

An Examination on the Positions of Frenet Ruled Surfaces Along Bertrand Pairs α and α^* According to Their Normal Vector Fields in E^3

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Abstract

In this paper we consider eight special Frenet ruled surfaces along to the Bertrand pairs $\{\alpha^*, \alpha\}$. First we define and find the parametric equations of Frenet ruled surfaces which are called *Bertrandian Frenet ruled surface*, along Bertrand curve α , in terms of the Frenet apparatus of Bertrand curve α . Later we find only one matrix gives us all sixteen positions of normal vector fields of eight Frenet ruled surfaces and *Bertrandian Frenet ruled surface* in terms of Frenet apparatus of Bertrand curve α too. Further using that matrix we have some results such as; normal ruled surface and *Bertrandian tangent*

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ruled surface have perpendicular normal vector fields along the curve $\varphi_2(s) = \alpha + \frac{\beta}{\beta k_1 - \lambda k_2} V_2$ or there are four pairs of Frenet ruled surface with perpendicular normal vector fields along the Bertrand pairs $\{\alpha^*, \alpha\}$.

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1 Introduction and Preliminaries

A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. A ruled surface is one which can be generated by the motion of a straight line in Euclidean 3-space, ([2], [3]). Choosing a directrix on the surface, i.e. a smooth unit speed curve $\alpha(s)$ orthogonal to the straight lines, and then choosing $v(s)$ to be unit vectors along the curve in the direction of the lines, the velocity vector α_s and v satisfy $\langle \alpha', v \rangle = 0$. The fundamental forms of the *B-scroll* with null directrix and Cartan frame in the Minkowskian 3-space is examined in [12]. The properties of the *B-scroll* are also examined in Euclidean 3-space and *n-space* and in Lorentzian 3-space and *n-space* with time-like directrix curve and null rulings ([9], [10], [11]). The properties of the *B-scroll* are also examined in Euclidean 3-space and *n-space* and in Lorentzian 3-space and *n-space* with time-like directrix curve and null rulings, ([5], [9], [10], [11], [17]). The Gauss map of B-scrolls has been examined in [1]. Deriving curves based on the other curves is a subject in geometry. Involute-evolute curves, Bertrand curves are this kind of curves. By using the similiar method we produce a new ruled surface based on the other ruled surface. *Involute B-scroll* is defined in [13]. The differential geometric elements of the *involute \tilde{D} scroll* are examined in [16]. It is well-known that, if a curve is differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve α , is called Frenet-Serret apparatus of the curves. Let Frenet vector fields be V_1, V_2, V_3 of α and let the first and second curvatures of the curve α be k_1 and k_2 , respectively. The quantities $\{V_1, V_2, V_3, D, k_1, k_2\}$ are collectively Frenet-Serret apparatus of the curves. Here Darboux vector D is the areal velocity vector of the Frenet frame of a space curve. It is named after Gaston Darboux who discovered it. For any unit speed curve α , in terms of the Frenet-Serret apparatus, the Darboux vector can be expressed as $D(s) = k_2(s)V_1(s) + k_1(s)V_3(s)$ where curvature functions are defined by $k_1 = \|V_1\|$ and $k_2 = \langle \dot{V}_2, V_3 \rangle$. The Darboux vector field of α and it has the following symmetrical properties $D \times V_1 = \dot{V}_1, D \times V_2 = \dot{V}_2, D \times V_3 = \dot{V}_3$,

[6]. Let a vector field be $\tilde{D}(s) = \frac{k_2}{k_1}(s)V_1(s) + V_3(s)$ along $\alpha(s)$ under the condition that $k_1(s) \neq 0$ and it is called the modified Darboux vector field of α , [8]. Also it is trivial that $\tilde{D}(s)' = \left(\frac{k_2}{k_1}\right)' V_1$. The Frenet formulae are also well known as

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Let $\alpha : I \rightarrow \mathbb{E}^3$ and $\alpha^* : I \rightarrow \mathbb{E}^3$ be the C^2 - class differentiable unit speed two curves and let $V_1(s), V_2(s), V_3(s)$ and $V_1^*(s), V_2^*(s), V_3^*(s)$ be the Frenet frames of the curves α and α^* , respectively. If the principal normal vector V_2 of the curve α is linearly dependent on the principal normal vector V_2^* of the curve α^* , then the pair $\{\alpha, \alpha^*\}$ are called Bertrand curve pair, [15],. Also α^* is called Bertrand mate. If the curve α^* is Bertrand mate of α , then we may write that $\alpha^*(s) = \alpha(s) + \lambda V_2(s)$. Differentiating, we get $\frac{ds}{ds^*} = \frac{1}{k_2\sqrt{\lambda^2 + \beta^2}}$. If the curve α^* is Bertrand mate $\alpha(s)$, then the angle between tangent vectors and distance between corresponding points of the Bertrand curve pair in \mathbb{E}^3 is constant, [15], [7],[14].

Theorem 1.1 *For a space curve $\alpha(s)$ with $k_2 \neq 0$, the curve $\alpha(s)$ is a Bertrand curve if and only if there exist nonzero real numbers λ and β such that constant $\lambda k_1 + \beta k_2 = 1$ for any $s \in I$. It follows from this fact that a circular helix is a Bertrand curve, [15],[7].*

Theorem 1.2 *Let $\alpha : I \rightarrow \mathbb{E}^3$ and $\alpha^* : I \rightarrow \mathbb{E}^3$ be the C^2 - class differentiable unit speed two curves and the quantities $\{V_1, V_2, V_3, \tilde{D}, k_1, k_2\}$ and $\{V_1^*, V_2^*, V_3^*, \tilde{D}^*, k_1^*, k_2^*\}$ are collectively Frenet-Serret apparatus of the curves α and the Bertrand mate α^* , respectively, then*

$$\begin{cases} V_1^* = \frac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2 + \beta^2}} & V_3^* = \frac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2 + \beta^2}}; \lambda k_2 > 0 \\ V_2^* = V_2 & \tilde{D}^* = \frac{k_1 \sqrt{\lambda^2 + \beta^2}}{(\beta k_1 - \lambda k_2)} \tilde{D}. \end{cases}$$

the first and second curvatures of the offset curve α^ are given by $k_1^* = \frac{\beta k_1 - \lambda k_2}{(\lambda^2 + \beta^2)k_2}$ and $k_2^* = \frac{1}{(\lambda^2 + \beta^2)k_2}$, respectively, [15].*

1.1 Normal vector fields of Frenet ruled surfaces

Definition 1.3 In the Euclidean 3 – space, let $\alpha(s)$ be the arclengthed curve. The equations

$$\begin{cases} \varphi_1(s, u_1) = \alpha(s) + u_1 V_1(s) \\ \varphi_2(s, u_2) = \alpha(s) + u_2 V_2(s) \\ \varphi_3(s, u_3) = \alpha(s) + u_3 V_3(s) \\ \varphi_4(s, u_4) = \alpha(s) + u_4 \tilde{D}(s) \end{cases}$$

are the parametrization of Frenet ruled surfaces which are called V_1 – scroll (tangent ruled surface), V_2 – scroll (normal ruled surface), V_3 – scroll (binormal ruled surface), Darboux ruled surface, respectively in [4].

Theorem 1.4 In the Euclidean 3 – space, let η_1, η_2, η_3 , and η_4 be the normal vector fields of ruled surfaces $\varphi_1, \varphi_2, \varphi_3$, and φ_4 , respectively, along the curve α . They can be expressed by the matrices product $[\eta] = [A][V]$ and have the following matrix;

$$[\eta] = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a & 0 & b \\ c & d & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where $a = \frac{-u_2 k_2}{\sqrt{(u_2 k_2)^2 + (1 - u_2 k_1)^2}}$, $b = \frac{(1 - u_2 k_1)}{\sqrt{(u_2 k_2)^2 + (1 - u_2 k_1)^2}}$, $c = \frac{-u_3 k_2}{\sqrt{(u_3 k_2)^2 + 1}}$ and $d = \frac{-1}{\sqrt{(u_3 k_2)^2 + 1}}$.

Proof. The normal vector fields η_1, η_2, η_3 , and η_4 of ruled surfaces $\varphi_1, \varphi_2, \varphi_3$, and φ_4 can be expressed as in the following four equalities $\eta_1 = -V_3$, $\eta_2 = \frac{-u_2 k_2 V_1 + (1 - u_2 k_1) V_3}{\sqrt{(u_2 k_2)^2 + (1 - u_2 k_1)^2}}$, $\eta_3 = \frac{-u_3 k_2 V_1 - V_2}{\sqrt{(u_3 k_2)^2 + 1}}$, and $\eta_4 = -V_2$. This completes the proof. ■

2 Bertrandian Frenet ruled surfaces of Bertrand curve

Definition 2.1 Let $\{\alpha^*, \alpha\}$ be Bertrand curve pair with $k_1 \neq 0$ and $k_2 \neq 0$. The parametrizations of Frenet ruled surfaces of Bertrand mate $\alpha^*(s)$ are as in the following way.

$$\begin{cases} \varphi_1^*(s, v_1) = \alpha^*(s) + v_1 V_1^*(s) = \alpha + \lambda V_2 + v_1 \frac{\beta V_1 + \lambda V_3}{\sqrt{\lambda^2 + \beta^2}}, \\ \varphi_2^*(s, v_2) = \alpha^*(s) + v_2 V_2^*(s) = \alpha + (\lambda + v_2) V_2, \\ \varphi_3^*(s, v_3) = \alpha^*(s) + v_3 V_3^*(s) = \alpha + \lambda V_2 + v_3 \left(\frac{-\lambda V_1 + \beta V_3}{\sqrt{\lambda^2 + \beta^2}} \right), \\ \varphi_4^*(s, v_4) = \alpha^*(s) + v_4 \tilde{D}^*(s) = \alpha + \lambda V_2 + v_4 \frac{k_1 \sqrt{\lambda^2 + \beta^2}}{(\beta k_1 - \lambda k_2)} \tilde{D}. \end{cases}$$

From Theorem 1.4, they are called **Bertrandian Tangent ruled surface**, **Bertrandian Normal ruled surface**, **Bertrandian Binormal ruled surface** and **Bertrandian Darboux ruled surface**, respectively. They are collectively **Bertrandian Frenet ruled surface**.

Theorem 2.2 In the Euclidean 3 – space , the normal vector fields $\eta_1^*, \eta_2^*, \eta_3^*$, and η_4^* of ruled surfaces $\varphi_1^*, \varphi_2^*, \varphi_3^*$, and φ_4^* , respectively, along the curve Bertrand mate α^* , can be expressed by the following matrix;

$$[\eta^*] = \begin{bmatrix} \eta_1^* \\ \eta_2^* \\ \eta_3^* \\ \eta_4^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a^* & 0 & b^* \\ c^* & d^* & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix} .$$

Proof. Substituing $a^* = \frac{-v_2k_2^*}{\sqrt{(v_2k_2^*)^2+(1-v_2k_1^*)^2}}$, $b^* = \frac{(1-v_2k_1^*)}{\sqrt{(v_2k_2^*)^2+(1-v_2k_1^*)^2}}$, $c^* = \frac{-v_3k_2^*}{\sqrt{(v_3k_2^*)^2+1}}$, and $d^* = \frac{-1}{\sqrt{(v_3k_2^*)^2+1}}$ we have the proof. ■

Theorem 2.3 In the Euclidean 3 – space, the product matrix of the position of the unit normal vector fields $\eta_1, \eta_2, \eta_3, \eta_4$ and $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ of Frenet ruled surfaces, along the Bertrand pairs α and α^* is

$$[\eta] [\eta^*]^T = \begin{bmatrix} \langle \eta_1, \eta_1^* \rangle & \langle \eta_1, \eta_2^* \rangle & \langle \eta_1, \eta_3^* \rangle & \langle \eta_1, \eta_4^* \rangle \\ \langle \eta_2, \eta_1^* \rangle & \langle \eta_2, \eta_2^* \rangle & \langle \eta_2, \eta_3^* \rangle & \langle \eta_2, \eta_4^* \rangle \\ \langle \eta_3, \eta_1^* \rangle & \langle \eta_3, \eta_2^* \rangle & \langle \eta_3, \eta_3^* \rangle & \langle \eta_3, \eta_4^* \rangle \\ \langle \eta_4, \eta_1^* \rangle & \langle \eta_4, \eta_2^* \rangle & \langle \eta_4, \eta_3^* \rangle & \langle \eta_4, \eta_4^* \rangle \end{bmatrix} \dots (I)$$

Proof. It is easy to say that $[\eta] [\eta^*]^T = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} [\eta_1^* \ \eta_2^* \ \eta_3^* \ \eta_4^*] .$ ■

Theorem 2.4 In the Euclidean 3 – space, the product matrix of the unit normal vector fields $\eta_1, \eta_2, \eta_3, \eta_4$ and $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ of Frenet ruled surfaces, along the Bertrand pairs α and α^* , can be given by the following matrix

$$[\eta] [\eta^*]^T = \begin{bmatrix} \frac{\beta}{\sqrt{\lambda^2+\beta^2}} & \frac{-\lambda a^* - \beta b^*}{\sqrt{\lambda^2+\beta^2}} & \frac{-\lambda c^*}{\sqrt{\lambda^2+\beta^2}} & 0 \\ \frac{a\lambda - b\beta}{\sqrt{\lambda^2+\beta^2}} & \frac{(a\beta + b\lambda)a^* + (-a\lambda + b\beta)b^*}{\sqrt{\lambda^2+\beta^2}} & \frac{(a\beta + b\lambda)c^*}{\sqrt{\lambda^2+\beta^2}} & 0 \\ \frac{c\lambda}{\sqrt{\lambda^2+\beta^2}} & \frac{c(\beta a^* - \lambda b^*)}{\sqrt{\lambda^2+\beta^2}} & \frac{cc^*\beta + dd^*\sqrt{\lambda^2+\beta^2}}{\sqrt{\lambda^2+\beta^2}} & -d \\ 0 & 0 & -d^* & 1 \end{bmatrix} \dots (II)$$

Proof. Let $[\eta] = [A][V]$ and $[\eta^*] = [A^*][V^*]$ hence

$$\begin{aligned} [\eta][\eta^*]^T &= [A][V]([A^*][V^*])^T \\ &= [A]([V][V^*]^T)[A^*]^T. \end{aligned}$$

$$[\eta][\eta^*]^T = \frac{1}{\sqrt{\lambda^2 + \beta^2}} [A] \begin{bmatrix} \beta & 0 & -\lambda \\ 0 & \sqrt{\lambda^2 + \beta^2} & 0 \\ \lambda & 0 & \beta \end{bmatrix} [A^*]^T$$

this product give us the result. ■

In the Euclidean 3 – space, the position of two surface, basicly, can be examined by the position of their unit normal vector fields. Using the equality of the last two matrice (I) and (II), we have sixteen interesting results according to the normal vector fields with the following results.

Corollary 2.1 *There are four pairs of normal vector fields perpendicular to each other of Frenet ruled surface along the Bertrand pairs $\{\alpha^*, \alpha\}$. Tangent ruled surface and Bertrandian Darboux ruled surface, Normal ruled surface and Bertrandian Darboux ruled surface, Darboux ruled surface and Bertrandian tangent ruled surface, Darboux ruled surface and Bertrandian normal ruled surface along the Bertrand pairs $\{\alpha^*, \alpha\}$ have perpendicular normal vector fields*

Proof. Since $\langle \eta_1, \eta_4^* \rangle = 0$ $\langle \eta_2, \eta_4^* \rangle = 0$ $\langle \eta_4, \eta_1^* \rangle = 0$ and $\langle \eta_4, \eta_2^* \rangle = 0$ it is trivial. ■

Corollary 2.2 *Tangent ruled surface and Bertrandian tangent ruled surface of Bertrand curve α have the normal vector fields under the condition $\cos \phi = \frac{1 - \lambda k_1}{\sqrt{k_2^2 \lambda^2 + (1 - \lambda k_1)^2}}$.*

Proof. Since $\langle \eta_1, \eta_1^* \rangle = \frac{\beta}{\sqrt{\lambda^2 + \beta^2}}$, it is trivial. ■

Corollary 2.3 *Tangent ruled surface and Bertrandian normal ruled surface of Bertrand curve α have perpendicular normal vector fields along the curve $\varphi_2^*(s) = \alpha(s) + \left(\lambda + \frac{\beta(\lambda^2 + \beta^2)k_2}{\lambda - \beta(\beta k_1 - \lambda k_2)} \right) V_2$.*

Proof. Since $\langle \eta_1, \eta_2^* \rangle = \frac{-\lambda a^* - \beta b^*}{\sqrt{\lambda^2 + \beta^2}}$ and using the orthogonality condition $\langle \eta_1, \eta_2^* \rangle = 0$, and also replacing k_1^* and k_2^* in $\frac{\lambda}{\beta} = \frac{(1 - v_2 k_1^*)}{v_2 k_2^*}$, we got the proof under the condition $v_2 = \frac{\beta(\lambda^2 + \beta^2)k_2}{\lambda - \beta(\beta k_1 - \lambda k_2)}$. ■

Corollary 2.4 *Tangent ruled surface and Bertrandian binormal ruled surface of Bertrand curve α have not perpendicular normal vector fields, except $v_3\lambda k_2^* = 0$, If $k_2^* = 0$ they have perpendicular normal vector fields.*

Proof. Since $\langle \eta_1, \eta_3^* \rangle = \frac{-\lambda c^*}{\sqrt{\lambda^2 + \beta^2}}$ and $\frac{v_3\lambda k_2^*}{\sqrt{\lambda^2 + \beta^2}\sqrt{(v_3k_2^*)^2 + 1}} = 0$, it is trivial. ■

Corollary 2.5 *Normal ruled surface and Bertrandian tangent ruled surface of Bertrand curve α have perpendicular normal vector fields along the curve $\varphi_2(s) = \alpha(s) + \frac{\beta}{\beta k_1 - \lambda k_2} V_2(s)$.*

Proof. Since $\langle \eta_2, \eta_1^* \rangle = \frac{a\lambda - b\beta}{\sqrt{\lambda^2 + \beta^2}}$, and $\langle \eta_2, \eta_1^* \rangle = 0$ it is trivial $u_2 = \frac{\beta}{\beta k_1 - \lambda k_2}$, this completes the proof. ■

Corollary 2.6 *Normal ruled surface and Bertrandian normal ruled surface of Bertrand curve α have perpendicular normal vector fields under the condition $v_2 = \frac{(u_2 k_2 \lambda + (1 - u_2 k_1) \beta)(\lambda^2 + \beta^2) k_2}{(\lambda - u_2) + (u_2 k_2 \lambda + (1 - u_2 k_1) \beta)(\beta k_1 - \lambda k_2)}$.*

Proof. Since $\langle \eta_2, \eta_2^* \rangle = \frac{(a\beta + b\lambda)a^* + (-a\lambda + b\beta)b^*}{\sqrt{\lambda^2 + \beta^2}}$, and $\langle \eta_2, \eta_2^* \rangle = 0$, we have the proof. For each parameter u_2 we have perpendicular normal vector fields along the curve with parameter v_2 and under the orthogonality condition $\langle \eta_2, \eta_2^* \rangle = 0$, we have the proof. ■

Corollary 2.7 *Normal ruled surface and Bertrandian binormal ruled surface have perpendicular normal vector fields along the curve $\varphi_2(s) = \alpha(s) + \lambda V_2(s)$.*

Proof. Since $\langle \eta_2, \eta_3^* \rangle = \frac{(a\beta + b\lambda)c^*}{\sqrt{\lambda^2 + \beta^2}}$, and under the condition $\frac{-v_3 k_2^* (a\beta + b\lambda)}{\sqrt{(v_3 k_2^*)^2 + 1}} = 0$, we have $-v_3 k_2^* (a\beta + b\lambda) = 0$ $k_1 \neq 0$, $(a\beta + b\lambda) = 0$, $\lambda = u_2$, this complete the proof. ■

Corollary 2.8 *Binormal ruled surface and Bertrandian tangent ruled surface of Bertrand curve α have not perpendicular normal vector fields since $u_3 k_2 \lambda \neq 0$. Also the angel ϕ between them is a nonzero function of the curvatures k_1 and k_2 of Bertrand curve α ; $\cos \phi = \frac{-u_3 k_2 \lambda}{\sqrt{\lambda^2 + \beta^2}\sqrt{(u_3 k_2)^2 + 1}}$.*

Proof. Since $\langle \eta_3, \eta_1^* \rangle = \frac{c\lambda}{\sqrt{\lambda^2 + \beta^2}}$ and $\frac{-u_3 k_2 \lambda}{\sqrt{(u_3 k_2)^2 + 1}} = 0$ it is trivial. ■

Corollary 2.9 *Binormal ruled surface and Bertrandian normal ruled surface of Bertrand curve α have perpendicular normal vector fields along the curve $\varphi_2^*(s) = \alpha + \left(\lambda + \frac{(\lambda^2 + \beta^2)\lambda k_2}{\lambda(\beta k_1 - \lambda k_2) - \beta} \right) V_2$, except $u_3 k_2 = 0$.*

Proof. Since $\langle \eta_3, \eta_2^* \rangle = \frac{(\beta a^* - \lambda b^*)c}{\sqrt{\lambda^2 + \beta^2}}$ and $\langle \eta_3, \eta_2^* \rangle = 0$ we have $\frac{-u_3 k_2}{\sqrt{(u_3 k_2)^2 + 1}} = 0$ or $\beta a^* - \lambda b^* = 0$. Hence $v_2 = \frac{(\lambda^2 + \beta^2)\lambda k_2}{\lambda(\beta k_1 - \lambda k_2) - \beta}$. ■

Corollary 2.10 *Binormal ruled surface and Bertrandian binormal ruled surface have perpendicular normal vector fields under the condition $v_3 = \frac{-(\lambda^2 + \beta^2)^{\frac{3}{2}}}{u_3 \beta}$.*

Proof. Since $\langle \eta_3, \eta_3^* \rangle = \frac{cc^* \beta + dd^* \sqrt{\lambda^2 + \beta^2}}{\sqrt{\lambda^2 + \beta^2}}$ and $\langle \eta_3, \eta_3^* \rangle = 0$, we have $cc^* \beta + dd^* \sqrt{\lambda^2 + \beta^2} = 0$ hence we have the proof. ■

Corollary 2.11 *Binormal ruled surface and Bertrandian Darboux ruled surface of Bertrand curve α have not perpendicular normal vector fields, the angle between η_3 and η_4^* is a non zero function $\cos \Omega = \frac{1}{\sqrt{(u_3 k_2)^2 + 1}} \neq 0$.*

Proof. Since $\langle \eta_3, \eta_4^* \rangle = -d$ and $\langle \eta_3, \eta_4^* \rangle = \frac{1}{\sqrt{(u_3 k_2)^2 + 1}} \neq 0$ it is trivial. ■

Corollary 2.12 *Darboux ruled surface and Bertrandian binormal ruled surface of Bertrand curve α have not perpendicular normal vector fields, and the angle between them is a nonzero function $\cos \psi = \frac{(\lambda^2 + \beta^2)k_2}{\sqrt{v_3^2 + ((\lambda^2 + \beta^2)k_2)^2}}$.*

Proof. Since $\langle \eta_4, \eta_3^* \rangle = -d^*$ and $\langle \eta_4, \eta_3^* \rangle = \frac{1}{\sqrt{(v_3 k_2^*)^2 + 1}} \neq 0$ it is trivial that $\langle \eta_4, \eta_3^* \rangle = \frac{(\lambda^2 + \beta^2)k_2}{\sqrt{v_3^2 + ((\lambda^2 + \beta^2)k_2)^2}}$. ■

Corollary 2.13 *Darboux ruled surface and Bertrandian Darboux ruled surface the normal vector fields of φ_4 and φ_4^* are parallel to each other since*

$$\langle \eta_4, \eta_4^* \rangle = 1.$$

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