Hindawi Publishing Corporation
Shock and Vibration
Volume 2014, Article ID 159213, 7 pages
http://dx.doi.org/10.1155/2014/159213

Research Article

An Efficient Analytical Method for Vibration Analysis of a Beam on Elastic Foundation with Elastically Restrained Ends

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Received 27 February 2014; Revised 22 April 2014; Accepted 30 April 2014; Published 14 May 2014

Academic Editor: Toshiaki Natsuki

An efficient analytical method for vibration analysis of a Euler-Bernoulli beam on elastic foundation with elastically restrained ends has been reported. A Fourier sine series with Stoke's transformation is used to obtain the vibration response. The general frequency determinant is developed on the basis of the analytical solution of the governing differential equation for all potential solution cases with rigid or restrained boundary conditions. Numerical analyses are performed to investigate the effects of various parameters, such as the springs at the boundaries to examine how the elastic foundation parameters affect the vibration frequencies.

1. Introduction

Beams resting on elastic foundations have wide application in engineering practice. The vibration analysis of beams is investigated using various elastic foundation models, such as, Vlasov, Pasternak, and Winkler models. A number of studies have been performed to predict the dynamic response of beams on elastic foundations with different boundary conditions.

Numerous works have been performed to explore the static deflection and vibration response of the beams resting on various elastic foundations. Chun [1] has investigated free vibration of hinged beam. Maurizi et al. [2] have considered the vibration frequencies for a beam with different boundary conditions. Vibration of beams on partial elastic foundations has been studied by Doyle and Pavlovic [3]. Laura et al. [4] have investigated beams which carry concentrated masses subject to an axial force. Abbas [5] has investigated vibration of Timoshenko beams with elastically restrained ends. Free vibration and stability behavior of uniform beams and columns with nonlinear elastic end rotational restraints has been considered by Rao and Naidu [6]. Free vibration behaviour of an Euler-Bernoulli beam resting on a variable Winkler foundation has been considered by Kacar et al. [7]. Civalek [8] has implemented differential quadrature and harmonic differential quadrature methods for buckling analysis of thin isotropic plates and elastic columns. H. K. Kim and M. S. Kim [9] have considered vibration of beams with generally restrained boundary conditions. A number of studies have been reported investigating the free vibration of beams on elastic foundation [10–25].

Although vibration analysis of beams on elastic foundation is a widely studied topic, there are only few papers that exist in the literature pertaining to the analysis of beams with elastically restrained ends. In this study, an efficient method is introduced for the analysis of the free vibration behavior of Euler-Bernoulli beams on an elastic foundation with elastic restraints. A Fourier sine series together with Stokes' transformation is used to evaluate the free vibration frequencies. The general frequency determinant is constructed by applying Stokes' transformation to the boundary conditions. Free vibration analyses of elastically supported beam on an elastic foundation are carried out and comparisons are made between with and without elastic foundation.

2. Application of Stokes' Transformation

Beam on elastic foundation with elastically restrained ends is depicted in Figure 1. Based on the Euler-Bernoulli beam

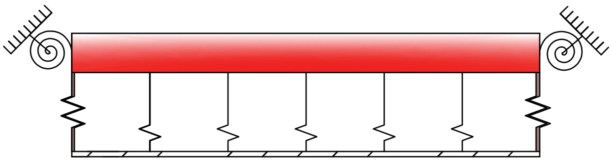


Figure 1: The beam on elastic foundation with elastically restrained ends.

theory, the equation of motion for a beam resting on a Winkler-type elastic foundation is given by

$$\rho A \frac{\partial^2 v(x,t)}{\partial t^2} + EI \frac{\partial^4 v(x,t)}{\partial x^4} + k_w v(x,t) = 0, \tag{1}$$

where EI is the flexural rigidity, ρ is the mass density, A is the cross-sectional area of the beam, and k_w is the stiffness of the foundation per unit length. Assuming harmonic vibration, the lateral displacement function $v(x,t)$ can be written in the form

$$v(x,t) = \psi(x)\cos(\omega t), \tag{2}$$

where $\psi(x)$ is the modal displacement function and ω is the natural frequency. The function $\psi(x)$ is described herein in three separate forms as follows:

$$\psi(x) = \begin{Bmatrix} \varphi_0 & x = 0 \\ \varphi_L & x = L \\ \displaystyle\sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{L}\right) & 0 < x < L \end{Bmatrix}. \tag{3}$$

The derivatives of $\psi(x)$ are based on Stokes' transformation:

$$\frac{d\psi(x)}{dx} = \frac{\varphi_L - \varphi_0}{L} + \sum_{m=1}^{\infty} \cos(\alpha_m x)\left(\frac{2\left((-1)^m \varphi_L - \varphi_0\right)}{L} + \alpha_m A_m\right), \tag{4}$$

$$\frac{d^2\psi(x)}{dx^2} = -\sum_{m=1}^{\infty} \alpha_m \sin(\alpha_m x)\left(\frac{2\left((-1)^m \varphi_L - \varphi_0\right)}{L} + \alpha_m A_m\right), \tag{5}$$

$$\frac{d^3\psi(x)}{dx^3} = \frac{\varphi_L'' - \varphi_0''}{L} + \sum_{m=1}^{\infty} \cos(\alpha_m x)\left(\frac{2\left((-1)^m \varphi_L'' - \varphi_0''\right)}{L} - \alpha_m^2\left(\frac{2\left((-1)^m \varphi_L - \varphi_0\right)}{L} + \alpha_m A_m\right)\right), \tag{6}$$

$$\frac{d^4\psi(x)}{dx^4} = -\sum_{m=1}^{\infty} \alpha_m \sin(\alpha_m x) \times \left(\frac{2\left((-1)^m \varphi_L'' - \varphi_0''\right)}{L} - \alpha_m^2\left(\frac{2\left((-1)^m \varphi_L - \varphi_0\right)}{L} + \alpha_m A_m\right)\right), \tag{7}$$

where

$$\alpha_m = \frac{m\pi}{L} \tag{8}$$

and lateral displacement function can be written as a sum of Fourier components; (3) and (7) are substituted into (1) to result in

$$-\sin(x\alpha_m)\left(LA_m\left(A\rho\omega^2 - EI\alpha_m^4 - k_w\right) + 2EI\alpha_m\left(\alpha_m^2\left(\varphi_0 - (-1)^m \varphi_L\right) + (-1)^m \varphi_L'' - \varphi_0''\right)\right) = 0. \tag{9}$$

By using above equation, the Fourier coefficient A_m can be written as follows:

$$A_m = \frac{2EI\alpha_m\left(\alpha_m^2\left(\varphi_0 - (-1)^m \varphi_L\right) + (-1)^m \varphi_L'' - \varphi_0''\right)}{L\left(-A\rho\omega^2 + EI\alpha_m^4 + k_w\right)} \tag{10}$$

and lateral displacement function can be written as a sum of Fourier components:

$$v(x,t) = \sum_{m=1}^{\infty} \frac{2EI\alpha_m\left(\alpha_m^2\left(\varphi_0 - (-1)^m \varphi_L\right) + (-1)^m \varphi_L'' - \varphi_0''\right)}{L\left(-A\rho\omega^2 + EI\alpha_m^4 + k_w\right)} \times \sin\left(\frac{m\pi x}{L}\right)\cos(\omega t). \tag{11}$$

When the stiffness of the foundation per unit length (k_w) is taken as zero, the above equation turns to be useful for a beam without elastic foundation [9].

3. The General Frequency Determinant for Different Boundary Conditions

The beam on elastic foundation is assumed to be elastically restrained by means of translational and rotational springs (see Figure 1). Then the boundary conditions are

$$\begin{aligned} S_0\varphi_0 &= -EI\frac{d^3 v(x,t)}{dx^3}, \quad x = 0, \\ S_L\varphi_L &= EI\frac{d^3 v(x,t)}{dx^3}, \quad x = L, \\ \Omega_0\frac{dv}{dx} &= EI\frac{d^2 v(x,t)}{dx^2}, \quad x = 0, \\ \Omega_L\frac{dv}{dx} &= -EI\frac{d^2 v(x,t)}{dx^2}, \quad x = L, \end{aligned} \tag{12}$$

where S_0, S_L are translational spring constants and Ω_0, Ω_L are rotational springs constants. The substitution of (4) and (6) into the above boundary conditions leads to four homogeneous equations

$$\begin{aligned}
&\left(-\overline{S}_0 - \sum_{m=1}^{\infty} \frac{2m^2\pi^2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\frac{\varphi_0}{L^2} \\
&\quad + \left(\sum_{m=1}^{\infty} \frac{2m^2(-1)^m\pi^2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\frac{\varphi_L}{L^2} \\
&\quad + \left(1 + \sum_{m=1}^{\infty} \frac{2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\varphi_0'' \\
&\quad - \left(1 + \sum_{m=1}^{\infty} \frac{2(-1)^m\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\varphi_L'' = 0, \\
&\left(\sum_{m=1}^{\infty} \frac{2m^2(-1)^m\pi^2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\frac{\varphi_0}{L^2} \\
&\quad - \left(\overline{S}_L + \sum_{m=1}^{\infty} \frac{2m^2\pi^2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\frac{\varphi_L}{L^2} \\
&\quad - \left(1 + \sum_{m=1}^{\infty} \frac{2(-1)^m\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\varphi_0'' \\
&\quad + \left(1 + \sum_{m=1}^{\infty} \frac{2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\varphi_L'' = 0, \\
&-\left(\overline{\Omega}_0 + 2\overline{\Omega}_0 \sum_{m=1}^{\infty} \frac{\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\frac{\varphi_0}{L^2} \\
&\quad + \left(\overline{\Omega}_0 + 2\overline{\Omega}_0 \sum_{m=1}^{\infty} \frac{(-1)^m\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\frac{\varphi_L}{L^2} \\
&\quad + \left(-1 + 2\overline{\Omega}_0 \sum_{m=1}^{\infty} \frac{m^2}{\pi^2\left(-m^4 - \delta + \lambda^4\right)}\right)\varphi_0'' \\
&\quad + \left(-2\overline{\Omega}_0 \sum_{m=1}^{\infty} \frac{m^2(-1)^m}{\pi^2\left(-m^4 - \delta + \lambda^4\right)}\right)\varphi_L'' = 0, \\
&\left(\overline{\Omega}_L + 2\overline{\Omega}_L \sum_{m=1}^{\infty} \frac{(-1)^m\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\frac{\varphi_0}{L^2} \\
&\quad - \left(\overline{\Omega}_L + 2\overline{\Omega}_L \sum_{m=1}^{\infty} \frac{\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}\right)\frac{\varphi_L}{L^2} \\
&\quad + \left(-2\overline{\Omega}_L \sum_{m=1}^{\infty} \frac{m^2(-1)^m}{\pi^2\left(-m^4 - \delta + \lambda^4\right)}\right)\varphi_0'' \\
&\quad + \left(-1 + 2\overline{\Omega}_L \sum_{m=1}^{\infty} \frac{m^2}{\pi^2\left(-m^4 - \delta + \lambda^4\right)}\right)\varphi_L'' = 0,
\end{aligned} \tag{13}$$

where

$$\begin{gathered}
\delta = \frac{L^4 k_w}{EI\pi^4}, \qquad \lambda^4 = \frac{\rho A L^4}{EI\pi^4}\omega^2, \\
\overline{S}_0 = \frac{S_0 L^3}{EI}, \qquad \overline{S}_L = \frac{S_L L^3}{EI}, \\
\overline{\Omega}_0 = \frac{\Omega_0 L}{EI}, \qquad \overline{\Omega}_L = \frac{\Omega_L L}{EI},
\end{gathered} \tag{14}$$

and one can obtain the following system of linear algebraic equations in matrix form to be solved for the constants $(\varphi_0/L^2, \varphi_L/L^2, \varphi_0'', \varphi_L'')$:

$$\begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & \varphi_{14} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & \varphi_{24} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} & \varphi_{34} \\ \varphi_{41} & \varphi_{42} & \varphi_{43} & \varphi_{44} \end{bmatrix} \begin{bmatrix} \dfrac{\varphi_0}{L^2} \\ \dfrac{\varphi_L}{L^2} \\ \varphi_0'' \\ \varphi_L'' \end{bmatrix} = 0. \tag{15}$$

The eigen values of the above systems of equations give the free vibration frequencies

$$\left|\varphi_{ij}\right| = 0; \quad (i, j = 1, 2, 3, 4), \tag{16}$$

where

$$\begin{aligned}
\varphi_{11} &= -\overline{S}_0 - \sum_{m=1}^{\infty} \frac{2m^2\pi^2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}, \\
\varphi_{12} &= \sum_{m=1}^{\infty} \frac{2m^2(-1)^m\pi^2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}, \\
\varphi_{13} &= 1 + \sum_{m=1}^{\infty} \frac{2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}, \\
\varphi_{14} &= -1 - \sum_{m=1}^{\infty} \frac{2(-1)^m\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}, \\
\varphi_{21} &= \sum_{m=1}^{\infty} \frac{2m^2(-1)^m\pi^2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}, \\
\varphi_{22} &= \overline{S}_L - \sum_{m=1}^{\infty} \frac{2m^2\pi^2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}, \\
\varphi_{23} &= -1 - \sum_{m=1}^{\infty} \frac{2(-1)^m\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}, \\
\varphi_{24} &= 1 + \sum_{m=1}^{\infty} \frac{2\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}, \\
\varphi_{31} &= -\overline{\Omega}_0 - 2\overline{\Omega}_0 \sum_{m=1}^{\infty} \frac{\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4}, \\
\varphi_{32} &= \overline{\Omega}_0 + 2\overline{\Omega}_0 \sum_{m=1}^{\infty} \frac{(-1)^m\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4},
\end{aligned}$$

$$\varphi_{33} = -1 + 2\overline{\Omega}_0 \sum_{m=1}^{\infty} \frac{m^2}{\pi^2\left(-m^4 - \delta + \lambda^4\right)},$$

$$\varphi_{34} = -2\overline{\Omega}_0 \sum_{m=1}^{\infty} \frac{m^2(-1)^m}{\pi^2\left(-m^4 - \delta + \lambda^4\right)},$$

$$\varphi_{41} = \overline{\Omega}_L + 2\overline{\Omega}_L \sum_{m=1}^{\infty} \frac{(-1)^m\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4},$$

$$\varphi_{42} = -\overline{\Omega}_L - 2\overline{\Omega}_L \sum_{m=1}^{\infty} \frac{\left(-\delta + \lambda^4\right)}{-m^4 - \delta + \lambda^4},$$

$$\varphi_{43} = -2\overline{\Omega}_L \sum_{m=1}^{\infty} \frac{m^2(-1)^m}{\pi^2\left(-m^4 - \delta + \lambda^4\right)},$$

$$\varphi_{44} = -1 + 2\overline{\Omega}_L \sum_{m=1}^{\infty} \frac{m^2}{\pi^2\left(-m^4 - \delta + \lambda^4\right)}. \quad (17)$$

Table 1: Verification of the proposed method for a beam with clamped-clamped ends.

Mode	Clamped-clamped Reference [26]	 Reference [27]	$\overline{S}_0 = \overline{S}_L = \overline{\Omega}_0 = \overline{\Omega}_L = 10000$ Present
	λ_i	λ_i	$\lambda_i \times \pi$
1	4.7324	4.7300	4.7490
2	7.8537	7.8532	7.8150
3	10.9958	10.9956	10.7992

Table 2: Verification of the proposed method for a beam with clamped-free ends.

Mode	Clamped-free Reference [26]	 Reference [27]	$\overline{S}_0 = \overline{\Omega}_0 = 10000$, $\overline{S}_L = \overline{\Omega}_L = 0$ Present
	λ_i	λ_i	$\lambda_i \times \pi$
1	1.9119	1.9119	1.9258
2	4.6965	4.6965	4.7236
3	7.8553	7.8552	7.8686

4. Numerical Results

In this section, the well-known problem of a beam on an elastic foundation is analyzed by the proposed method. To calculate the frequency parameters (λ_i), we solve (16). Firstly, in order to verify the accuracy of the proposed formulation, a comparison of the frequency results with the same results available from other numerical methods for the beams with classical supporting conditions is carried out. It is interesting to note that restrained boundary conditions will degenerate into the classical ones, provided that proper values are given to spring parameters in (16).

4.1. Validation of the Proposed Method. In this subsection, it is desired to evaluate the accuracy of the proposed method when applied to some special cases of the model. To the authors' knowledge, the differential transform method [26, 27] has been used for vibration analysis of beam with rigid boundary conditions. As aforementioned the proposed method can be used to determine the vibration frequencies of a beam in various classical supporting conditions, as well as any desired boundary conditions. It should be noted that, by letting $\overline{S}_0 = \infty$, $\overline{S}_L = \infty$, $\overline{\Omega}_0 = \infty$, and $\overline{\Omega}_L = \infty$, (16) will automatically degenerate into the beam clamped at both ends. When the spring parameters are used to represent a cantilever beam, the frequency determinant is written in the same order by letting $\overline{S}_0 = \infty$, $\overline{\Omega}_0 = \infty$, $\overline{S}_L = 0$, and $\overline{\Omega}_L = 0$. Frequencies of a simply supported beam can be achieved by using the values $\overline{S}_0 = \infty$, $\overline{S}_L = \infty$, $\overline{\Omega}_0 = 0$, and $\overline{\Omega}_L = 0$ in (16).

In the numerical verifications, the frequency parameters are calculated by the present approach using the first 50 terms of the infinite series. Elastic spring parameters are taken as $\overline{S}_0 = \overline{S}_L = \overline{\Omega}_0 = \overline{\Omega}_L = 10000$ for the beam with clamped-clamped ends and $\overline{S}_0 = \overline{\Omega}_0 = 10000$, $\overline{S}_L = \overline{\Omega}_L = 0$ for the cantilever beam. For convenience, the following reference parameter is introduced:

$$K_0 = \frac{\delta}{\pi^4}. \quad (18)$$

The elastic foundation parameter is taken as ($K_0 = 1/\pi^4$) in Tables 1 and 2. In the second validation analysis, a simple supported beam is considered. In Table 3, vibration frequency parameters are given for various values of the (K_0) parameter ($K_0 = 10, 50, 100, 500, 1000, 2000$).

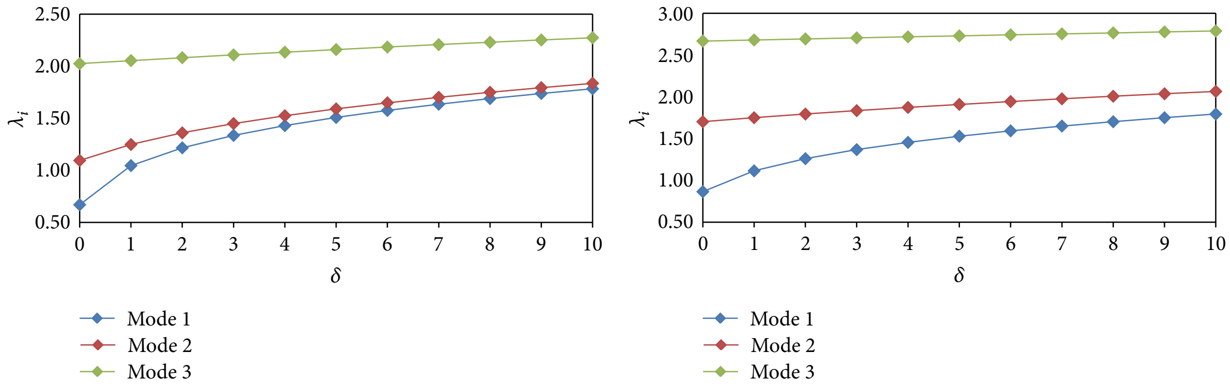
The most important observation from Tables 1, 2, and 3 is due to the fact that all frequency parameters of the beam are calculated by using the first 50 terms of the infinite series. Improvement in accuracy can be gained by increasing the terms of infinite series. However, as seen from Tables 1, 2, and 3, the present results seem to be more acceptable.

After verifying correctness of proposed method, the effects of different spring parameters on the lateral vibration of a beam are discussed. The effect of the spring parameter on the vibration responses of beam is demonstrated in Figure 2. The results in Figure 2 are calculated by using (16) for the values of ($\delta = 1, 2, 3, \ldots, 10$). One can observe that the first three frequency parameters are increased by considering the effects of the elastic foundation parameter. There is an abrupt change in the first modes when the foundation parameter varies from 0 to 1. On the other hand, with the consideration of the elastic foundation parameter, all vibration frequencies of the beam become dependent on the spring parameters.

4.2. Beam on Elastic Foundation with Elastically Restrained Ends. For comparison purposes, the variation in the ratio of vibration frequencies of beam with embedding elastic medium to that without embedding medium with different elastic foundation parameters is plotted for the first three modes. To compare the results of analytical analysis with no elastic foundation case, a λ_i/λ_0 ratio is considered. The index

TABLE 3: Verification of the first three frequency parameters for a simple supported beam.

K_0	Mode 1			Mode 2			Mode 3		
	Reference [26]	Reference [27]	Present ($\lambda_1 \times \pi$)	Reference [26]	Reference [27]	Present ($\lambda_2 \times \pi$)	Reference [26]	Reference [27]	Present ($\lambda_3 \times \pi$)
10	3.219	3.219	3.215	6.293	6.293	6.265	9.427	9.428	9.338
50	3.484	3.484	3.480	6.333	6.333	6.305	9.439	9.440	9.351
100	3.748	3.748	3.746	6.382	6.382	6.358	9.454	9.454	9.371
500	4.944	4.944	4.943	6.736	6.736	6.716	9.571	9.571	9.489
1000	5.756	5.756	5.755	7.112	7.112	7.095	9.710	9.710	9.633
2000	6.767	6.767	6.767	7.724	7.724	7.710	9.972	9.972	9.901



(a) $\overline{S}_0 = 10, \overline{S}_L = 10, \overline{\Omega}_0 = 10^5$, and $\overline{\Omega}_L = 10^5$

(b) $\overline{S}_0 = 10^5, \overline{S}_L = 10, \overline{\Omega}_0 = 10^5$, and $\overline{\Omega}_L = 10$

FIGURE 2: The effects of different spring parameters on the vibration frequencies.

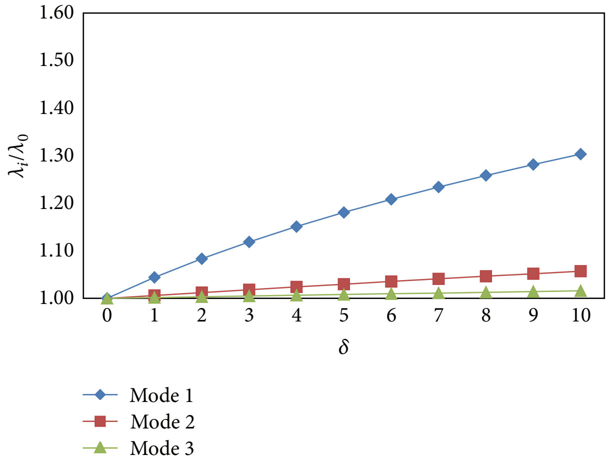


FIGURE 3: Effect of elastic foundation parameter on the vibration frequencies for $\overline{S}_0 = 10^5, \overline{S}_L = 10^5, \overline{\Omega}_0 = 10^5$, and $\overline{\Omega}_L = 10^5$.

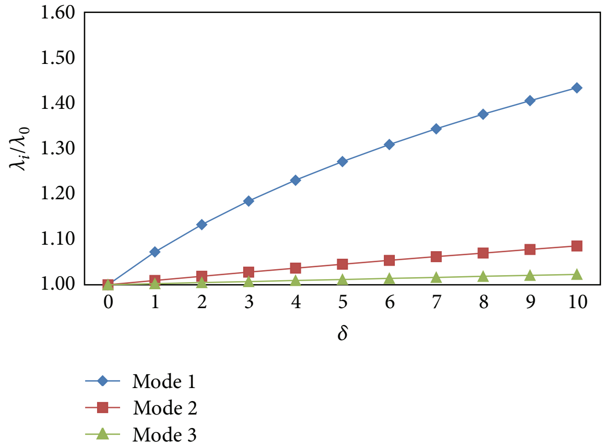


FIGURE 4: Effect of elastic foundation parameter on the vibration frequencies for $\overline{S}_0 = 10^5, \overline{S}_L = 10^5, \overline{\Omega}_0 = 10$, and $\overline{\Omega}_L = 10$.

(i) denotes the mode number and (0) denotes the case without elastic foundation.

Analyses are performed to investigate the effects of elastic foundation parameter to examine how it affects the vibration frequencies of the system. The results of analysis for both cases are depicted in Figures 3–6. Fixing the spring parameters ($\overline{S}_0 = 10^5, \overline{S}_L = 10^5, \overline{\Omega}_0 = 10^5$, and $\overline{\Omega}_L = 10^5$) and varying the elastic foundation parameter (δ) result in a significant change in the vibration frequencies (see Figure 3). For the case in hand, changing the elastic foundation parameter (δ) from 0 to 10 results in an increase in the first mode of about 30 percent, whereas second and third modes are not much affected from the existence of the elastic foundation parameter, as can be noted from the figure.

TABLE 4: The first three frequency parameters for different boundary conditions (λ_1, λ_2, and λ_3).

δ	$\overline{S}_0 = \overline{S}_L = 10^5, \overline{\Omega}_0 = \overline{\Omega}_L = 10^5$			$\overline{S}_0 = \overline{S}_L = 10^5, \overline{\Omega}_0 = \overline{\Omega}_L = 10$		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
Reference [9]	1.51746	2.51718	3.52107	1.32704	2.25310	3.20403
1	1.58444	2.53271	3.52678	1.42308	2.27465	3.21160
2	1.64386	2.54796	3.53247	1.50286	2.29560	3.21912
3	1.69746	2.56294	3.53813	1.57165	2.31599	3.22659
4	1.74642	2.57766	3.54376	1.63243	2.33586	3.23400
5	1.79157	2.59213	3.54936	1.68709	2.35523	3.24137
6	1.83355	2.60637	3.55494	1.73690	2.37414	3.24869
7	1.87282	2.62038	3.56049	1.78276	2.39260	3.25595
8	1.90977	2.63416	3.56602	1.82534	2.41065	3.26317
9	1.94469	2.64770	3.57152	1.86512	2.42830	3.27034
10	1.97783	2.66110	3.57699	1.90252	2.44558	3.27747

TABLE 5: The first three frequency parameters for different boundary conditions (λ_1, λ_2, and λ_3).

δ	$\overline{S}_0 = \overline{S}_L = 10, \ \overline{\Omega}_0 = \overline{\Omega}_L = 10^5$			$\overline{S}_0 = 10^5, \overline{S}_L = 10, \ \overline{\Omega}_0 = 10^5, \overline{\Omega}_L = 10$		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
Reference [9]	0.66997	1.09617	2.02895	0.86875	1.71112	2.68259
1	1.04696	1.25031	2.05824	1.11930	1.75897	2.69544
2	1.21809	1.36226	2.08634	1.26610	1.80321	2.70812
3	1.33764	1.45191	2.11334	1.37453	1.84441	2.72062
4	1.43170	1.52748	2.13934	1.46208	1.88303	2.73295
5	1.51019	1.59326	2.16443	1.53623	1.91940	2.74512
6	1.57806	1.65177	2.18867	1.60098	1.95382	2.75712
7	1.63816	1.70465	2.21214	1.65870	1.98651	2.76897
8	1.69228	1.75302	2.23488	1.71096	2.01766	2.78068
9	1.74166	1.79769	2.25695	1.75883	2.04743	2.79223
10	1.78717	1.83926	2.27838	1.80308	2.07596	2.80365

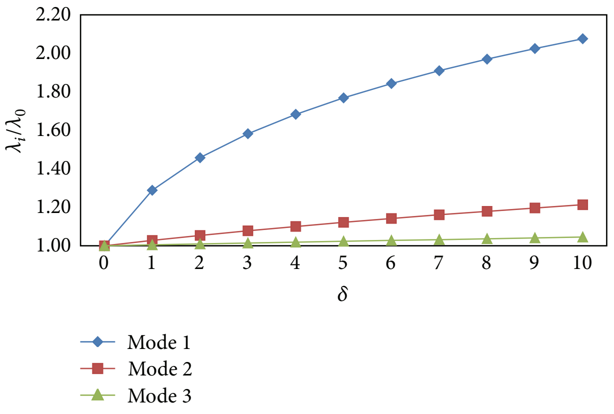


FIGURE 5: Effect of elastic foundation parameter on the vibration frequencies for $\overline{S}_0 = 10^5$, $\overline{S}_L = 10$, $\overline{\Omega}_0 = 10^5$, and $\overline{\Omega}_L = 10$.

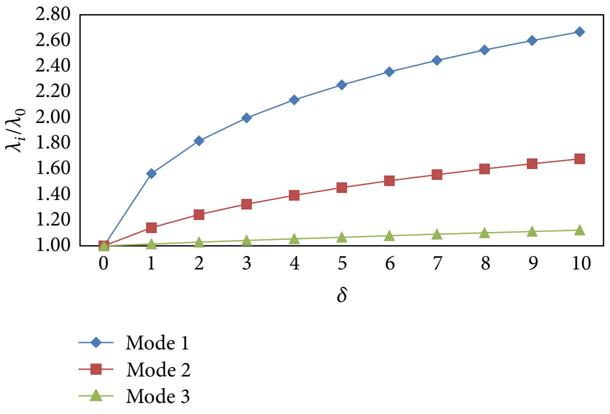


FIGURE 6: Effect of elastic foundation parameter on the vibration frequencies for $\overline{S}_0 = 10$, $\overline{S}_L = 10$, $\overline{\Omega}_0 = 10^5$, and $\overline{\Omega}_L = 10^5$.

The enhancement of the first mode is observed for the different boundary conditions as presented in Figures 4 and 5. For a beam with boundary conditions $\overline{S}_0 = 10^5$, $\overline{S}_L = 10^5$, $\overline{\Omega}_0 = 10$, and $\overline{\Omega}_L = 10$, as the elastic foundation parameter (δ) changes from 0 to 10, the first frequency parameter increases by about 45, as can be noted from Figure 4. Finally, for a beam with boundary conditions $\overline{S}_0 = 10$, $\overline{S}_L = 10$, $\overline{\Omega}_0 = 10^5$, and $\overline{\Omega}_L = 10^5$, this enhancement reaches about 260 percent, as can be noted from Figure 6.

The first three frequency parameters obtained from the analysis and predicted by the suggested formulas are presented as shown in Tables 4 and 5. It can be seen from the tables that, by increasing the the elastic foundation parameter, the first three vibration frequencies increase. It can be noted that the first frequency increase is more than the others. As is obvious from the tables, the proposed analytical method offers acceptable results. This parametric study points out to the possibility of enhancing the frequency parameters of beams with restrained boundary conditions.

5. Conclusion

On the basis of Euler-Bernoulli beam theory, the free vibration response of a beam resting on a Winkler-type elastic foundation with restrained boundary conditions has been investigated. A simplified analytical method is developed, which can be used for a beam with any types of boundary conditions. The general frequency determinant is calculated by a combination of Fourier series expansion and Stokes' transformation. The influence of the elastic foundation and spring parameters on the natural frequencies is examined in some numerical examples. The results of the present analytical method demonstrate good agreement with the results of other methods.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] K. R. Chun, "Free vibration of a beam with one end spring-hinged and the other free," *Journal of Applied Mechanics, Transactions ASME*, vol. 39, no. 4, pp. 1154–1155, 1972.

[2] M. J. Maurizi, R. E. Rossi, and J. A. Reyes, "Vibration frequencies for a uniform beam with one end spring-hinged and subjected to a translational restraint at the other end," *Journal of Sound and Vibration*, vol. 48, no. 4, pp. 565–568, 1976.

[3] P. F. Doyle and M. N. Pavlovic, "Vibration of beams on partial elastic foundations," *Earthquake Engineering Structural Dynamics*, vol. 10, no. 5, pp. 663–674, 1982.

[4] P. A. A. Laura, P. Verniere de Irassar, and G. M. Ficcadenti, "A note on transverse vibrations of continuous beams subject to an axial force and carrying concentrated masses," *Journal of Sound and Vibration*, vol. 86, no. 2, pp. 279–284, 1983.

[5] B. A. H. Abbas, "Vibrations of Timoshenko beams with elastically restrained ends," *Journal of Sound and Vibration*, vol. 97, no. 4, pp. 541–548, 1984.

[6] G. V. Rao and N. R. Naidu, "Free vibration and stability behaviour of uniform beams and columns with non-linear elastic end rotational restraints," *Journal of Sound and Vibration*, vol. 176, no. 1, pp. 130–135, 1994.

[7] A. Kacar, H. T. Tan, and M. O. Kaya, "A note free vibration analysis of beams on variable Winkler elastic foundation by using the differential transform method," *Mathematical and Computational Applications*, vol. 16, pp. 773–783, 2001.

[8] Ö. Civalek, "Application of differential quadrature (DQ) and harmonic differential quadrature (HDQ) for buckling analysis of thin isotropic plates and elastic columns," *Engineering Structures*, vol. 26, no. 2, pp. 171–186, 2004.

[9] H. K. Kim and M. S. Kim, "Vibration of beams with generally restrained boundary conditions using fourier series," *Journal of Sound and Vibration*, vol. 245, no. 5, pp. 771–784, 2001.

[10] M. Hetenyi, "Beams and plates on elastic foundation," *Applied Mechanics Reviews*, vol. 19, pp. 95–102, 1966.

[11] C. Miranda and K. Nair, "Finite beams on elastic foundation," *Journal Structural Division*, vol. 92, pp. 2131–2142, 1966.

[12] D. Z. Yankelevsky and M. Eisenberger, "Analysis of a beam column on elastic foundation," *Computers and Structures*, vol. 23, no. 3, pp. 351–356, 1986.

[13] M. Eisenberger and J. Clastornik, "Vibrations and buckling of a beam on a variable winkler elastic foundation," *Journal of Sound and Vibration*, vol. 115, no. 2, pp. 233–241, 1987.

[14] M. A. De Rosa, "Stability and dynamics of beams on Winkler elastic foundations," *Earthquake Engineering & Structural Dynamics*, vol. 18, no. 3, pp. 377–388, 1989.

[15] J. Wang, "Vibration of stepped beams on elastic foundations," *Journal of Sound and Vibration*, vol. 149, no. 2, pp. 315–322, 1991.

[16] Y. C. Lai, B. Y. Ting, W.-S. Lee, and B. R. Becker, "Dynamic response of beams on elastic foundation," *Journal of Structural Engineering ACSE*, vol. 118, no. 3, pp. 853–858, 1992.

[17] D. Zhou, "A general solution to vibrations of beams on variable winkler elastic foundation," *Computers and Structures*, vol. 47, no. 1, pp. 83–90, 1993.

[18] D. Thambiratnam and Y. Zhuge, "Free vibration analysis of beams on elastic foundation," *Composite Structure*, vol. 60, pp. 971–980, 1996.

[19] D. N. Paliwal and R. K. Pandey, "The free vibration of a cylindrical shell on an elastic foundation," *Journal of Vibration and Acoustics, Transactions of the ASME*, vol. 120, no. 1, pp. 63–71, 1998.

[20] P. Gülkan and B. N. Alemdar, "Exact finite element for a beam on a two-parameter elastic foundation: a revisit," *Structural Engineering and Mechanics*, vol. 7, no. 3, pp. 259–276, 1999.

[21] K. Al-Hosani, S. Fadhil, and A. El-Zafrany, "Fundamental solution and boundary element analysis of thick plates on Winkler foundation," *Computers and Structures*, vol. 70, no. 3, pp. 325–336, 1999.

[22] J.-H. Yin, "Closed-form solution for reinforced Timoshenko beam on elastic foundation," *Journal of Engineering Mechanics*, vol. 126, no. 8, pp. 868–874, 2000.

[23] J.-H. Yin, "Comparative modeling study of reinforced beam on elastic foundation," *Journal of Geotechnical and Geoenvironmental Engineering*, vol. 126, no. 3, pp. 265–271, 2000.

[24] D. M. Santee and P. B. Gonçalves, "Oscillations of a beam on a non-linear elastic foundation under periodic loads," *Shock and Vibration*, vol. 13, no. 4-5, pp. 273–284, 2006.

[25] P. A. A. Laura and R. H. Gutierrez, "Analysis of vibrating Timoshenko beams using the method of differential quadrature," *Shock and Vibration*, vol. 1, pp. 89–93, 1993.

[26] A. Kacar, H. T. Tan, and M. O. Kaya, "Free vibration analysis of beams on variable winkler elastic foundation by using the differential transform method," *Mathematical Computational Applications*, vol. 16, pp. 773–783, 2011.

[27] M. Balkaya, M. O. Kaya, and A. Sağlamer, "Analysis of the vibration of an elastic beam supported on elastic soil using the differential transform method," *Archive of Applied Mechanics*, vol. 79, no. 2, pp. 135–146, 2009.