), \sin f(v), 0)$$

is referred to as a right conoid if it is generated by the translational motion of a straight line (ruling) that intersects a fixed straight line, called the axis, while preserving a constant orthogonality with respect to a given plane [5]. Geometrically, this construction implies that each ruling remains parallel to a fixed plane and simultaneously intersects the axis, yielding a distinguished subclass of ruled surfaces with notable symmetry and structural simplicity. Without loss of generality, one may assume that the prescribed plane is the xy -plane and that the axis coincides with the z -axis. Under these assumptions, the parametrization of the right conoid reduces to

$$r(u, v) = \begin{pmatrix} u \cos f(v) \\ u \sin f(v) \\ h(v) \end{pmatrix}$$

where $f(v)$ and $h(v)$ are smooth functions. This representation highlights the intrinsic geometry of right conoids, where the directional field of the rulings is governed by $f(v)$, while the function $h(v)$ determines the profile curve along the axis. Throughout this study, the function $f(v)$ will be taken as $f(v) = v$. Thus the right conoid is given as

$$r(u, v) = (u \cos v, u \sin v, h(v)).$$

By straightforward computations, the first fundamental form, the second fundamental form, and the unit normal vector of the right conoid are obtained as follows

$$I = du^2 + (u^2 + (h'(v))^2) dv^2,$$

$$II = -\frac{2h'(v)}{\sqrt{u^2 + (h'(v))^2}} du dv + \frac{uh''(v)}{\sqrt{u^2 + (h'(v))^2}} dv^2,$$

$$\mathbf{N} = \frac{(h'(v) \sin v, -h'(v) \cos v, u)}{\sqrt{u^2 + (h'(v))^2}}$$

and the mean and Gaussian curvatures are obtained as follows

$$H(u, v) = \frac{uh''(v)}{2(u^2 + (h'(v))^2)^{3/2}},$$

$$K(u, v) = -\frac{(h'(v))^2}{(u^2 + (h'(v))^2)^2}$$

where $u^2 + (h'(v))^2 > 0$.

3. Main Theorems and Examples

Theorem 3.1. Let $r(u, v) = (u \cos v, u \sin v, h(v))$ be a right conoid surface in $\mathbb{R}^3$. Suppose that $H(u, v)$ denotes the mean curvature of the right conoid surface and is chosen so that h depends only on v , where $A(u, v) = c(u) + 2u \int H(u, v) dv$. Then, for $|A(u, v)| < 1$, $u > 0$ and

$$\frac{\partial}{\partial u} \left[\frac{uA(u, v)}{\sqrt{1 - A(u, v)^2}} \right] = 0,$$

there exists a family of conoid surfaces whose profile curves are given by

$$\alpha(v, A(u, v), c_1) = \left(0, 0, \int \frac{uA(u, v)}{\sqrt{1 - (A(u, v))^2}} dv + c_1 \right)$$

where $c_1 \in \mathbb{R}$.

Proof. Consider the mean curvature of the right conoid surface

$$H(u, v) = \frac{uh''(v)}{2(u^2 + (h'(v))^2)^{3/2}}.$$

Let

$$p(v) = h'(v).$$

Then, for the above equation can be rewritten as

$$\frac{p'(v)}{(u^2 + p(v)^2)^{3/2}} = \frac{2H(u, v)}{u}$$

where $u \neq 0$ and yields

$$\frac{d}{dv} \left(\frac{h'(v)}{\sqrt{u^2 + (h'(v))^2}} \right) = 2uH(u, v).$$

Integrating with respect to v

$$\frac{h'(v)}{\sqrt{u^2 + (h'(v))^2}} = c(u) + 2u \int H(u, v) dv.$$

If we define

$$A(u, v) = c(u) + 2u \int H(u, v) dv$$

then, for $u > 0$, we obtain

$$\frac{h'(v)}{\sqrt{u^2 + (h'(v))^2}} = A(u, v)$$

which gives

$$h'(v) = \frac{uA(u, v)}{\sqrt{1 - A(u, v)^2}}.$$

Finally, for $|A(u, v)| < 1$, we get

$$h(v) = \int \frac{uA(u, v)}{\sqrt{1 - (A(u, v))^2}} dv + c_1$$

where c_1 is an arbitrary constant of integration. A compatibility condition is, however, required. Since $h = h(v)$, the quantity

$$\frac{uA(u, v)}{\sqrt{1 - (A(u, v))^2}}$$

must be independent of u . Therefore, such a function $h(v)$ exists only for those functions $H(u, v)$ for which this compatibility condition is satisfied. In that case, $h(v)$ is obtained by integrating the above expression. $\square$

Corollary 3.1. *A right conoid surface is minimal, then, M is either a plane or a helicoid.*

Example 3.1. *Consider a right conoid surface with the given mean curvature*

$$H(u, v) = \frac{u}{2(u^2 + v^2)^{3/2}}$$

in $\mathbb{R}^3$. By considering Theorem 3.1 and $c(u) = 0$, we obtain $h(v) = \frac{v^2}{2} + c_1$ and the parametrization of the surface as follows

$$r(u, v) = \left(u \cos v, u \sin v, \frac{v^2}{2} + c_1 \right).$$

For $c_1 = 0$ and the domain

$$\begin{cases} 1 < u < 3 \\ -3 < v < 3 \end{cases}$$

the surface is shown in [Figure 3.1](#).

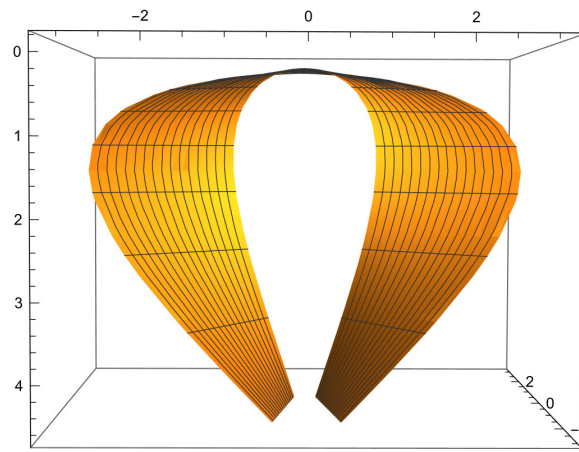


Figure 3.1: The right conoid surface having mean $H(u, v) = \frac{u}{2(u^2 + v^2)^{3/2}}$.

Example 3.2. Consider a right conoid surface with the given mean curvature

$$H(u, v) = \frac{ue^v}{2(u^2 + e^{2v})^{3/2}}$$

in $\mathbb{R}^3$. Using Theorem 3.1 and $c(u) = 0$, we obtain $h(v) = e^v + c_1$ and the parametrization of the surface as follows

$$r(u, v) = (u \cos v, u \sin v, e^v + c_1).$$

For $c_1 = 0$ and the domain

$$\begin{cases} 1 < u < 8 \\ -3 < v < 3 \end{cases}$$

the surface is shown in Figure 3.2.

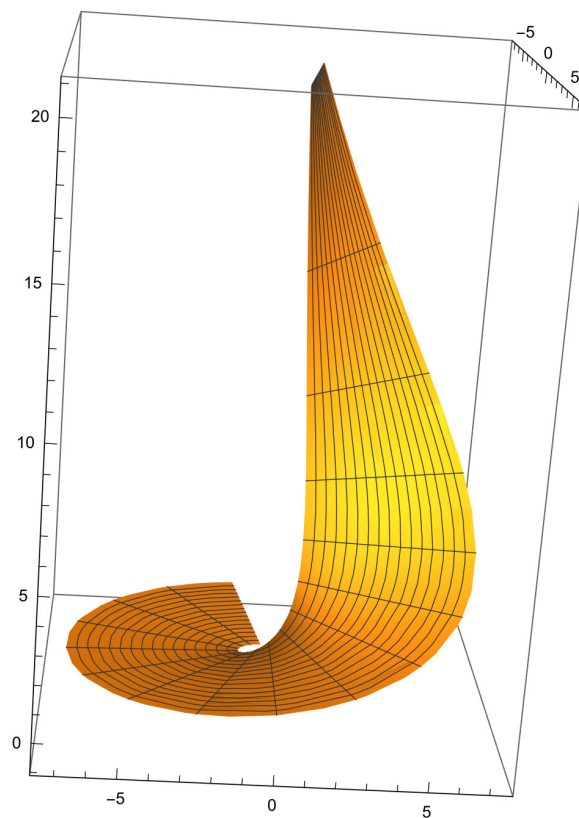


Figure 3.2: The right conoid surface having mean $H(u, v) = \frac{ue^v}{2(u^2 + e^{2v})^{3/2}}$.

Theorem 3.2. Let $r(u, v) = (u \cos v, u \sin v, h(v))$ be a right conoid surface in $\mathbb{R}^3$. Suppose that $K(u, v)$ denotes the mean curvature of the right conoid surface and $\frac{\partial}{\partial u} \left[\frac{-(2K(u, v)u^2 + 1) \pm \sqrt{1 + 4K(u, v)u^2}}{2K(u, v)} \right] = 0$. Then, for $-\frac{1}{4u^2} \leq K(u, v) < 0$, there exists a family of conoid surfaces whose profile curves are given by

$$\alpha(v, K(u, v), c) = \left(0, 0, \pm \int \sqrt{\frac{-(2K(u, v)u^2 + 1) \pm \sqrt{1 + 4K(u, v)u^2}}{2K(u, v)}} dv + c \right)$$

where c is arbitrary constant.

Proof. Consider the Gaussian curvature of the right conoid surface

$$K(u, v) = -\frac{(h'(v))^2}{(u^2 + (h'(v))^2)^2}.$$

Let

$$p(v) = h'(v).$$

Then the above equation can be rewritten as

$$-K(u, v) = \frac{p(v)^2}{(u^2 + p(v)^2)^2}.$$

Applying the substitution

$$y(v) = p(v)^2$$

yields

$$-Ky^2 - (2Ku^2 + 1)y - Ku^4 = 0.$$

For $-\frac{1}{4u^2} \leq K(u, v) < 0$, from the above equation, it follows that

$$y = \frac{-(2Ku^2 + 1) \pm \sqrt{(2Ku^2 + 1)^2 - 4K^2u^4}}{2K}.$$

Since, $p(v) = h'(v)$ and $y(v) = p(v)^2$, it follows that

$$p^2 = (h'(v))^2 = \frac{-(2Ku^2 + 1) \pm \sqrt{1 + 4Ku^2}}{2K}$$

and therefore

$$h'(v) = \pm \sqrt{\frac{-(2Ku^2 + 1) \pm \sqrt{1 + 4Ku^2}}{2K}}.$$

Integrating with respect to v yields

$$h(v) = \pm \int \sqrt{\frac{-(2K(u, v)u^2 + 1) \pm \sqrt{1 + 4K(u, v)u^2}}{2K(u, v)}} dv + c \quad (3.1)$$

where c is an arbitrary constant of integration. A compatibility condition is, however, required. Since $h = h(v)$, the quantity

$$\frac{-(2Ku^2 + 1) \pm \sqrt{1 + 4Ku^2}}{2K}$$

must be independent of u . Therefore, such a function $h(v)$ exists only for those functions $K(u, v)$ for which this compatibility condition is satisfied. In that case, $h(v)$ is obtained by integrating the above expression. □

Corollary 3.2. A right conoid surface is developable, then, M is a plane.

Example 3.3. Consider a conoid surface with the given Gaussian curvature

$$K(u, v) = -\frac{v^2}{(u^2 + v^2)^2}$$

in $\mathbb{R}^3$. By considering Theorem 3.2, we obtain the following integral to determine $h(v)$

$$I(v) = \int \sqrt{\frac{-\left(2\left(-\frac{v^2}{(u^2+v^2)^2}\right)u^2+1\right) + \sqrt{4\left(-\frac{v^2}{(u^2+v^2)^2}\right)u^2+1}}{2\left(-\frac{v^2}{(u^2+v^2)^2}\right)}} dv.$$

Since the integrand contains a square root, the expression must be analyzed separately according to the admissible ranges of the variables u and v . A case-by-case examination shows that the integral is given by

$$I(v) = \begin{cases} \int |v| dv, & |v| \leq |u|, \\ \int \frac{u^2}{|v|} dv, & |v| > |u|. \end{cases} \quad (3.2)$$

Hence, the value of the integral depends on the relative magnitudes of $|u|$ and $|v|$ yielding the above piecewise expression. By the definition of a right conoid surface, the smooth function h depends only on v . It follows from Equation (3.2) that

$$h(v) = \frac{v^2}{2} + c.$$

Thus, we obtain the parametrization of the surface as follows

$$r(u, v) = \left(u \cos v, u \sin v, \frac{v^2}{2} + c \right).$$

For $c = 0$ and the domain

$$\begin{cases} -8 < u < -4 \\ 0 < v < 4 \end{cases}$$

the surface is shown in Figure 3.3.

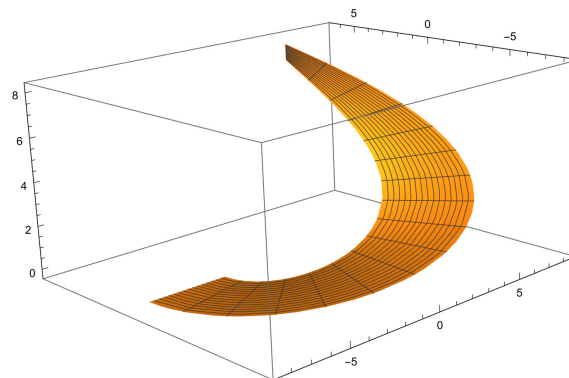


Figure 3.3: The right conoid surface having Gaussian curvature $K(u, v) = -\frac{v^2}{(u^2+v^2)^2}$.

Example 3.4. Consider a conoid surface with the given Gaussian curvature

$$K(u, v) = -\frac{e^{2v}}{(u^2 + e^{2v})^2}$$

in $\mathbb{R}^3$. By considering Theorem 3.2, we obtain the following integral to determine $h(v)$

$$I(v) = \int \sqrt{\frac{-\left(2\left(-\frac{e^{2v}}{(u^2+e^{2v})^2}\right)u^2+1\right) + \sqrt{1-\frac{4u^2e^{2v}}{(u^2+e^{2v})^2}}}{2\left(-\frac{e^{2v}}{(u^2+e^{2v})^2}\right)}} dv.$$

Since the integrand contains a square root, the expression must be analyzed separately according to the admissible ranges of the variables u and v . A case-by-case examination shows that the integral is given by

$$I(v) = \begin{cases} e^v + c, & e^v \leq |u|, \\ -\frac{u^2}{e^v} + c, & e^v > |u|. \end{cases} \quad (3.3)$$

Hence, the integral takes the following piecewise form, depending on $|u|$ and e^v . By the definition of a right conoid surface, the smooth function h depends only on v . It follows from Equation (3.3) that

$$h(v) = e^v + c.$$

Thus, we obtain the parametrization of the surface as follows

$$r(u, v) = (u \cos v, u \sin v, e^v + c).$$

For $c = 0$ and the domain

$$\begin{cases} 2 < u < 10 \\ 0 < v < \ln u \end{cases}$$

the surface is shown in Figure 3.4.

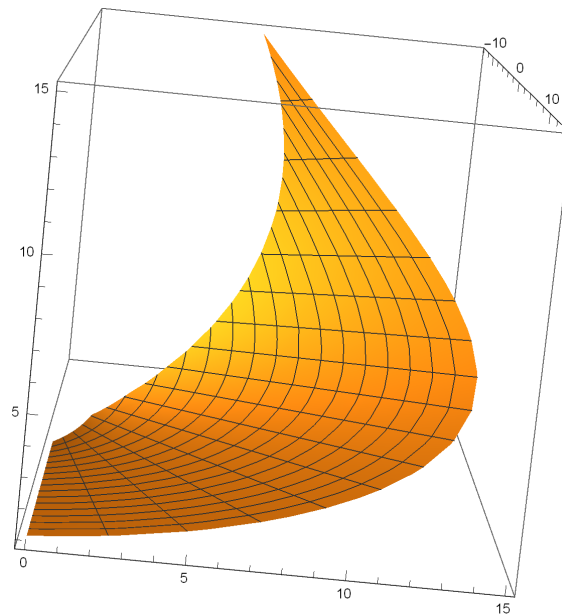


Figure 3.4: The right conoid surface having Gaussian curvature $K(u, v) = -\frac{e^{2v}}{(u^2 + e^{2v})^2}$.

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