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On the classification of framed rectifying curves in Euclidean space

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There are many studies on regular rectifying curves in classical differential geometry, and important results have been obtained. However, these studies are limited for a smooth curve with singular points. To examine such curves and surfaces, the concept of framed curve, which is the general form of regular and Legendre curves, is used. Framed curves are defined as curves that have a moving frame with singular points in Euclidean space. We investigate framed rectifying curves via the dilation of framed curves on S^2 in $\mathbb{R}^3$. Moreover, the result of dilation of framed curves is the framed rectifying curve or not. We give some classifications for the dilation of framed curves. Finally, we give some related examples with their figures.

KEYWORDS

dilation of framed curves, framed curves, framed rectifying curves, framed spherical curves

MSC CLASSIFICATION

53A04; 58K05; 58K30

1 | INTRODUCTION

The curves theory and some special curves are important in classical differential geometry. There have been many studies on non-regular curves in recent years, and they have contributed to singularity theory. Framed curves are one of them. Framed curves are defined as curves that have a moving frame with singular points in an Euclidean space. Framed curves are important in literature as they are generalizations of both regular curves and Legendre curves. Also, since framed curves may have singular points, they contribute to the singularity theory. There are many articles concerning the framed curves and Legendre curves (for instance, previous works[1–7]).

The concept of the rectifying curve, the position vector is always known as the curves lying in the rectifying plane.[8] Some characterizations and classifications about regular rectifying curves are investigate in various articles.[81–17] In Wang et al,[6] the concept of framed rectifying curve including both regular and non-regular rectifying curves is introduced. Also, similar to regular curves, framed rectifying curves arising via dilation of framed spherical curves are introduced. However, in Deshmukh et al,[12] it was explained that this dilation cannot always be a rectifying curve for regular curves.

Inspired by these situations, in this article, we give some classifications for the dilation of framed curves. In addition, this classification, which can be used for both regular and non-regular rectifying curves, is given with some conditions. In the final part, interesting examples and figures supporting the theory are given.

2 | BASIC DEFINITIONS AND FRAMED CURVES

Let $\mathbb{R}^3$ be the three-dimensional Euclidean space equipped with the inner product $\langle a, b\rangle = a_1b_1 + a_2b_2 + a_3b_3$ where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$. The norm of a is defined by $\|a\| = \sqrt{\langle a, a\rangle}$. The vector product is defined of a and b by

$$a \wedge b = (a_2b_3 - a_3b_2, a_1b_3 - a_1b_3, a_1b_2 - a_2b_1),$$

on $\mathbb{R}^3$.

A framed curve is defined as a smooth curve with singular points, in detail see Honda and Takahashi.[1] Now consider any $\gamma : I \to \mathbb{R}^3$ curve that can have a singular point. The set

$$\Delta_2 = \{\mu = (\mu_1, \mu_2) \in \mathbb{R}^3 \times \mathbb{R}^3 | \langle \mu_i, \mu_j\rangle = \delta_{ij}, i, j = 1, 2\},$$

has a smooth three-dimensional manifold structure. This structure is defined as the “Stiefel manifold” in manifold theory. Let $\mu = (\mu_1, \mu_2) \in \Delta_2$. A unit vector is defined by $\nu = \mu_1 \times \mu_2$.

After this preparation, the definition of the framed curve can be given:

Definition 1 (**Framed curve**). $(\gamma, \mu) : I \to \mathbb{R}^3 \times \Delta_2$ is called a framed curve if $\langle \gamma'(s), \mu_j(s)\rangle = 0$ for each $s \in I$ and $j = 1, 2$. Moreover, $\gamma : I \to \mathbb{R}^3$ is called a framed base curve if there exists $\mu : I \to \Delta_2$ such that (γ, μ) is a framed curve.[1]

Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve and $\nu = \mu_1 \times \mu_2$. The derivative formula is given by

$$\begin{pmatrix} \nu'(s) \\ \mu_1'(s) \\ \mu_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & -m(s) & -n(s) \\ m(s) & 0 & l(s) \\ n(s) & -l(s) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \mu_1(s) \\ \mu_2(s) \end{pmatrix},$$

where $l(s) = \langle \mu_1'(s), \mu_2(s)\rangle$, $m(s) = \langle \nu(s), \mu_1'(s)\rangle$ and $n(s) = \langle \nu(s), \mu_2'(s))\rangle$. Obviously, if $m(s) = n(s) = 0$, then $\nu'(s) = 0$. Throughout this study, we accept the case $\nu'(s) \neq 0$. Moreover, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that

$$\gamma'(s) = \alpha(s)\nu(s).$$

As can be seen, although the framed base curve has singular points, the frame can be defined since μ can be defined. Hence, the α function is important to characterize singular points. In addition, s_0 is a singular point of γ if and only if $\alpha(s_0) = 0$. Clearly, if $\alpha(s) = 0$ for each $s \in I$, then $\gamma(s)$ is a point.[1] Consequently, $(l(s), m(s), n(s), \alpha(s))$ are called the framed curvature of γ.

Also, as in regular curves, many adapted frames can be obtained for the general frame. These frames are available in the literature such as rotated, reflected, and generalized, etc. It is used adapted frames for simplicity in addition the general frame. In this study, an adapted frame which is a more general form of the Frenet-Serret frame is used. Generalized vectors are obtained with the help of an adapted frame in Wang et al[6] and can be obtained as follows:

$(\overline{\mu_1}(s), \overline{\mu_2}(s)) \in \Delta_2$ is defined by

$$\begin{pmatrix} \overline{\mu_1}(s) \\ \overline{\mu_2}(s) \end{pmatrix} = \begin{pmatrix} \cos\varphi(s) & -\sin\varphi(s) \\ \sin\varphi(s) & \cos\varphi(s) \end{pmatrix} \begin{pmatrix} \mu_1(s) \\ \mu_2(s) \end{pmatrix}.$$

Then, $(\gamma, \overline{\mu_1}, \overline{\mu_2}) : I \to \mathbb{R}^3 \times \Delta_2$ also a framed curve and

$$\bar{\nu}(s) = \nu(s). \tag{2.1}$$

Let's assume that $\varphi : I \to \mathbb{R}$ is a smooth function that satisfies $m(s)\sin\varphi(s) + n(s)\cos\varphi(s) = 0$. Therefore, this equation can be rearranged with the $m(s) = -p(s)\cos\varphi(s)$ and $n(s) = p(s)\sin\varphi(s)$ equations.[6] Hence, we get

$$\begin{aligned} \nu'(s) &= -m(s)\mu_1(s) - n(s)\mu_2(s), \\ &= p(s)[\cos\varphi(s)\mu_1(s) - \sin\varphi(s)\mu_2(s)], \\ &= p(s)\overline{\mu_1}(s), \\ \overline{\mu_1}'(s) &= -(l(s) - \varphi'(s))\sin\varphi(s)\mu_1(s) \\ &\quad + (l(s) - \varphi'(s))\cos\varphi(s)\mu_2(s) + (m(s)\cos\varphi(s) - n(s)\sin\varphi(s))\nu(s), \\ &= -p(s)\nu(s) + (l(s) - \varphi'(s))\overline{\mu_2}(s), \\ \overline{\mu_2}'(s) &= -(l(s) - \varphi'(s))\cos\varphi(s)\mu_1(s) \\ &\quad + (l(s) - \varphi'(s))\sin\varphi(s)\mu_2(s) + (m(s)\sin\varphi(s) + n(s)\cos\varphi(s))\nu(s), \\ &= -(l(s) - \varphi'(s)\overline{\mu_1}(s). \end{aligned} \tag{2.2}$$

$\{\nu, \overline{\mu_1}, \overline{\mu_2}\}$ is called an adapted frame along the framed base curve $\gamma(s)$. Consequently, according to 2.2, the derivative formula is given by

$$\begin{pmatrix} \nu'(s) \\ \overline{\mu}_1'(s) \\ \overline{\mu}_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & p(s) & 0 \\ -p(s) & 0 & q(s) \\ 0 & -q(s) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \overline{\mu_1}(s) \\ \overline{\mu_2}(s) \end{pmatrix}.$$

The vectors $\nu(s), \overline{\mu_1}(s), \overline{\mu_2}(s)$ are called the generalized (tangent, principle normal, and binormal) vector of the framed curve, respectively, where

$$p(s) = \|\nu'(s)\| > 0$$

and

$$q(s) = l(s) - \theta'(s).$$

The functions $(p(s), q(s), \alpha(s))$ are defined to as the framed curvature of $\gamma(s)$.[6]

3 | FRAMED RECTIFYING CURVES VIA DILATION OF FRAMED BASE CURVES ON S^2

Definition 2. For the $(\gamma, \overline{\mu_1}, \overline{\mu_2}) : I \to \mathbb{R}^3 \times \Delta_2$ framed curve, if the position vector of the γ framed base curve satisfies the $\gamma(s) = \delta(s)\nu(s) + \epsilon(s)\overline{\mu_2}(s)$ equation for some functions $\delta(s)$ and $\epsilon(s)$, it is called a framed rectifying curve.[6]

Before introducing the notion of framed rectifying curve via dilation of framed spherical curves, the following definition can be given:

Definition 3. If the framed base curve γ of framed curve $(\gamma, \overline{\mu_1}, \overline{\mu_2}) : I \to \mathbb{R}^3 \times \Delta_2$ is a curve on S^2, it is called a framed spherical curve.[6]

A classification for regular rectifying curves is described in Chen and Dillen,[9] but it is not sufficient for the non-regular case. Inspired by this, in Wang et al,[6] the classification of framed curves is given by Wang, Pei, and Gao in Theorem 4. Hence, in Wang et al,[6] a new classification was introduced, which is important since it is used for both regular and non-regular rectifying curves:

Theorem 1. *Let $(\gamma, \overline{\mu_1}, \overline{\mu_2}) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve with $p(s) > 0$. Then γ is a framed rectifying curve if and only if*

$$\gamma(s) = \rho\sqrt{\left(\tan^2\left(\int |\omega'(s)|ds + c\right) + 1\right)}\omega(s), \tag{3.1}$$

where $\omega(s)$ is a framed curve on S^2 and c is a constant, ρ is a positive number.[6]

Remark 1. Framed rectifying curves is given by the dilated curves equation (3.1) with constant c and $\rho > 0$ where $\omega(s)$ is a framed curve on S^2. This equation can be written in the form $\gamma(s) = \rho \sec\left(\int |\omega'(s)|ds + c\right)\omega(s)$. However, for regular case, Deshmukh, Chen and Alshammari showed that the dilation obtained with every spherical curve cannot be a rectifying curve.[12] Since, framed curves are generalized in regular curves, if the arc of the great circle $\omega(s) = (\sin s, 0, \cos s)$ given by using Equation (3.1),

$$\gamma(s) = \rho\left(\frac{\sin s}{\cos(s+c)}, 0, \frac{\cos s}{\cos(s+c)}\right). \tag{3.2}$$

Taking the derivative of Equation (3.2), we get

$$\gamma'(s) = \rho\frac{1}{\cos^2(s+c)}\left(\sin(s+c)\sin s + \cos(s+c)\cos s, 0, \sin(s+c)\cos s - \cos(s+c)\sin s\right). \tag{3.3}$$

Since $\gamma'(s) = \alpha_\gamma(s)\nu_\gamma(s)$, we have

$$\alpha_\gamma(s) = \rho\frac{1}{\cos^2(s+c)}$$

and

$$\nu_\gamma(s) = \left(\sin(s+c)\sin s + \cos(s+c)\cos s, 0, \sin(s+c)\cos s - \cos(s+c)\sin s\right). \tag{3.4}$$

By editing the equation (3.4), we get

$$\nu_\gamma(s) = (\cos c, 0, \sin c). \tag{3.5}$$

Consequently, we have $\nu'_\gamma(s) = 0$. Hence, it contradicts our acceptance and on the other hand definition of framed rectifying curve requires that its curvature is positive. That is, γ cannot be a framed rectifying curve. Consequently, some properties will be given for this dilation.

Proposition 1. *Let $\omega(s)$ framed curve in S^2 for $\forall s \in I$. Then, $\{\omega, \nu_\omega, \omega \wedge \nu_\omega\}$ is an orthonormal system. Then we have,*

$$p_\omega = \sqrt{\alpha_\omega{}^2 + h^2}$$

where framed curvature p_ω, α_ω and $h = \langle \omega \wedge \nu_\omega, \nu_\omega{}' \rangle$.

Proof. Let $\omega(s)$ framed curve in unit sphere S^2. Therefore, we have $\langle \omega(s), \omega(s) \rangle = 1$. By taking the derivative of last equation and since ω is a framed curve, we have:

$$\alpha_\omega \langle \omega, \nu_\omega \rangle = 0. \tag{3.6}$$

Hence, if $\alpha_\omega \langle \omega, \nu_\omega \rangle = 0$ for each $s \in I$, it is either $\alpha_\omega = 0$ or $\langle \omega, \nu_\omega \rangle = 0$. If $\alpha_\omega = 0$ for each $s \in I$, then ω is a point. Hence, since is a framed curve on S^2, $\langle \omega, \nu_\omega \rangle = 0$ for each $s \in I$. Therefore, $\{\omega, \nu_\omega, \omega \wedge \nu_\omega\}$ is an orthonormal system. Let us suppose that $h = \langle \omega \wedge \nu_\omega, \nu_\omega{}' \rangle$. Consequently, we get

$$\nu_\omega{}' = (-\alpha_\omega)\omega + h(\omega \wedge \nu_\omega).$$

Since $\nu_\omega{}' = p_\omega \overline{\mu_1}_\omega$, we have

$$\overline{\mu_1}_\omega = -\frac{\alpha_\omega}{p_\omega}\omega + \frac{h}{p_\omega}\omega \wedge \nu_\omega,$$

$$p_\omega = \sqrt{\alpha_\omega{}^2 + h^2}.$$

□

Remark 2. If we consider C^∞ category, then the statement “functions $f(s)g(s) = 0$ for all $s \in I$, then $f(s) = 0$ for all $s \in I$ or $g(s) = 0$ for all $s \in I$” is wrong. Then there exist counter-examples. Hence, let's assume that f and g functions as real analytic functions. Similarly, in Proposition 1, α_ω and $\langle \omega, \nu_\omega \rangle$ are accepted as real analytical function.

Example 1. The spherical nephroid $\omega : \mathbb{R} \to S^2 \subset \mathbb{R}^3$ is defined by

$$\omega(s) = \frac{1}{4}\left(3\cos s - \cos 3s, 3\sin s - \sin 3s, 2\sqrt{3}\cos s\right).$$

See Figures 1 and 2. Then,

$$\nu_\omega = \frac{1}{2}\left(\sqrt{3}\cos 2s, \sqrt{3}\sin 2s, -1\right),$$

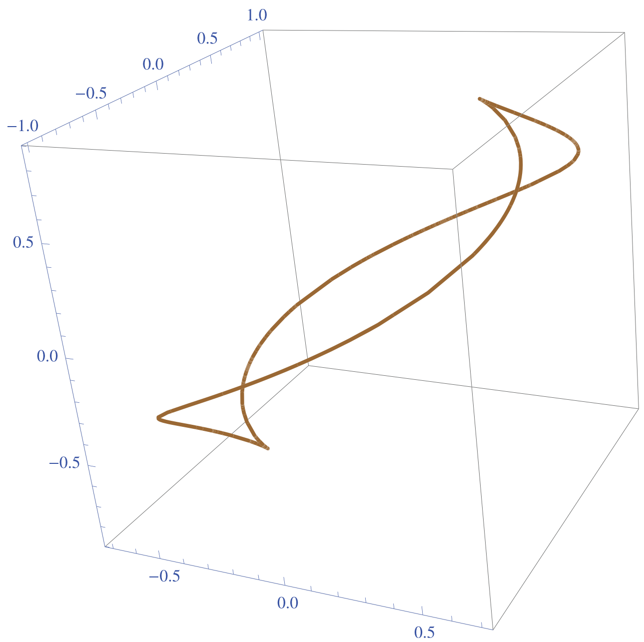

FIGURE 1 Spherical nephroid [Colour figure can be viewed at wileyonlinelibrary.com]

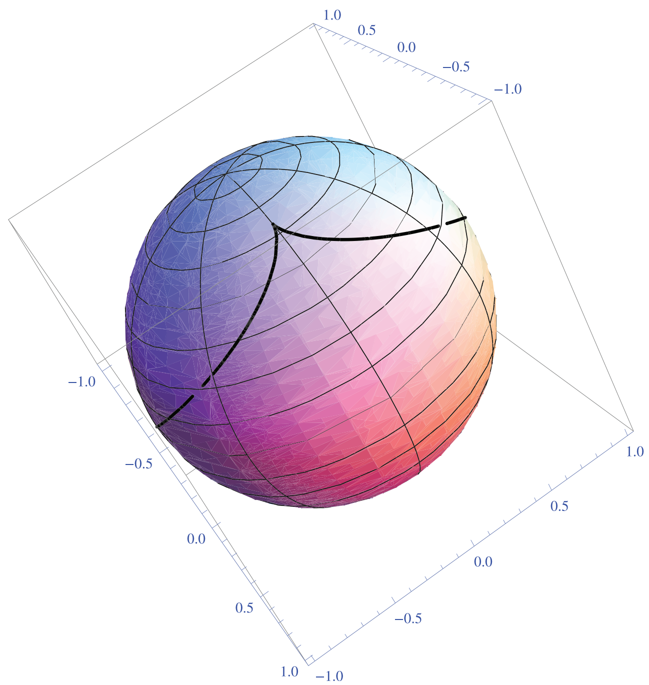

FIGURE 2 Spherical nephroid on S^2 [Colour figure can be viewed at wileyonlinelibrary.com]

and

$$\alpha_\omega = \sqrt{3}\sin s.$$

By necessary calculations, we have $\langle \omega, \nu_\omega \rangle = 0$. On the other hand, we get

$$\omega \wedge \nu_\omega = \left(-\frac{1}{2}\sin s - \sin s\cos^2 s, \cos s - \sin^2 s\cos s, -\frac{\sqrt{3}}{2}\sin s \right),$$

$$\nu'_\omega = \left(-\sqrt{3}\sin 2s, \sqrt{3}\cos 2s, 0 \right).$$

Since $h = \langle \omega \wedge \nu_\omega, \nu'_\omega \rangle$, we get

$$h = \sqrt{3}\cos s.$$

Consequently, we have

$$p_\omega = \sqrt{\alpha_\omega{}^2 + h^2} = \sqrt{3}.$$

Theorem 2. *Let $\gamma(s) = \rho\sqrt{\left(\tan^2\left(\int|\omega'(s)|ds + c\right) + 1\right)}\omega(s)$ be a dilation of γ where $\omega(s)$ be a framed curve on S^2. Then we have*

$$p_\gamma(s) = \cos\left(\int |\omega'(s)|ds + c\right)\sqrt{|p_\omega{}^2 - \alpha_\omega{}^2|},$$

$$q_\gamma(s) = \sin\left(\int |\omega'(s)|ds + c\right)\sqrt{|p_\omega{}^2 - \alpha_\omega{}^2|},$$

$$\alpha_\gamma(s) = \rho\sec^2\left(\int |\omega'(s)|ds + c\right)\alpha_\omega(s),$$

$$\nu_\gamma(s) = \sin\left(\int |\omega'(s)|ds + c\right)\omega(s) + \cos\left(\int |\omega'(s)|ds + c\right)\nu_\omega(s),$$

$$\overline{\mu_1}_\gamma(s) = \omega(s) \wedge \nu_\omega(s),$$

$$\overline{\mu_2}_\gamma(s) = \cos\left(\int |\omega'(s)|ds + c\right)\omega(s) - \sin\left(\int |\omega'(s)|ds + c\right)\nu_\omega(s)$$

where p_ω and $\alpha_\omega > 0$ is the framed curvature of framed curve and $\{p_\gamma, q_\gamma, \alpha_\gamma, \nu_\gamma, \overline{\mu_1}_\gamma, \overline{\mu_2}_\gamma\}$ is the framed apparatus of γ.

Proof. Let the expansion of the γ framed curve be in the form of

$$\gamma(s) = \rho\sec\left(\int |\omega'(s)|ds + c\right)\omega(s) \tag{3.7}$$

where ρ is a positive number, $\omega(s)$ is a framed curve on S^2. By differentiating (3.7), we get

$$\gamma'(s) = \frac{\rho\sin\left(\int|\omega'(s)|ds + c\right)}{\cos^2\left(\int|\omega'(s)|ds + c\right)}|\omega'(s)|\omega(s) + \frac{\rho}{\cos\left(\int|\omega'(s)|ds + c\right)}\omega'(s). \tag{3.8}$$

Since γ and ω are framed curves, we have $\gamma'(s) = \alpha_\gamma(s)\nu_\gamma(s)$ and $\omega'(s) = \alpha_\omega(s)\nu_\omega(s)$. Consequently, from (3.8), we get

$$\alpha_\gamma(s)\nu_\gamma(s) = \frac{\rho\sin\left(\int|\omega'(s)|ds + c\right)}{\cos^2\left(\int|\omega'(s)|ds + c\right)}|\alpha_\omega(s)|\omega(s) + \frac{\rho}{\cos\left(\int|\omega'(s)|ds + c\right)}\alpha_\omega(s)\nu_\omega(s). \tag{3.9}$$

Suppose that $\alpha_\omega(s) > 0$. Therefore, according to (3.9), we get

$$\alpha_\gamma(s) = \rho\sec^2\left(\int|\omega'(s)|ds + c\right)\alpha_\omega(s), \tag{3.10}$$

$$\nu_\gamma(s) = \sin\left(\int|\omega'(s)|ds + c\right)\omega(s) + \cos\left(\int|\omega'(s)|ds + c\right)\nu_\omega(s). \tag{3.11}$$

By differentiating (3.11), we get

$$p_\gamma(s)\overline{\mu_1}_\gamma(s) = \cos\left(\int|\omega'(s)|ds + c\right)(h\omega \wedge \nu_\omega) \tag{3.12}$$

where $h = \langle\omega \wedge \nu_\omega, \nu_\omega{}'\rangle$. Consequently, from (3.12), we have

$$p_\gamma(s) = \cos\left(\int|\omega'(s)|ds + c\right)\sqrt{|p_\omega{}^2 - \alpha_\omega{}^2|},$$
$$\overline{\mu_1}_\gamma(s) = \omega(s) \wedge \nu_\omega(s).$$

Now, by using $\overline{\mu_2}_\gamma(s) = \nu_\gamma(s) \wedge \overline{\mu_1}_\gamma(s)$, we get

$$\overline{\mu_2}_\gamma(s) = \cos\left(\int|\omega'(s)|ds + c\right)\omega(s) - \sin\left(\int|\omega'(s)|ds + c\right)\nu_\omega(s). \tag{3.13}$$

By differentiating (3.13), we have

$$-q_\gamma(s)\overline{\mu_1}_\gamma(s) = -\sin\left(\int|\omega'(s)|ds + c\right)(h\omega \wedge \nu_\omega)$$

and it leads to

$$q_\gamma(s) = \sin\left(\int|\omega'(s)|ds + c\right)\sqrt{|p_\omega{}^2 - \alpha_\omega{}^2|}.$$

□

Remark 3. If $\alpha_\omega < 0$ in Theorem 2, since the ν_ω of the ω framed curve will change, the similar equations are obtained in Theorem 2.

Corollary 1. *Let $\omega(s)$ be a framed curve on S^2 and let $\gamma(s) = \rho\sec\left(\int|\omega'(s)|ds + c\right)\omega(s)$ be a dilation of γ. According to Equation (3.10) in Theorem 2, if s_0 is a singular point of ω, then also s_0 is a singular point of γ.*

Remark 4. Since by Theorem 4 in Wang et al,[6] the framed curve $\gamma(s) = \rho\sec\left(\int|\omega'(s)|ds + c\right)\omega(s)$ is a framed rectifying curve that requires $p_\gamma > 0$. Therefore, we have the the following corollary according to Theorem 2.

Corollary 2. *Let $\omega(s)$ be a framed curve on S^2 that is not an arc of the great circle, then $\gamma(s) = \rho\sec\left(\int|\omega'(s)|ds + c\right)\omega(s)$ is a framed rectifying curve.*

Example 2. The spherical cardioid $\omega : [0, 4\pi) \to \mathbb{R}^3$ is defined by

$$\omega(s) = \frac{1}{3}\left(2\cos s - \cos 2s, 2\sin s - \sin 2s, 2\sqrt{2}\cos\frac{s}{2}\right).$$

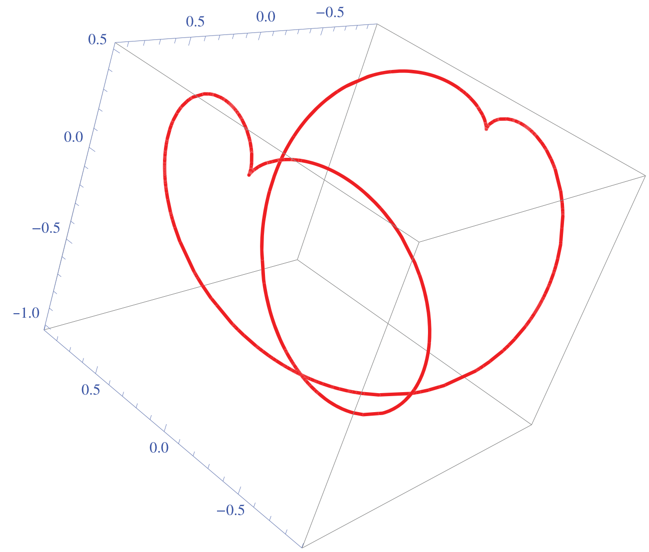

FIGURE 3 Spherical cardioid [Colour figure can be viewed at wileyonlinelibrary.com]

See Figure 3. Then we get

$$v_\omega(s) = \left(-2\sqrt{2}\cos\frac{s}{2} + \frac{8\sqrt{2}}{3}\cos^3\frac{s}{2}, 2\sqrt{2}\sin\frac{s}{2} - \frac{8\sqrt{2}}{3}\sin^3\frac{s}{2}, -\frac{1}{3}\right)$$

and

$$\alpha_\omega(s) = \sqrt{2}\sin\frac{s}{2}.$$

By a calculation, we get

$$\|\omega'(s)\| = \sqrt{2}\sin\frac{s}{2}$$

and

$$\int \|\omega'(s)\|ds + c = -2\sqrt{2}\cos\frac{s}{2} + c.$$

Suppose that $\rho = 1$ and $c = 0$, then according to Theorem 2, we have

$$\begin{aligned}\gamma(s) = (&\frac{2}{3}\sec\left(-2\sqrt{2}\cos\frac{s}{2}\right)\cos s - \frac{1}{3}\sec\left(-2\sqrt{2}\cos\frac{s}{2}\right)\cos 2s,\\ &\frac{2}{3}\sec\left(-2\sqrt{2}\cos\frac{s}{2}\right)\sin s - \frac{1}{3}\sec\left(-2\sqrt{2}\cos\frac{s}{2}\right)\sin 2s,\\ &\frac{2\sqrt{2}}{3}\sec\left(-2\sqrt{2}\cos\frac{s}{2}\right)\cos\frac{s}{2}).\end{aligned} \tag{3.14}$$

See Figure 4. On the other hand, we have

$$\begin{aligned}v_\gamma(s) = \Bigg(&\frac{2}{3}\sin\left(-2\sqrt{2}\cos\frac{s}{2}\right)\cos s - \frac{1}{3}\sin\left(-2\sqrt{2}\cos\frac{s}{2}\right)\cos 2s,\\ &-2\sqrt{2}\cos\left(-2\sqrt{2}\cos\frac{s}{2}\right)\cos\frac{s}{2} + \frac{8\sqrt{2}}{3}\cos\left(-2\sqrt{2}\cos\frac{s}{2}\right)\cos^3\frac{s}{2},\\ &\frac{2}{3}\sin\left(-2\sqrt{2}\cos\frac{s}{2}\right)\sin s - \frac{1}{3}\sin\left(-2\sqrt{2}\cos\frac{s}{2}\right)\sin 2s,\\ &2\sqrt{2}\cos\left(-2\sqrt{2}\cos\frac{s}{2}\right)\sin\frac{s}{2} - \frac{8\sqrt{2}}{3}\cos\left(-2\sqrt{2}\cos\frac{s}{2}\right)\sin^3\frac{s}{2},\\ &\frac{2\sqrt{2}}{3}\sin\left(-2\sqrt{2}\cos\frac{s}{2}\right)\cos\frac{s}{2} - \frac{1}{3}\cos\left(-2\sqrt{2}\cos\frac{s}{2}\right)\Bigg).\end{aligned} \tag{3.15}$$

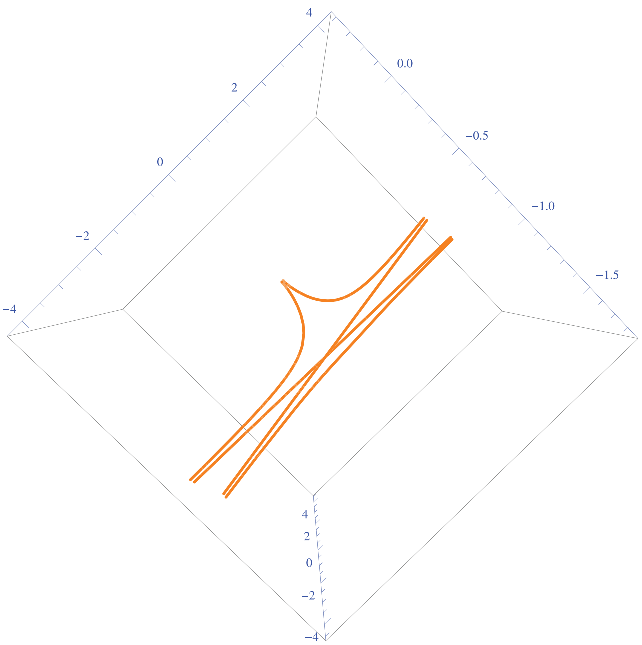

FIGURE 4 Framed rectifying curve via dilation of spherical cardioid [Colour figure can be viewed at wileyonlinelibrary.com]

Moreover, since $p_\omega = \|v_\omega\| = \sqrt{2}$, we find

$$\alpha_\gamma(s) = \sqrt{2}\sin\frac{s}{2}\sec^2\left(-2\sqrt{2}\cos\frac{s}{2}\right),$$
$$p_\gamma = \sqrt{2}\cos\frac{s}{2}\cos\left(-2\sqrt{2}\cos\frac{s}{2}\right),$$
$$q_\gamma = \sqrt{2}\cos\frac{s}{2}\sin\left(-2\sqrt{2}\cos\frac{s}{2}\right).$$

4 | CONCLUSION

In this study, the concept of framed rectifying curve is expanded, and new results have been obtained. Inspired by this study, the relationships of framed rectifying curves with centrode and extremal curves can be examined.

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CONFLICT OF INTERESTS

The authors declare no potential conflict of interests.

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None reported.

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