



# On Associated Curves of a Framed Curve in Euclidean 4-space

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## Abstract

In this paper, we define some new curves, called associated curves, for a singular curve in Euclidean 4-Space by using its framed frame. We give necessary and sufficient conditions for these associated curves to be helices. Furthermore, we introduce a new type of rectifying curve derived from the associated curve and give its characterization.

## 1. Introduction

In differential geometry, the theory of curves represents a significant field of study. Among the various types of curves, helices, slant helices, and rectifying curves are of particular interest due to their distinctive geometric properties. Additionally, associated curves derived from a given curve have been extensively investigated. Among these associated curves, the most extensively studied include Bertrand curve couples, Mannheim partner curves, spherical indicatrices, and involute-evolute curve couples (see [1–14]). There are numerous studies in the literature on Frenet curves, which are useful for investigating the geometric properties of regular curves. However, in recent years, curves with singular points, for which the Frenet frame cannot be constructed, have frequently appeared in various applications. Honda et al. introduced the concept of framed curves to explore the geometric properties of curves with singular points [15]. Framed curves are a natural generalization of Frenet curves. Following this, Honda et al. presented the idea of framed immersion [16]. For more details on the notion of framed curves and framed surfaces, one can refer to [17–25].

In 2003, Chen characterized rectifying curves as those whose position vectors lie within their rectifying planes. Considering these characterizations, he demonstrated that the position vector of a rectifying curve aligns with the Darboux vector, interpreting these curves kinematically as those whose position vectors indicate the instantaneous rotation axis at each point along the curve [3, 4]. After that, in 2004, İlarıslan and Nešović defined normal curves as those whose position vectors lie in the orthogonal complement of their tangent vector field in  $\mathbb{R}_1^3$  [26]. Then, Önder et al. introduced the notion of  $B_2$ -slant helices in  $\mathbb{E}^4$ , defining them as curves whose second binormal vector  $B_2$  maintains a constant angle with a fixed direction and outlined their fundamental properties [27]. After that, Choi and Kim introduced three new types of curves: principal (binormal)-direction curves, principal (binormal)-donor curves, and PD-rectifying curves. They provided characterizations of the general and slant helices in terms of their associated curves, and presented a valuable method for deriving general and slant helices from a planar curve. Additionally, a novel characterization of Bertrand curves was proposed, using the PD-rectifying curve [7]. Subsequently, Choi et al. extended this concept by defining principal (binormal)-directional and principal (binormal)-donor curves for Frenet curves in Minkowski space  $\mathbb{E}_1^3$  [8]. Recently, Macit et al. introduced  $W$ -direction curves,  $W$ -rectifying curves and  $V$ -direction curves in  $\mathbb{E}^3$ , as well as principal-direction curves,  $B_1$ -direction curves,  $B_2$ -direction curves, and  $B_2$ -rectifying curves in  $\mathbb{E}^4$  [11].

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In this paper, we define  $\eta_1$ -direction curve,  $\eta_2$ -direction curve and  $\eta_3$ -direction curve in  $\mathbb{E}^4$ . After that we give relationships between framed associated curves and slant helices. Finally we introduce  $\eta_3$ -rectifying curve and give characterization about this curve.

## 2. Preliminaries

Let  $\mathbb{R}^4$  be four-dimensional Euclidean space equipped with the standard inner product:

$$\langle x, y \rangle = \sum_{i=1}^4 x_i y_i,$$

where  $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$ . The norm of a vector  $x \in \mathbb{R}^4$  is given by  $\|x\| = \sqrt{\langle x, x \rangle}$ . The vector product in  $\mathbb{R}^4$  is defined by

$$x \times y \times z = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where  $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4), z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4$  and  $e_1, e_2, e_3, e_4$  are the canonical basis vectors of  $\mathbb{R}^4$ .

Theorem 2.1. Consider a unit-speed curve  $\beta : I \rightarrow \mathbb{R}^4$ . The Frenet frame vectors corresponding to this curve are given by the following expressions [28]:

$$T = \beta', N = \frac{\beta''}{\|\beta''\|}, B_2 = -\frac{\beta' \times \beta'' \times \beta'''}{\|\beta' \times \beta'' \times \beta'''\|}, B_1 = B_2 \times T \times N.$$

Theorem 2.2. Consider a unit-speed curve  $\beta : I \rightarrow \mathbb{R}^4$ . The curvatures corresponding to this curve are given by the following expressions [28]:

$$k_1 = \|\beta''\|, k_2 = \frac{\langle B_1, \beta'''\rangle}{k_1}, k_3 = \frac{\langle B_2, \beta^{(4)}\rangle}{k_1 k_2}.$$

Definition 2.3. A unit-speed curve  $\beta : I \rightarrow \mathbb{R}^4$  is classified as a general helix if its unit tangent vector makes a constant angle with a fixed direction [29].

We define a 6-dimensional smooth manifold as:

$$\Delta_3 = \left\{ \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \mid \langle \mu_i, \mu_j \rangle = \delta_{ij}, i, j = 1, 2, 3 \right\}.$$

A unit vector  $\nu$  is then defined by  $\nu = \mu_1 \times \mu_2 \times \mu_3$ , satisfying  $\det(\nu, \mu_1, \mu_2, \mu_3) = 1$  and  $(\nu, \mu) \in \Delta_4$ .

Definition 2.4. A map  $(\gamma, \mu) : I \rightarrow \mathbb{R}^4 \times \Delta_3$  is called a framed curve if  $\langle \gamma'(s), \mu_i(s) \rangle = 0$  for all  $s \in I$  and  $i = 1, 2, 3$ .  $\gamma : I \rightarrow \mathbb{R}^4$  is called as a framed base curve if there exists  $\mu : I \rightarrow \Delta_3$  such that  $(\gamma, \mu)$  is a framed curve, [?].

By following similar way as the curvatures of regular curve, we can define smooth function for framed curve. Given a framed base curve  $(\gamma, \mu)$  with an associated moving frame  $\{\nu(s), \mu(s)\}$ , the Frenet-Serret type formula is given by

$$\begin{bmatrix} \mu_1'(s) \\ \mu_2'(s) \\ \mu_3'(s) \\ \nu'(s) \end{bmatrix} = \begin{bmatrix} 0 & f(s) & g(s) & h(s) \\ -f(s) & 0 & j(s) & k(s) \\ -g(s) & -j(s) & 0 & l(s) \\ -h(s) & -k(s) & -l(s) & 0 \end{bmatrix} \begin{bmatrix} \mu_1(s) \\ \mu_2(s) \\ \mu_3(s) \\ \nu(s) \end{bmatrix},$$

where  $f(s), g(s), h(s), j(s), k(s), l(s)$  are smooth curvature functions. Moreover, there exists a smooth function  $\alpha(s)$  such that  $\nu'(s) = \alpha(s)\nu(s)$ .  $(f, g, h, j, k, l, \alpha)$  are called the framed curvatures of  $\gamma$ . A point  $s_0 \in I$  is a singular point if  $\alpha(s_0) = 0$ .

Theorem 2.5. Let  $(f, g, h, j, k, l, \alpha)$  be smooth functions. There exists a framed curve  $(\gamma, \mu)$  with these framed curvatures [?].

Theorem 2.6. Given framed curves  $(\gamma, \mu)$  and  $(\tilde{\gamma}, \tilde{\mu})$ , they are congruent as framed curves, if their curvatures coincide [?].

By using Euler angles  $\theta, \varphi, \psi$ , one can define a new orthonormal frame  $\eta = (\eta_1, \eta_2, \eta_3)$  by

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = A(\theta, \varphi, \psi) \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix},$$

where  $A(\theta, \varphi, \psi)$  is a rotation matrix depending on the Euler angles. It follows that the new vector  $\tilde{\nu} = \eta_1 \times \eta_2 \times \eta_3 = \nu$ , so the pair  $(\gamma, \eta)$  is still a framed curve.

Assume that the following relations exist between the Euler angles and the framed curvatures;

$$\frac{\tan \psi}{\cos \theta} = l \sin \varphi - k \cos \varphi$$

and

$$h = \cot \theta (l \cos \varphi + k \sin \varphi),$$

then, the adapted frame  $\{\nu(s), \eta_1(s), \eta_2(s), \eta_3(s)\}$  along  $\gamma(s)$  is given by

$$\begin{bmatrix} \nu'(s) \\ \eta_1'(s) \\ \eta_2'(s) \\ \eta_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & p(s) & 0 & 0 \\ -p(s) & 0 & q(s) & 0 \\ 0 & -q(s) & 0 & r(s) \\ 0 & 0 & -r(s) & 0 \end{bmatrix} \begin{bmatrix} \nu(s) \\ \eta_1(s) \\ \eta_2(s) \\ \eta_3(s) \end{bmatrix}, \quad (2.1)$$

where  $(p(s), q(s), r(s), \alpha(s))$  are framed curvature of  $\gamma(s)$ , which relate to the original ones as follows:

$$\begin{aligned} p &= -h \sec \theta \sec \psi, \\ q &= -(j - \varphi') \sin \theta - \psi', \\ r &= \frac{\cos \theta}{\cos \psi} (j - \varphi'), \\ f &= -\sin \varphi (\theta' - r \sin \psi), \\ g &= -\cos \varphi (\theta' - r \sin \psi), \\ j &= r \frac{\cos \psi}{\cos \theta} + \theta'. \end{aligned}$$

The vectors  $\nu, \eta_1, \eta_2, \eta_3$  are referred to as the generalized tangent, generalized principal normal, generalized first binormal, and generalized second binormal vectors of framed curve, respectively.

**Definition 2.7.** Let  $(\gamma, \eta) : I \rightarrow \mathbb{R}^4 \times \Delta_3$  be framed curve with non-zero framed curvatures and  $\{\nu, \eta_1, \eta_2, \eta_3\}$  be its Framed frame.

$\gamma$  is called framed slant helix if its generalized normal vector makes a constant angle with a fixed direction.

$\gamma$  is called framed  $\eta_2$ -slant helix if its generalized first binormal vector makes a constant angle with a fixed direction.

$\gamma$  is called framed  $\eta_3$ -slant helix if its generalized second binormal vector makes a constant angle with a fixed direction [20].

### 3. Framed Associated Curves

In this section, we characterize the framed associated curves with non-zero framed curvatures in  $\mathbb{R}^4$ .

**Definition 3.1.** Let  $(\gamma, \eta) : I \rightarrow \mathbb{R}^4 \times \Delta_3$  be framed curve with non-zero framed curvatures and  $\{\nu, \eta_1, \eta_2, \eta_3\}$  be its Framed frame.

An integral curve of the generalized principal normal vector field of  $\gamma$  is called the  $\eta_1$ -direction curve of  $\gamma$ .

An integral curve of the generalized first binormal vector field of  $\gamma$  is called the  $\eta_2$ -direction curve of  $\gamma$ .

An integral curve of the generalized second binormal vector field of  $\gamma$  is called the  $\eta_3$ -direction curve of  $\gamma$ .

$\eta_1, \eta_2$  and  $\eta_3$  never vanish at any point. So, associated curves of framed curve are regular curves.

**Theorem 3.2.** Let  $(\gamma, \eta) : I \rightarrow \mathbb{R}^4 \times \Delta_3$  be framed curve with non-zero framed curvatures and let  $\bar{\gamma}$  denote the  $\eta_1$ -direction curve of  $\gamma$ . Then, the curvatures of  $\bar{\gamma}$  are given by:

$$\begin{aligned} \bar{k}_1 &= \sqrt{p^2 + q^2}, \\ \bar{k}_2 &= \frac{\sqrt{p^2 q^2 r^2 + q^4 r^2 + (qp' - pq')^2}}{p^2 + q^2}, \\ \bar{k}_3 &= \frac{\sqrt{p^2 + q^2} \left( q(-p'(2rq' + qr') + qrp'') + p(q^2 r^3 + 2rq'^2 + q(q'r' - rq'')) \right)}{p^2 q^2 r^2 + q^4 r^2 + (qp' - pq')^2}. \end{aligned}$$

**Proof.** Let  $\{\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2, \bar{k}_1, \bar{k}_2, \bar{k}_3\}$  denote the Frenet apparatus of the curve  $\bar{\gamma}$ . By the definition of the  $\eta_1$ -direction curve, we can write  $\eta_1 = \bar{\gamma}' = \bar{T}$ . By differentiating the equation, we get  $\bar{T}' = \eta_1' = -p\nu + q\eta_2$ . Therefore, the first curvature  $\bar{k}_1$  of  $\bar{\gamma}$  can be expressed as  $\bar{k}_1 = \sqrt{p^2 + q^2}$ . Using Theorem 2.1, the Frenet frame vector field of  $\bar{\gamma}$  can be computed as

$$\begin{aligned} \bar{N} &= \frac{1}{\sqrt{p^2 + q^2}} (\eta_2 q - p\nu), \\ \bar{B}_1 &= \frac{(-q^2 p' + pqq')\nu - (pqp' - p^2 q')\eta_2 + (p^2 qr + q^3 r)\eta_3}{\sqrt{p^2 + q^2} \sqrt{p^2 q^2 r^2 + q^4 r^2 + (qp' - pq')^2}}, \\ \bar{B}_2 &= -\frac{(q^2 r)\nu + (pqr)\eta_2 + (qp' - pq')\eta_3}{\sqrt{p^2 q^2 r^2 + q^4 r^2 + (qp' + pq')^2}}. \end{aligned}$$

Therefore, by applying Theorem 2.2, other curvatures of  $\bar{\gamma}$  can be computed as

$$\begin{aligned}\bar{k}_2 &= \frac{\sqrt{p^2q^2r^2 + q^4r^2 + (qp' - pq')^2}}{p^2 + q^2}, \\ \bar{k}_3 &= \frac{\sqrt{p^2 + q^2} \left( q(-p'(2rq' + qr') + qrp'') + p \left( q^2r^3 + 2rq'^2 + q(q'r' - rq'') \right) \right)}{p^2q^2r^2 + q^4r^2 + (qp' - pq')^2}.\end{aligned}$$

□

Theorem 3.3. Let  $(\gamma, \eta) : I \rightarrow \mathbb{R}^4 \times \Delta_3$  be framed curve with non-zero framed curvatures and let  $\hat{\gamma}$  denote the  $\eta_2$ -direction curve of  $\gamma$ . Then, the curvatures of  $\hat{\gamma}$  are given by:

$$\begin{aligned}\hat{k}_1 &= \sqrt{q^2 + r^2}, \\ \hat{k}_2 &= \frac{\sqrt{p^2q^2(q^2 + r^2) + (rq' - qr')^2}}{q^2 + r^2}, \\ \hat{k}_3 &= \frac{\sqrt{q^2 + r^2} \left( p^3q^2r + qp'(rq' - qr') + p \left( r(2q'^2 - qq'') + q(2q'r' + qr'') \right) \right)}{p^2q^2(q^2 + r^2) + (rq' - qr')^2}.\end{aligned}$$

Proof. Let  $\{\hat{T}, \hat{N}, \hat{B}_1, \hat{B}_2, \hat{k}_1, \hat{k}_2, \hat{k}_3\}$  denote the Frenet apparatus of the curve  $\hat{\gamma}$ . According to definition of the  $\eta_2$ -direction curve, we can write  $\eta_1 = \hat{\gamma}' = \hat{T}$ . By differentiating the equation, we get  $\hat{T}' = \eta_2' = -q\eta_1 + r\eta_3$ . Therefore, the first curvature  $\hat{k}_1$  of  $\hat{\gamma}$  can be expressed as  $\hat{k}_1 = \sqrt{q^2 + r^2}$ . Using Theorem 2.1, the Frenet frame vector field of  $\hat{\gamma}$  can be computed as

$$\begin{aligned}\hat{N} &= \frac{-q\eta_1 + r\eta_3}{\sqrt{q^2 + r^2}}, \\ \hat{B}_1 &= \frac{(pq^3 + pqr^2)v - (r^2q' + qrr')\eta_1 - (qrq' - q^2r')\eta_3}{\sqrt{q^2 + r^2} \sqrt{(p^2q^4 + p^2q^2r^2 + (-rq' + qr')^2)}}, \\ \hat{B}_2 &= -\frac{(-rq' + qr')v - pqr\eta_1 - pq^2\eta_3}{\sqrt{(p^2q^4 + p^2q^2r^2 + (-rq' + qr')^2)}}.\end{aligned}$$

Therefore, by applying Theorem 2.2, other curvatures of  $\hat{\gamma}$  can be computed as

$$\begin{aligned}\hat{k}_2 &= \frac{\sqrt{p^2q^2(q^2 + r^2) + (rq' - qr')^2}}{q^2 + r^2}, \\ \hat{k}_3 &= \frac{\sqrt{q^2 + r^2} \left( p^3q^2r + qp'(rq' - qr') + p \left( r(2q'^2 - qq'') + q(2q'r' + qr'') \right) \right)}{p^2q^2(q^2 + r^2) + (rq' - qr')^2}.\end{aligned}$$

□

Theorem 3.4. Let  $(\gamma, \eta) : I \rightarrow \mathbb{R}^4 \times \Delta_3$  be framed curve with non-zero framed curvatures and let  $\tilde{\gamma}$  denote the  $\eta_3$ -direction curve of  $\gamma$ . Then, the curvatures of  $\tilde{\gamma}$  are given by:

$$\begin{aligned}\tilde{k}_1 &= \text{sgn}(r)r, \\ \tilde{k}_2 &= \text{sgn}(q)q, \\ \tilde{k}_3 &= \text{sgn}(r)p.\end{aligned}$$

Proof. Let  $\{\tilde{T}, \tilde{N}, \tilde{B}_1, \tilde{B}_2, \tilde{k}_1, \tilde{k}_2, \tilde{k}_3\}$  denote the Frenet apparatus of the curve  $\tilde{\gamma}$ . Since  $\tilde{\gamma}' = \tilde{T} = \eta_3$ , it follows that  $\tilde{T}' = \eta_3' = -r\eta_2$ . Thus  $\tilde{k}_1(t) = |r(t)|$ . By using Theorem 1 2.1, the remaining Frenet frame vector field of  $\tilde{\gamma}$  can be computed as

$$\begin{aligned}\tilde{N} &= -\text{sgn}(r)\eta_2, \\ \tilde{B}_1 &= \text{sgn}(q)\text{sgn}(r)\eta_1, \\ \tilde{B}_2 &= -\text{sgn}(q)v\end{aligned}$$

and using Theorem 2.2, the remaining curvatures of  $\tilde{\gamma}$  are

$$\begin{aligned}\tilde{k}_2 &= \text{sgn}(q)q, \\ \tilde{k}_3 &= \text{sgn}(r)p.\end{aligned}$$

□

Theorem 3.5. Let  $(\gamma, \eta) : I \rightarrow \mathbb{R}^4 \times \Delta_3$  be framed curve.  $\gamma$  is a framed slant helix if and only if  $\eta_1$ -direction curve  $\tilde{\gamma}$  is a general helix.

Proof. Let  $\{v, \eta_1, \eta_2, \eta_3\}$  be the framed frame of  $\gamma$ . From the definition of the  $\eta_1$ -direction curve, it follows  $\tilde{\gamma}'(t) = \bar{T}(t) = \eta_1(t)$ . Therefore  $\gamma$  is a framed slant helix iff  $\langle \eta_1, u \rangle = \cos \phi$  where  $\phi$  is constant and  $u$  is constant unit vector field iff  $\langle \bar{T}, u \rangle = \cos \phi$  iff  $\tilde{\gamma}$  is a general helix  $\square$

Theorem 3.6. Let  $(\gamma, \eta) : I \rightarrow \mathbb{R}^4 \times \Delta_3$  be framed curve.  $\gamma$  is a framed  $\eta_2$ -slant helix if and only if  $\eta_2$ -direction curve  $\hat{\gamma}$  is a general helix.

Proof. Let  $\{v, \eta_1, \eta_2, \eta_3\}$  be the framed frame of  $\gamma$ . From the definition of the  $\eta_2$ -direction curve, it follows  $\hat{\gamma}'(t) = \hat{T}(t) = \eta_3(t)$ . Therefore  $\gamma$  is a framed  $\eta_2$ -slant helix iff  $\langle \eta_2, v \rangle = \cos \phi$  where  $\phi$  is constant and  $v$  is constant unit vector field iff  $\langle \hat{T}, v \rangle = \cos \phi$  iff  $\hat{\gamma}$  is a general helix  $\square$

Theorem 3.7. Let  $(\gamma, \eta) : I \rightarrow \mathbb{R}^4 \times \Delta_3$  be framed curve.  $\gamma$  is a framed  $\eta_3$ -slant helix if and only if  $\eta_3$ -direction curve  $\tilde{\gamma}$  is a general helix.

Proof. Let  $\{v, \eta_1, \eta_2, \eta_3\}$  be the framed frame of  $\gamma$ . From the definition of the  $\eta_3$ -direction curve, it follows  $\tilde{\gamma}'(t) = \tilde{T}(t) = \eta_3(t)$ . Therefore  $\gamma$  is a framed  $\eta_3$ -slant helix iff  $\langle \eta_3, v \rangle = \cos \phi$  where  $\phi$  is constant and  $v$  is constant unit vector field iff  $\langle \tilde{T}, v \rangle = \cos \phi$  iff  $\tilde{\gamma}$  is a general helix  $\square$

Definition 3.8. Let  $(\gamma, \eta)$  denote a Framed curve with framed frame  $\{v, \eta_1, \eta_2, \eta_3\}$  and non-zero  $\{p, q, r\}$  and  $\tilde{\gamma}$  denotes the framed  $\eta_3$ -direction curve of  $\gamma$ .  $\tilde{\gamma}$  is defined as a  $\eta_3$ -rectifying curve if its position vector always lies in the orthogonal complement of the generalized normal vector of  $\gamma$ . This means that the position vector can be written as

$$\tilde{\gamma} = \lambda_1 \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3 \tag{3.1}$$

where  $\lambda_1, \lambda_2, \lambda_3$  are differentiable functions.

Theorem 3.9. Let  $(\gamma, \eta)$  denote a Framed curve with and non-zero curvatures.  $\tilde{\gamma}$  is congruent to a  $\eta_3$ -rectifying curve if and only if there exists a constant  $c$  such that the following differential equation holds

$$-sgn(r) \left( \frac{sgn(q) c \tilde{k}_3}{sgn(r) \tilde{k}_2} \right)' + \left( s - sgn(q) c \int \frac{\tilde{k}_1 \tilde{k}_3}{\tilde{k}_2} ds \right) \tilde{k}_1 = 0. \tag{3.2}$$

Proof. Assume that  $\tilde{\gamma}$  is congruent to a  $\eta_3$ -rectifying. Then by definition,  $\tilde{\gamma}$  is the  $\eta_3$ -direction curve of the curve  $\gamma$ . Therefore, according to Theorem 3.4, we get  $v = -sgn(q) \tilde{B}_2$ ,  $\eta_2 = -sgn(r) \tilde{N}$ ,  $\eta_3 = \tilde{T}$ . By substituting these expression into the equation (3.1) and differentiating, we obtain the following equations

$$\begin{aligned} \lambda_3' + sgn(r) \lambda_2 \tilde{k}_1 &= 1, \\ -sgn(r) \lambda_2' + \lambda_3 \tilde{k}_1 &= 0 \\ sgn(q) \lambda_1 \tilde{k}_3 - sgn(r) \lambda_2 \tilde{k}_2 &= 0 \\ sgn(q) \lambda_1' &= 0. \end{aligned}$$

From the last equations, it follows that  $\lambda_1 = c$ , where  $c \in \mathbb{R}$ . Substituting this into the third equation leads to  $\lambda_2 = \frac{sgn(q) c \tilde{k}_3}{sgn(r) \tilde{k}_2}$ .

Furthermore, using this in the first equation yields  $\lambda_3 = s - sgn(q) c \int \frac{\tilde{k}_1 \tilde{k}_3}{\tilde{k}_2} ds$ . Finally, substituting these results into the second equation gives

$$-sgn(r) \left( \frac{sgn(q) c \tilde{k}_3}{sgn(r) \tilde{k}_2} \right)' + \left( s - sgn(q) c \int \frac{\tilde{k}_1 \tilde{k}_3}{\tilde{k}_2} ds \right) \tilde{k}_1 = 0.$$

Conversely, assume that the curvatures of  $\tilde{\gamma}$  satisfy the condition given in the equation (3.2). Let us consider the vector

$$X = \tilde{\gamma} + sgn(q) c \tilde{B}_2 + \frac{sgn(q) c \tilde{k}_3}{\tilde{k}_2} \tilde{N} + \left( s - sgn(q) c \int \frac{\tilde{k}_1 \tilde{k}_3}{\tilde{k}_2} ds \right) \tilde{T}.$$

By differentiating, we find  $X' = 0$ , which implies that  $X$  is constant vector field. Thus  $\tilde{\gamma}$  is congruent to a  $\eta_3$ -rectifying curve.  $\square$

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