

On 2-algebras: Crossed R -modules and categorical R -algebras

Zekeriya ARVASI^{1*} and Elif ILGAZ ÇAĞLAYAN²

¹Department of Mathematics and Computer Science, Faculty of Arts and Science, Eskişehir Osmangazi University, Eskişehir, Turkey
ORCID iD: <https://orcid.org/XXXXXXXXXX>

²Department of Mathematics, Faculty of Arts and Science, Bilecik Şeyh Edebali University, Bilecik, Turkey

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Abstract: In this work, we describe the category of categorical R -algebras and show that a categorical R -algebra is a category object in $\mathcal{C} = \text{Alg}_R$. By using this property of categorical R -algebras, we can give an equivalency between the category of categorical R -algebras and the category of crossed R -modules and also the category of simplicial R -algebras.

Key words: Category Object, Monoid Object, Categorical R -algebra, Crossed R -module, Simplicial R -algebra.

Introduction

The category objects (or internal categories) are a generalisation of the notion of small category, and are defined with respect to a fixed arbitrary category. If the arbitrary category is taken to be the category of sets then one recovers the theory of small categories.

The idea is that given a category \mathcal{C} , we obtain the definition of a 'category in \mathcal{C} ' by expressing the definition of a usual (small) category completely in terms of commutative diagrams and then interpreting those diagrams in \mathcal{C} .

Category objects were introduced by Ehresmann [9] in the 1960s, and by now they are an important part of category theory [5]. Some other works about internal categories can be found in [12] and [4].

Crossed module was defined as a model of homotopy 2-types by J.H.C. Whitehead, [14]. In [6, 7], crossed modules were considered as 2-dimensional groups. The commutative algebra version of this construction has been adapted by T.Porter, [1, 11].

Crossed modules have been applied in many areas such that group presentations, algebraic K-theory and also homological algebra. In [8, 13], crossed module theory has been analysed with detail. For the (commutative) algebraic version of crossed modules can be seen in [1, 10, 11]. Also, Arvasi and Odabaş have given computing crossed modules in [2] and [3].

In this paper, we give a notion called categorical R -algebra and show that a categorical R -algebra is a category object in $\mathcal{C} = \text{Alg}_R$ and also a monoid object in $\mathcal{C} = \mathbf{Cat}$. After thinking a categorical R -algebra as a category object and a monoid object, we give some equivalencies about crossed R -modules and simplicial R -algebras.

*Correspondence: zarvasi@ogu.edu.tr

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1 **1. Preliminaries**

2 We recall some standart definitions will be used in this work. See [1, 4, 10, 11, 13].

3 **The category of category objects ($\text{Cat}(\mathcal{C})$.)**

4 Let \mathcal{C} be any category. A *category object* in \mathcal{C} consists of:

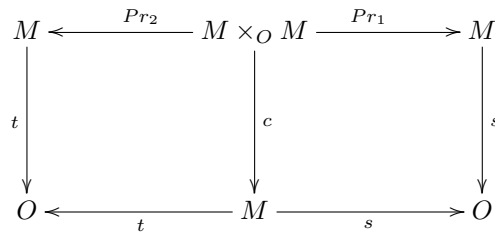
- 5 • An object of object O
- 6 • An object of morphism M

7 together with

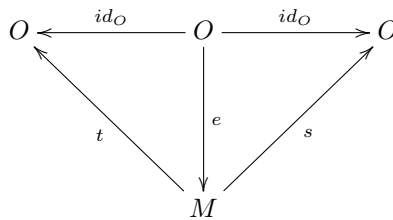
- 8 • source and target morphisms $s, t : M \rightarrow O$
- 9 • an identity morphism $e : O \rightarrow M$
- 10 • a composition morphism $c : M \times_O M \rightarrow M$

11 such that the following diagrams commute, expressing the usual category laws:

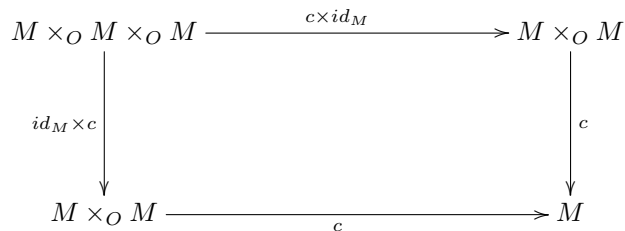
(i) laws specifying the source and target of composite morphisms



(ii) laws specifying the source and target of identity morphism



(iii) the associative law for composition of morphisms



(iv) the left and right unit laws for composition of morphisms

$$\begin{array}{ccccc}
 O \times_O M & \xrightarrow{e \times id_M} & M \times_O M & \xleftarrow{id_M \times e} & M \times_O O \\
 & \searrow Pr_2 & \downarrow c & \swarrow Pr_1 & \\
 & & M & &
 \end{array}$$

The pullback $M \times_O M$ is defined via the square:

$$\begin{array}{ccc}
 M \times_O M & \xrightarrow{\rho_2} & M \\
 \rho_1 \downarrow & & \downarrow s \\
 M & \xrightarrow{t} & O
 \end{array}$$

1 We denote this category object with $C = (O, M, s, t, e, c)$.

2 A *functor* from C to D consists of

- 3 • a morphism $F_0 : O \rightarrow O'$
- 4 • a morphism $F_1 : M \rightarrow M'$

5 such that

$$s^C o = m s^D, t^C o = m t^D, e^C m = o e^D \text{ and } c^C m = (m \sqcap m) c^D.$$

6 where C, D are category objectd in any category \mathcal{C} .

7 Thus we have a category of category objects in \mathcal{C} and this category is denoted by $\mathbf{Cat}(\mathcal{C})$.

8 **Monoidal category**

9 A *monoidal category* is a category \mathcal{C} equipped with

- 10 • a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ out of the product category of \mathcal{C} with itself, called tensor product,
- 11 • an object I called the unit object
- a natural isomorphism

$$\alpha : ((-) \otimes (-)) \otimes (-) \xrightarrow{\cong} (-) \otimes ((-) \otimes (-))$$

with components of the form

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (y \otimes z)$$

12 called the associator,

- a natural isomorphism

$$\lambda : (1 \otimes (-)) \xrightarrow{\cong} (-)$$

with components of the form

$$\lambda_x : (1 \otimes x) \xrightarrow{\cong} x$$

13 called the left unitor, and

- a natural isomorphism

$$\rho : ((-) \otimes 1) \xrightarrow{\cong} (-)$$

with componets of the form

$$\rho_x : (x \otimes 1) \xrightarrow{\cong} x$$

called the right unitor

such that the axioms known as triangle and pentagon axioms are satisfied.

Examples:

(i) $\mathcal{C} = \mathbf{Set}$ is a cartesian monoidal category with the cartesian product and one- element sets serving as the unit.

(ii) $\mathcal{C} = \mathbf{Cat}$ is a cartesian monoidal category with the product category and the category with one object and only its identity morphism serving as the unit.

(iii) $\mathcal{C} = \mathbf{Mod}_R$ (R is a commutative ring) is a monoidal category with the tensor product of modules \otimes_R serving as the monoidal product and the ring R serving as the unit.

(iv) $\mathcal{C} = \mathbf{Alg}_R$ (R is a commutative ring) is a monoidal category with the tensor product of algebras as the monoidal product and R as the unit.

The category of monoid objects ($\mathbf{Mon}(\mathcal{C})$)

A *monoid object* in a monoidal category $(\mathcal{C}, \otimes, I)$ is an object E in \mathcal{C} together with two morphisms

$$\begin{aligned} \mu : E \otimes E &\longrightarrow E && \text{called multiplication,} \\ \eta : I &\longrightarrow E && \text{called unit,} \end{aligned}$$

such that the following diagrams commutative:

(i)

$$\begin{array}{ccccc} (E \otimes E) \otimes E & \xrightarrow{\alpha} & E \otimes (E \otimes E) & \xrightarrow{1 \otimes \mu} & E \otimes E \\ \downarrow \mu \otimes 1 & & & & \downarrow \mu \\ E \otimes E & \xrightarrow{\mu} & & & E \end{array}$$

(ii)

$$\begin{array}{ccccc} I \otimes E & \xrightarrow{\eta \otimes 1} & E \otimes E & \xleftarrow{1 \otimes \eta} & E \otimes I \\ & \searrow \lambda & \downarrow \mu & \swarrow \rho & \\ & & E & & \end{array}$$

In the above notions, I is the unit element and α, λ and ρ are respectively the associativity, the left identity and right identity of the monoidal category \mathcal{C} .

Examples:

(a) A monoid object in the category of sets $\mathcal{C} = \mathbf{Sets}$ is a monoid.

(b) A monoid object in the category of R -modules $\mathcal{C} = \mathbf{Mod}_R$ is an R -algebra.

(c) A monoid object in the category of simplicial R -modules $\mathcal{C} = \mathbf{sMod}_R$ is a simplicial R -algebra.

(d) A monoid object in the category of categories $\mathcal{C} = \mathbf{Cat}$ is a small category \mathcal{C} with two functors $\mu : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and $\eta : \mathcal{C}^{\otimes 0} \rightarrow \mathcal{C}$ where $\mathcal{C}^{\otimes 0}$ is a category with one object and only its identity morphism.

Let C and D be monoid objects in \mathcal{C} . A *monoid morphism* from C to D in \mathcal{C} is a morphism $C \xrightarrow{\varphi} D$ that is compatible with the multiplication and unit, that is the diagrams

$$\begin{array}{ccc} C \otimes C & \xrightarrow{\mu^C} & C \\ \varphi \otimes \varphi \downarrow & & \downarrow \varphi \\ D \otimes D & \xrightarrow{\mu^D} & D \end{array}$$

and

$$\begin{array}{ccc} I & \xrightarrow{\eta^C} & C \\ & \searrow \eta^D & \downarrow \varphi \\ & & D \end{array}$$

are commutative. ($\mu^C \varphi = (\varphi \otimes \varphi) \mu^D$ and $\eta^C \varphi = \eta^D$)

Thus, we have a category of monoid objects which is denoted by $\mathbf{Mon}(\mathcal{C})$.

Crossed modules

In this paper, all algebras will be commutative. Also, we accept that k is a commutative ring, R is a k -algebra with identity.

A crossed module, (C, R, ∂) , (or shortly crossed R -module) consists of an R -algebra C and a morphism $\partial : C \rightarrow R$ with actions R on C , written $(r, c) \mapsto r \cdot c$ for $r \in R$, $c \in C$, satisfying the following conditions:

(CM1)

$$\partial(r \cdot c) = r \partial(c)$$

for all $r \in R$ and $c \in C$.

(CM2)

$$\partial(c) \cdot c' = cc'$$

for all $c, c' \in C$.

A morphism, $(\alpha, \beta) : V \rightarrow V'$ of crossed modules consists of morphisms $\alpha : C \rightarrow C'$ and $\beta : R \rightarrow R'$ such that

$$\begin{array}{ll} (i) & \partial \beta = \alpha \partial' \\ (ii) & \alpha(r \cdot c) = \alpha(c) \cdot \beta(r) \end{array}$$

for all $c \in C$, $r \in R$ where $V = (C, R, \partial)$ and $V' = (C', R', \partial')$ are crossed modules.

Thus, we have a category of crossed modules from the above definitions and it is denoted by \mathbf{XMod} .

Examples:

(a) Let A be an R -algebra and I be an ideal of A . Then (I, A, i) is a crossed module with the multiplication action of A on I . Conversely, we induce an ideal from a given crossed module. Indeed, for a given crossed module (C, R, ∂) , $\partial(C)$ is an ideal of R .

(b) Let M be an R -module. Then $(M, R, 0)$ is a crossed module. Conversely, for a crossed module (C, R, ∂) , one can get that $\ker \partial$ is an $R/\partial(C)$ -module.

(c) Let $\partial : C \rightarrow R$ be a surjective R -algebra homomorphism. Define the action of R on C by $r \cdot c = \tilde{r}c$ where $\tilde{r} \in \partial^{-1}(r)$. Then, (C, R, ∂) is a crossed module with the defined action.

(d) Let C be an R -algebra such that $\text{Ann}(C) = 0$ or $C^2 = C$ then $(C, \mu(C), \partial)$ is a crossed module, where $\mu(C)$ is the algebra of multipliers of C and ∂ is the canonical homomorphism.

7 Simplicial algebras

Let R be a commutative ring with identity. We will use the term *commutative algebra* to mean a commutative algebra over R .

A *simplicial (commutative) algebra* \mathbf{E} consists of a family of algebras $\{E_n\}$ together with face and degeneracy maps $d_i = d_i^n : E_n \rightarrow E_{n-1}$, $0 \leq i \leq n$, ($n \neq 0$) and $s_i = s_i^n : E_n \rightarrow E_{n+1}$, $0 \leq i \leq n$, satisfying the usual simplicial identities. It can be completely described as a functor $\Delta^{op} \rightarrow \mathbf{Alg}_R$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \dots < n\}$ and increasing maps. We have a category of simplicial algebras the above definitions and it is denoted by \mathbf{sAlg}_R . We have for each $k \geq 0$ a subcategory $\Delta_{\leq k}$ determined by the objects $[j]$ of Δ with $j \leq k$. A k -truncated simplicial commutative algebra is a functor from $(\Delta_{\leq k}^{op})$ to \mathbf{Alg}_R .

Consider the product $\Delta \times \Delta$ whose objects are pairs $([p], [q])$ and whose maps are pairs of weakly increasing maps. A functor $(\Delta \times \Delta)^{op} \rightarrow \mathbf{Alg}_R$ is called a *bisimplicial algebra* with value in \mathbf{Alg}_R . To give this functor is equivalent with giving for each (p, q) algebra $E_{p,q}$ and homomorphisms

$$\begin{aligned} d_i^h : E_{p,q} &\rightarrow E_{p-1,q} \\ s_i^h : E_{p,q} &\rightarrow E_{p+1,q} & i : 0, \dots, p \\ d_j^v : E_{p,q} &\rightarrow E_{p,q-1} \\ s_j^v : E_{p,q} &\rightarrow E_{p,q+1} & j : 0, \dots, q \end{aligned}$$

such that the maps d_i^h, s_i^h commute with d_j^v, s_j^v and d_i^h, s_i^h (resp. d_j^v, s_j^v) satisfy the usual simplicial identities.

Here d_i^h, s_i^h denote the horizontal operators and d_j^v, s_j^v denote the vertical operators. We have a category of bisimplicial algebras from the above definitions and it is denoted by $\mathbf{s}^2 \mathbf{Alg}_R$.

19 2. Categorical R-algebras as category objects

In this section, we give a notion called categorical R -algebra and show that a categorical R -algebra is a category object in $\mathcal{C} = \mathbf{Alg}_R$.

Definition 2.1 A *categorical R -algebra* is a (small) category \mathcal{C} equipped with

- R -algebras $O = \text{Ob}\mathcal{C}$ and $M = \text{Mor}\mathcal{C}$;
- a functor

$$\mu^{\mathcal{C}} : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C}$$

produced by the multiplication morphisms

$$\mu^O : O \times O \rightarrow O$$

and

$$\mu^M : M \times M \rightarrow M.$$

1 If \mathcal{C} and \mathcal{D} are categorical R -algebras, a morphism $\mathcal{C} \xrightarrow{\varphi} \mathcal{D}$ is a functor. Then one has a category, of
 2 course. It will usually be denoted by \mathbf{cAlg}_R .

Lemma 2.2 *Let \mathcal{C} be a categorical R -algebra.*

(a) *The source, target and identity morphisms*

$$M \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \\ \xleftarrow{e} \end{array} O$$

and the composition morphism

$$c : M \times_O M \rightarrow M$$

are R -algebra morphisms.

(b) If $\mathcal{C}^{\otimes 0}$ is a small category with one object and only its identity morphism, then

$$\eta^{\mathcal{C}} : \mathcal{C}^{\otimes 0} \rightarrow \mathcal{C}$$

3 is a functor.

Proof (a) From $\mathcal{C} \otimes \mathcal{C} \xrightarrow{\mu^{\mathcal{C}}} \mathcal{C}$ is a functor, we have the following commutative diagrams:

$$\begin{array}{ccc} M \otimes M & \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \\ \xleftarrow{e} \end{array} & O \otimes O \\ \alpha \downarrow & & \downarrow \beta \\ M & \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \\ \xleftarrow{e} \end{array} & O \end{array}$$

where $\alpha = \text{Mor}\mu^{\mathcal{C}}$ and $\beta = \text{Ob}\mu^{\mathcal{C}}$, that is

$$\begin{aligned} \alpha s &= s\beta \\ \alpha t &= t\beta \\ \beta e &= e\alpha \end{aligned}$$

and

$$\begin{array}{ccc} \mathcal{M} \times_O \mathcal{M} & \xrightarrow{c} & \mathcal{M} \\ \alpha \times \alpha \downarrow & & \downarrow \alpha \\ M \times_O M & \xrightarrow{c} & M \end{array}$$

that is,

$$c\alpha = (\alpha \times \alpha)c.$$

4 where $\mathcal{M} = \text{Mor}(\mathcal{C} \otimes \mathcal{C})$.

5 So, we can write

$$\begin{aligned} \gamma s &= \alpha s \\ &= s\beta \\ &= (s \otimes s)\delta \end{aligned}$$

1 and

$$\begin{aligned}\gamma t &= \alpha t \\ &= t\beta \\ &= (t \otimes t)\delta\end{aligned}$$

2 as well as

$$\begin{aligned}\delta e &= \beta e \\ &= e\alpha \\ &= (e \otimes e)\gamma.\end{aligned}$$

3 where $\gamma = \mu^M$ and $\delta = \mu^O$.

By considering the canonical isomorphism

$$\xi : (M \times_O M) \otimes (M \times_O M) \longrightarrow (M \otimes M) \times_O (M \otimes M)$$

4 we also have

$$\begin{aligned}\varepsilon c &= \xi(\gamma \times \gamma)c \\ &= \xi(\alpha \times \alpha)c \\ &= \xi c\alpha \\ &= (c \otimes c)\alpha.\end{aligned}$$

5 where $\varepsilon = \mu^{M \times_O M}$. Thus, s, t, e and c are R -algebra homomorphisms.

6 (b) According to (a), the structure morphisms s, t, e and c are R -algebra homomorphisms. So, we have

$$\begin{aligned}\eta^M s &= \eta^O = s^{\otimes 0} \eta^O = s\eta^O, \\ \eta^M t &= \eta^O = t^{\otimes 0} \eta^O = t\eta^O, \\ \eta^O e &= \eta^M = e^{\otimes 0} \eta^M = e\eta^M\end{aligned}$$

7 and

$$\begin{aligned}(\eta^M \times_O \eta^M)c &= \eta^{M_t \Gamma_s M} c \\ &= c^{\otimes 0} \eta^M \\ &= c\eta^M,\end{aligned}$$

8 that is, η^C with $Ob(\eta^C) := \eta^O$ and $Mor(\eta^C) := \eta^M$ is a functor. □

9 Thus, a categorical R -algebra is a category with R -algebras O and M such that the source and target
10 maps $s, t : M \longrightarrow O$, the identity map $e : O \longrightarrow M$, and the composition map $c : M \times_O M \longrightarrow M$ are
11 R -algebra morphisms.

12 **Proposition 2.3** *If \mathcal{C} is a categorical R -algebra, then*

(i) *the composition can be given as*

$$\begin{aligned}c : M \times_O M &\longrightarrow M \\ (m, n) &\longmapsto m - (et)(m) + n\end{aligned}$$

1 with an inverse of m up to composition is $(et)(m) - m + (es)(m)$.

2
3 (ii) $KertKers = \{0\}$.

4 **Proof** (i) $t(m) = s(n)$ where $m, n \in M$ are composable morphisms. As c and e are R -algebra homomor-
5 phisms, we can write

$$\begin{aligned} c(m, n) &= c(m + 0, 0 + n) \\ &= c(m + 0, (et)(m) - (et)(m) + n) \\ &= c(m, (et)(m)) + c(0, -(et)(m) + n) \\ &= m - (et)(m) + n. \end{aligned}$$

On the other hand, as

$$\begin{aligned} t((et)(m) - m + (es)(m)) &= t(et)(m) - t(m) + t(es)(m) = s(m) \\ s((et)(m) - m + (es)(m)) &= s(et)(m) - s(m) + s(es)(m) = t(m) \end{aligned}$$

also

$$c(m, (et)(m) - m + (es)(m)) = m - (et)(m) + (et)(m) - m + (es)(m) = (es)(m)$$

while

$$c((et)(m) - m + (es)(m), m) = (et)(m) - m + (es)(m) - (es)(m) + m = (et)(m),$$

6 we can get that $(et)(m) - m + (es)(m)$ is the inverse of m .

7 (ii) Let $m \in Kert$, $n \in Kers$. Then we have $t(m) = 0 = s(n)$ and according to (a) we can write

$$\begin{aligned} mn &= m - (et)(m) + n \\ &= m - (es)(n) + n \\ &= n - (et)(m) + m \\ &= n - (es)(n) + m \\ &= nm. \end{aligned}$$

8 Thus, we have $KertKers = \{0\}$. □

9 **Lemma 2.4** If $M \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \\ \xrightarrow{e} \end{array} O$ is retraction (i.e, $se = te = id$) and $KersKert = \{0\}$, then there exists

10 a categorical R -algebra \mathcal{C} with $Ob\mathcal{C} := O$, $Mor\mathcal{C} := M$, and categorical structure morphisms s , t , e .

11 **Proof** We have composite $c(m, n) := m - (et)(m) + n$, for elements $m, n \in M$ with $t(m) = s(n)$. On the
12 other hand, for $m, n, m', n' \in M$ with $t(m) = s(n)$ and $t(m') = s(n')$

$$\begin{aligned} c(m, n)c(m', n') &= (m - (et)(m) + n)(m' - (et)(m') + n') \\ &= (m - (et)(m))m' + nm' + (m - (et)(m))(n' - (es)(n')) + n(n' - (es)(n')) \\ &= mm' - (et)(m)m' + nm' + mn' - (et)(m)n' - m(es)(n') + (et)(mm') + nn' \\ &\quad - n(es)(n') \\ &= c(mm', nn') + A + B \end{aligned}$$

1 where

$$\begin{aligned} A &= mn' - (et)(m)n' - m(es)(n') + (et)(m)(es)(n') \\ &= (m - (et)(m))(n' - (es)(n')) \in KertKers \end{aligned}$$

2 and

$$\begin{aligned} B &= nm' - (es)(n)m' - n(et)(m') + (es)(n)(et)(m') \\ &= (n - (es)(n))(m' - (et)(m')) \in KersKert. \end{aligned}$$

Since $KersKert = \{0\}$, $A = B = 0$. So,

$$c(m, n)c(m', n') = c(mm', nn') = c((m, n)(m', n')).$$

3 That is, the morphism (R -module homomorphism) c is an R -algebra homomorphism. Now, we check that the
4 category axioms given in definition of category object in \mathcal{C} .

5 (i) We have

$$\begin{aligned} (sc)(m, n) &= s(m - (et)(m) + n) \\ &= s(m) - s(et)(m) + s(n) \\ &= s(m) - s(n) + s(n) \\ &= s(m) \end{aligned}$$

6 and

$$\begin{aligned} (tc)(m, n) &= t(m - (et)(m) + n) \\ &= t(m) - t(et)(m) + t(n) \\ &= t(m) - t(m) + t(n) \\ &= t(n) \end{aligned}$$

7 for all $m, n \in M$ with $t(m) = s(n)$.

8 (ii) The identities $se = te = id$ are given by assumption.

9 (iii) We have

$$\begin{aligned} c(k, c(m, n)) &= c(k, m - (et)(m) + n) \\ &= k - (et)(k) + m - (et)(m) + n \\ &= k - (es)(m) + m - (es)(n) + n \\ &= c(k - (es)(m) + m, n) \\ &= c((k, m), n) \end{aligned}$$

10 for all $k, m, n \in M$ with $t(k) = s(m)$ and $t(m) = s(n)$, so the composition is associative.

11 (iv) We have

$$\begin{aligned} c((es)(m), m) &= (es)(m) - (et)(es)(m) + m \\ &= (es)(m) - (es)(m) + m \\ &= m \end{aligned}$$

1 and

$$\begin{aligned} c(m, (et)(m)) &= m - (et)(m) + (et)(m) \\ &= m \end{aligned}$$

2 for $m \in M$.

3 Thus, \mathcal{C} with $Ob\mathcal{C} := O$, $Mor\mathcal{C} := M$ and $s^{\mathcal{C}} = s$, $t^{\mathcal{C}} = t$, $e^{\mathcal{C}} = e$, $c^{\mathcal{C}} = c$ is a category object in \mathbf{Alg}_R .
4 □

5 **Corollary 2.5** *A categorical R -algebra is a category object in $\mathcal{C} = \mathbf{Alg}_R$.*

Proof Let \mathcal{C} be categorical R -algebra. Then \mathcal{C} is a (small) category such that O and M are R -algebras. According to lemma 2.2 (a), the categorical structure morphisms s, t, e and c are R -algebra morphisms. On the other hand, from the Proposition 2.3, the composition morphism is defined as

$$\begin{aligned} c: M \times_O M &\longrightarrow M \\ (m, n) &\longmapsto m - (et)(m) + n. \end{aligned}$$

6 We will show that this composition map satisfy the diagrams of the definition of category object. The proofs
7 of (i)-(iii)-(iv) are similar to the proofs given in the lemma 2.4. So we only show that (ii).

8 (ii) For $m : x \rightarrow y$, we have

$$\begin{aligned} c(e(x), m) &= e(x) - es(m) + m \\ &= e(x) - e(x) + m \\ &= m \end{aligned}$$

9 and

$$\begin{aligned} c(m, e(y)) &= m - es(e(y)) + e(y) \\ &= m - e(y) + e(y) \\ &= m. \end{aligned}$$

10 □

11 3. Equivalence between categorical R -algebras and crossed R -modules

12 In this section, we show that the categories \mathbf{XMod} and \mathbf{cAlg}_R are equivalent. By using the corollary 2.5, we
13 can show this equivalency. Of course, this is the similar way of Porter works [11, 12].

Now, we remember that for the given crossed module (C, R, ∂) , the semidirect product $C \rtimes R$ is formed using the action of R on C the crossed module provides. Hence, we have

$$(c, r)(c', r') = (cc' + cr' + rc', rr')$$

14 for $(c, r), (c', r') \in C \rtimes R$.

Proposition 3.1 *Given a crossed module (C, R, ∂) , we have a categorical R -algebra \mathcal{C} , in which the objects and morphisms are given by*

$$Ob\mathcal{C} := R \text{ and } Mor\mathcal{C} := C \rtimes R,$$

1 source, target and identity morphisms (R -module homomorphisms) are given by

$$\begin{aligned} s(c, r) &= \partial(c) + r \\ t(c, r) &= r \\ e(r) &= (0, r) \end{aligned}$$

for $(c, r) \in C \rtimes R$ and $r \in \text{Ob}\mathcal{C}$, the R -algebra of composable morphisms is

$$\{(c_2, \partial(c_1) + r_1), (c_1, r_1)\} \in \text{Mor}\mathcal{C} \times \text{Mor}\mathcal{C} \mid c_1, c_2 \in C, r_1 \in R\}$$

and the composition in \mathcal{C} is given by

$$c((c_2, \partial(c_1) + r_1), (c_1, r_1)) := (c_2 + c_1, r_1)$$

2 for $c_1, c_2 \in C, r_1 \in R$.

3 **Proof** Since $\text{Ob}\mathcal{C} = R$ and $\text{Mor}\mathcal{C} = C \rtimes R$ are R -algebras, firstly we show that s, t and e are R -algebra
4 homomorphisms. We get

$$\begin{aligned} s((c, r)(c', r')) &= s(cc' + cr' + rc', rr') \\ &= \partial(cc' + cr' + rc') + rr' \\ &= \partial(cc') + \partial(cr') + \partial(rc') + rr' \end{aligned}$$

5

$$\begin{aligned} s(c, r)s(c', r') &= (\partial(c) + r)(\partial(c') + r') \\ &= \partial(c)\partial(c') + \partial(c)r' + r\partial(c') + rr' \\ &= \partial(cc') + \partial(cr') + \partial(rc') + rr' \end{aligned}$$

6 that is $s((c, r)(c', r')) = s(c, r)s(c', r')$ and

$$\begin{aligned} t((c, r)(c', r')) &= t(cc' + cr' + rc', rr') \\ &= rr' \\ &= t(c, r)t(c', r') \end{aligned}$$

for $(c, r), (c', r') \in \text{Mor}\mathcal{C}$ as well as

$$e(rr') = (0, rr') = (0, r)(0, r') = e(r)e(r')$$

7 for $r, r' \in \text{Ob}\mathcal{C}$.

8 The R -algebra of composable morphisms can be computed as follows:

$$\begin{aligned} \text{Mor}\mathcal{C} \times_{\mathcal{O}} \text{Mor}\mathcal{C} &= \{(c_2, r_2), (c_1, r_1)\} \in \text{Mor}\mathcal{C} \times \text{Mor}\mathcal{C} \mid t(c_2, r_2) = s(c_1, r_1)\} \\ &= \{(c_2, r_2), (c_1, r_1)\} \in \text{Mor}\mathcal{C} \times \text{Mor}\mathcal{C} \mid r_2 = \partial(c_1) + r_1\} \\ &= \{(c_2, \partial(c_1) + r_1), (c_1, r_1)\} \in \text{Mor}\mathcal{C} \times \text{Mor}\mathcal{C} \mid c_1, c_2 \in C, r_1 \in R\}. \end{aligned}$$

1 Thus c is an R -algebra homomorphism since

$$\begin{aligned}
 & c(((c_2, \partial(c_1) + r_1), (c_1, r_1))((c'_2, \partial(c'_1) + r'_1), (c'_1, r'_1))) \\
 = & c((c_2, \partial(c_1) + r_1)(c'_2, \partial(c'_1) + r'_1), (c_1, r_1)(c'_1, r'_1)) \\
 = & c((c_2c'_2 + c_2(\partial(c'_1) + r'_1) + (\partial(c_1) + r_1)c'_2, (\partial(c_1) + r_1)(\partial(c'_1) + r'_1)), \\
 & (c_1c'_1 + c_1r'_1 + r_1c'_1, r_1r'_1)) \\
 = & c((c_2c'_2 + c_2c'_1 + c_2r'_1 + c_1c'_2 + r_1c'_2, \partial(c_1c'_1) + \partial(c_1r'_1) + \partial(r_1c'_1) + r_1r'_1), \\
 & (c_1c'_1 + c_1r'_1 + r_1c'_1, r_1r'_1)) \\
 = & (c_2c'_2 + c_2c'_1 + c_2r'_1 + c_1c'_2 + r_1c'_2 + c_1c'_1 + c_1r'_1 + r_1c'_1, r_1r'_1)
 \end{aligned}$$

2 and on the other hand

$$\begin{aligned}
 & c((c_2, \partial(c_1) + r_1), (c_1, r_1))c((c'_2, \partial(c'_1) + r'_1), (c'_1, r'_1)) \\
 = & (c_1 + c_2, r_1)(c'_1 + c'_2, r'_1) \\
 = & (c_1c'_1 + c_1c'_2 + c_2c'_1 + c_2c'_2 + r_1c'_1 + r_1c'_2 + c_1r'_1 + c_2r'_1, r_1r'_1).
 \end{aligned}$$

3 Now, we have to check that \mathcal{C} satisfies the axioms for a category object in $\mathcal{C} = \mathbf{Alg}_R$ given in its definition.

(i) We have

$$(se)(r) = s(e(r)) = s(0, r) = \partial(0) + r = r$$

and

$$(te)(r) = t(e(r)) = t(0, r) = r$$

4 for $r \in Ob\mathcal{C}$.

5 (ii) Given a pair of composable morphisms $((c_2, \partial(c_1) + r_1), (c_1, r_1))$ in \mathcal{C} , we have

$$\begin{aligned}
 (sc)((c_2, \partial(c_1) + r_1), (c_1, r_1)) &= s(c_2 + c_1, r_1) \\
 &= \partial(c_2 + c_1) + r_1 \\
 &= \partial(c_2) + \partial(c_1) + r_1 \\
 &= s(c_2, \partial(c_1) + r_1)
 \end{aligned}$$

6 and

$$\begin{aligned}
 (tc)((c_2, \partial(c_1) + r_1), (c_1, r_1)) &= t(c_2 + c_1, r_1) \\
 &= r_1 \\
 &= t(c_1, r_1)
 \end{aligned}$$

7 (iii) We have

$$\begin{aligned}
 & c((c_3, \partial(c_2) + \partial(c_1) + r_1), c((c_2, \partial(c_1) + r_1), (c_1, r_1))) \\
 = & c((c_3, \partial(c_2) + \partial(c_1) + r_1), (c_2 + c_1, r_1)) \\
 = & (c_3 + c_2 + c_1, r_1) \\
 = & c((c_3 + c_2, \partial(c_1) + r_1), (c_1, r_1)) \\
 = & c(c(c_3, \partial(c_2) + \partial(c_1) + r_1), (c_2, \partial(c_1) + r_1), (c_1, r_1))
 \end{aligned}$$

1 for $c_1, c_2, c_3 \in C, r_1 \in R$.

2 (iv) We get

$$\begin{aligned} c((es)(c, r), (c, r)) &= c(e(\partial(c) + r), (c, r)) \\ &= c((0, \partial(c) + r), (c, r)) \\ &= (0 + c, r) = (c, r) \end{aligned}$$

3 and φ

$$\begin{aligned} c((c, r), (et)(c, r)) &= c((c, r), (0, r)) \\ &= (c + 0, r) = (c, r) \end{aligned}$$

4 for $(c, r) \in Mor\mathcal{C}$.

5 Thus, \mathcal{C} is a category object in \mathbf{Alg}_R . □

Corollary 3.2 *We have a functor*

$$F : \mathbf{XMod} \longrightarrow \mathbf{cAlg}_R.$$

Proposition 3.3 *For every categorical R -algebra \mathcal{C} , we have a crossed module (C, R, ∂) with given by*

$$C := Kert, \quad R := Ob\mathcal{C}$$

6 *structure morphism $\partial := s|_{Kert}$, where the action of the R on C is given by $r \cdot c := e(r)c$ for $r \in R, c \in C$.*

7 **Proof** Since $Mor\mathcal{C}$ is an R -algebra and s is an R -algebra homomorphism, $Kert$ is an R -algebra and $s|_{Kert}$
8 is an R -algebra homomorphism. There is a well defined $Ob\mathcal{C}$ -action on $Kert$ because of e is an R -algebra
9 homomorphism. Now, we will show that (CM1) and (CM2).

10 (CM1) We have

$$\begin{aligned} s(r \cdot c) &= s(e(r)c) \\ &= s(e(r))s(c) \\ &= rs(c) \end{aligned}$$

for $r \in R = Ob\mathcal{C}$ and all $c \in Mor\mathcal{C}$, and hence in particular

$$(s|_{Kert})(r \cdot c) = r(s|_{Kert(c)})$$

11 for $r \in Ob\mathcal{C}, c \in Kert$.

12 (CM2) We have

$$\begin{aligned} s|_{Kert}(n) \cdot m &= s(n) \cdot m \\ &= e(s(n))m \\ &= nm \end{aligned}$$

13 for all $n, m \in Kert$. □

Corollary 3.4 *We have a functor*

$$G : \mathbf{cAlg}_R \longrightarrow \mathbf{XMod}.$$

1 **Theorem 3.5** *The categories \mathbf{XMod} and \mathbf{cAlg}_R are equivalent.*

2 **Proof** For the functors $\mathbf{XMod} \xrightarrow{F} \mathbf{cAlg}_R$ and $\mathbf{cAlg}_R \xrightarrow{G} \mathbf{XMod}$, we show that $GF \cong id_{\mathbf{cAlg}_R}$ and
 3 $FG \cong id_{\mathbf{XMod}}$. Let (C, R, ∂) be a crossed module. We have

$$\begin{aligned} FG(C, R, \partial) &= F(R, C \rtimes R, s, t, e, c) \\ &= (Kert, R, s|_{Kert}) \end{aligned}$$

4 and

$$\begin{aligned} Kert &= \{(c, r) \mid t(c, r) = 0\} \\ &= \{(c, r) \mid r = 0\} \\ &= \{(c, 0) \mid c \in C\} \\ &= C \rtimes \{0\}. \end{aligned}$$

Also

$$\begin{aligned} R \times Kert &\longrightarrow R \\ (r, (c, 0)) &\longmapsto r \cdot (c, 0) = e(r)(c, 0) = (0, r)(c, 0) = (r \cdot c, 0) \end{aligned}$$

and

$$s|_{Kert}(c, 0) = s(c, 0) = \partial(c) + 0 = \partial(c).$$

5 That is we obtain $FG(C, R, \partial) = (C \rtimes \{0\}, R, s|_{Kert}) \cong (C, R, \partial) = id_{\mathbf{XMod}}(C, R, \partial)$.

6 On the other hand, let $C = (ObC, MorC, s, t, e, c)$ be a categorical R -algebra. We have

$$\begin{aligned} GF(C) &= G(Kert, ObC, s|_{Kert}) \\ &= (ObC, Kert \rtimes ObC, s, t, e, c) \end{aligned}$$

7 and for $(m, o) \in Kert \rtimes ObC$ we can write

$$\begin{aligned} s(m, o) &= s|_{Kert}(m) + o = s(m) + o \\ t(m, o) &= o \\ e(o) &= (0, o). \end{aligned}$$

Now, we must show $Kert \rtimes ObC \cong MorC$. Firstly, we define a function such that

$$\begin{aligned} \varphi : MorC &\longrightarrow Kert \rtimes ObC \\ m &\longmapsto (m - (et)(m), t(m)). \end{aligned}$$

8 Since

$$\begin{aligned} t(m - (et)(m)) &= t(m) - t(et)(m) \\ &= t(m) - t(m) = 0 \end{aligned}$$

1 $m - (et)(m) \in Kert$, φ is well defined. Also since for $m, n \in MorC$

$$\begin{aligned}\varphi(mn) &= (mn - (et)(mn), t(mn)) \\ &= (mn - (et)(m)(et)(n), t(mn))\end{aligned}$$

2 and

$$\begin{aligned}\varphi(m)\varphi(n) &= (m - (et)(m), t(m))(n - (et)(n), t(n)) \\ &= ((m - (et)(m))(n - (et)(n)) + (m - (et)(m))t(n) + t(m)(n - (et)(n)), t(m)t(n)) \\ &= (mn - m(et)(n) - (et)(m)n + (et)(m)(et)(n) + (m - (et)(m))e(t(n)) + e(t(m))(n - (et)(n)), t(mn)) \\ &\quad (\cdot\text{-action of } ObC \text{ on } Kert) \\ &= (mn - (et)(m)(et)(n), t(mn)),\end{aligned}$$

φ is a homomorphism. Also, we have an anti-homomorphism

$$\begin{aligned}\varphi^{-1} : Kert \times ObC &\longrightarrow MorC \\ (m, o) &\longmapsto e(o) + m\end{aligned}$$

3 and since

$$\begin{aligned}\varphi\varphi^{-1}(m, o) &= \varphi(\varphi^{-1}(m, o)) \\ &= \varphi(e(o) + m) \\ &= (e(o) + m - (et)(e(o) + m), t(e(o) + m)) \\ &= (e(o) + m - e(te)(o) - (et)(m), (te)(o) + t(m)) \\ &= (m, o) \quad (\because te = id \text{ and } m \in Kert)\end{aligned}$$

4 and

$$\begin{aligned}\varphi^{-1}\varphi(m) &= \varphi^{-1}(\varphi(m)) \\ &= \varphi^{-1}(m - (et)(m), t(m)) \\ &= e(t(m)) + m - (et)(m) \\ &= m,\end{aligned}$$

5 φ is an isomorphism. Thus $GF \cong id_{cAlg_R}$. □

6 4. Categorical R-algebras as monoid objects

7 **Proposition 4.1** *A categorical R-algebra is a monoid object in $\mathcal{C} = Cat$.*

8 **Proof** Let \mathcal{C} be a categorical R-algebra. Then O and M are R-algebras, that is, we have

$$\begin{aligned}(1 \otimes \mu^A)\mu^A &= (\mu^A \otimes 1)\mu^A, \\ (\eta^A \otimes 1)\mu^A &= Pr_2 \text{ and } (1 \otimes \eta^A)\mu^A = Pr_1,\end{aligned}$$

9
10 for $A \in \{O, M\}$, cf. definition of monoid object in \mathcal{C} . Furthermore, by the definition of a categorical R-algebra,

1 we have a functor μ^C given by $Ob(\mu^C) := \mu^O$ and $Mor(\mu^C) := \mu^M$. Additionally, lemma 2.2 (b) tells us that
 2 there is a functor η^C where $Ob(\eta^C) := \eta^O$, $Mor(\eta^C) := \eta^M$. This implies

$$\begin{aligned} (1 \otimes \mu^C)\mu^C &= (\mu^C \otimes 1)\mu^C, \\ (\eta^C \otimes 1)\mu^C &= Pr_2 \text{ and } (1 \otimes \eta^C)\mu^C = Pr_1, \end{aligned}$$

3 that is, C together with the functors μ^C and η^C is an monoid object in **Cat**. □

4 **Corollary 4.2** *The followings are equivalent:*

- (1) a crossed R -module (C, R, ∂)
- (2) a categorical R -algebra
- (3) a monoid object in Cat
- (4) a simplicial R -algebra whose Moore complex is of lenght 1.

Proof (1) \iff (2) \iff (3) are clear from the Theorem 3.5 and Proposition 4.1. Now we will show that
 (3) \iff (4). We obtain a simplicial R -module by taking the nerve. This simplicial R -module is a simplicial
 R -algebra because the category is monoid object in Cat . Its Moore complex is

$$\dots 1 \longrightarrow 1 \longrightarrow C \longrightarrow R,$$

5 which is of lenght 1.

Suppose that there is a simplicial algebra **E** whose Moore complex is of lenght 1, that is

$$\dots 1 \longrightarrow 1 \longrightarrow \ker d_1 \longrightarrow E_0,$$

6 By choosing $C = E_1$ and $R = \text{image of } E_0 \text{ in } E_1$ by the degeneracy map and the structural morphisms $s = d_1$
 7 and $t = d_0$, we get a categorical R -algebra. From the relations between face and degeneracy maps, we have
 8 $se = id = te$. On the other hand, to prove $KersKert = \{0\}$ it is sufficient to see that for $x \in Kerd_1$ and
 9 $y \in Kerd_0$ the element $[s_0(x), s_0(y) - s_1(y)]$ of E_2 is in fact in $Kerd_1 \cap Kerd_2$ and its image by d_0 is $[x, y]$.
 10 As $Kerd_1 \cap Kerd_2 = 0$, we have $[Kerd_0, Kerd_1] = 0$.

11 So, from the Lemma 2.4, we have a categorical R -algebra and use of the previous equivalence gives the
 12 monoid object in Cat with $O = E_0$ and $M = E_1$. □

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