Received: 5 December 2023 | Revised: 2 April 2024 | Accepted: 5 April 2024

DOI: 10.1002/mma.10149

RESEARCH ARTICLE

WILEY

On special curves in Lie groups with Myller configuration

Zehra İşbilir[1] | **Bahar Doğan Yazıcı**[2] | **Murat Tosun**[3]

[1]Department of Mathematics, Düzce University, Düzce, Turkey

[2]Department of Mathematics, Bilecik Şeyh Edebali University, Bilecik, Turkey

[3]Department of Mathematics, Sakarya University, Sakarya, Turkey

Correspondence
Murat Tosun, Department of Mathematics, Sakarya University, Sakarya, Türkiye.
Email: tosun@sakarya.edu.tr

Communicated by: W. Sprößig

In this study, we determine a new type comprehensive frame, which is called the generalized Frenet-type frame in three-dimensional Lie groups with Myller configurations, and it includes several special and classical type frames for Euclidean 3-space and three-dimensional Lie groups. After constructing this new comprehensive frame, we obtain derivative formulas with the help of the Lie curvature. In addition, we define some special type curves. The geometry of versor fields along a curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations is a generalization of the usual theory of curves. Since this particular relationship, the osculating-type and rectifying-type curves with Frenet-type frame in three-dimensional Lie groups with Myller configurations include some special cases for osculating and rectifying curves in different spaces.

KEYWORDS

Frenet-type frame, Lie groups, Myller configuration, osculating-type curves, rectifying-type curves, versor field

MSC CLASSIFICATION

53A04, 22E15

1 | INTRODUCTION

In classical differential geometry, the theory of curves is a quite fundamental work frame for lots of researchers. Moving frames have been an important concept in differential geometry from the investigation of the Frenet (or Frenet–Serret) frame [1, 2], which is established for only regular curves with non-zero curvature condition. In the existing literature, there are many studies with respect to the Frenet frames for regular space curves. In the Euclidean 3-space E_3, every unit speed curve $C : I \rightarrow E_3$ can be associated with the orthogonal unit vector fields tangent vector field T, principal normal vector field N, and binormal vector field B. The planes spanned by $\{T, N\}$, $\{T, B\}$, and $\{N, B\}$ are called the osculating plane, rectifying plane, and normal plane at each point of the curve C, respectively [3]. Osculating, rectifying, and normal curves, which are determined in Euclidean [3–5] and Minkowski spaces studies [6–8] with different dimensions and some characterizations of them, are examined in several studies. The curve C for which the position vector of the curve C always lies in their rectifying plane are called rectifying curves [4]. With the same logic, if the position vector of the curve C always lies in its osculating plane, the curve is called an osculating curve [3], and also, if the position vector of the curve C always lies in its normal plane, the curve is called a normal curve [3, 7, 8].

Let us determine some versor fields (viz., a vector field) such as tangent, principal normal, and binormal and also some plane fields such as rectifying, osculating, and normal planes along the curve C. A versor field and a plane field are represented by $(C, \overline{\xi})$ and (C, π), respectively. The pair $\{(C, \overline{\zeta}), (C, \pi)\}$ for $\overline{\zeta} \in \pi$ is called a Myller configuration, which is denoted by $\mathfrak{M}(C, \overline{\zeta}, \pi)$, in the Euclidean space E_3. In 1960, Miron examined some studies with respect to pair $\{(C, \overline{\xi}), (C, \pi)\}$ [9].

The versor field $(C, \overline{\xi})$ can be represented in an invariant frame, which is an orthonormal and positively oriented frame. The moving equations, invariants, and fundamental theorem of the Frenet frame of the versor field $(C, \overline{\xi})$ are given in [10]. Additionally, if the plane π is tangent to the curve C, then we get a tangent Myller configuration which is denoted by $\mathfrak{M}_t(C, \overline{\zeta}, \pi)$ [10, 11]. Then, the geometry of tangent Myller configuration $\mathfrak{M}_t(C, \overline{\zeta}, \pi)$ is a special case of the general Myller configuration $\mathfrak{M}(C, \overline{\zeta}, \pi)$. The geometry of versor fields along a curve with Frenet-type frame in Myller configuration for E_3 is a generalization of the usual theory of curves with Frenet frame in classical Euclidean space.[1]

Macsim et al. studied some special curves and characterizations of them [13] in a Myller configuration for E_3 such as rectifying-type [11] and Bertrand-type curves [14]. Owing to the special relation that exists between the Frenet-type frame in Myller configuration for E_3 and the classical Frenet frame in E_3, Macsim et al. determined the rectifying-type curves with Frenet-type frame in Myller configuration for E_3, and due to the natural construction of the Myller configuration, the rectifying curves with classical Frenet frame in E_3 is one of the special cases of rectifying-type curves with Frenet-type frame in Myller configuration for E_3. With the same motivation, Macsim et al. introduced the Bertrand-type curves in Myller configuration for E_3 in [14]. Similarly, because of the relationship mentioned above, Bertrand curves with Frenet frame in E_3 is one of the special cases of Bertrand-type curves with Frenet-type frame in Myller configuration for E_3. Then, versor fields along a curve in a four-dimensional Lorentz space are studied by Heroiu [12]. Also, osculating-type curves with Frenet-type frame in Myller configuration for E_3 is studied [15]. In a similar manner, osculating curves with classical Frenet frame in E_3 is one of the particular cases of osculating-type curves with Frenet-type frame in Myller configuration for E_3. Doğan Yazıcı and Tosun introduced the osculating curves in Myller configuration with quasi-type frame in [16].

One of the most fundamental and significant notions for mathematics to physics is Lie groups, which are separated as Abelian Lie groups, $SO(3)$ and S^3 . Lie groups contribute to many work frames with respect to the theory and application in mechanics and physics. Then, incompressible inviscid fluid motion and rigid body motion correspond to the geodesic flow of left or right invariant metric have been determined on a Lie group [17–19]. Çöken and Çiftçi scrutinized the degenerate semi-Riemannian geometry of Lie group in [20]. Moreover, Çiftçi examined the generalization of Lancret's theorem and general helices in [21]. Moreover, characterization of rectifying, osculating, and normal curves in three-dimensional compact Lie groups [22] were introduced. Okuyucu et al. investigated the spinor Frenet equations in the three-dimensional Lie groups [23]. Mak [24] introduced the natural and conjugate mates of a Frenet curve in three-dimensional Lie group. Moreover, Doğan Yazıcı et al. introduced framed curves in the three-dimensional Lie groups [25]. Other interesting studies on Lie groups with curve, surface, and quaternion approaches are available in [26–30].

In this paper, our motivation is to bring together the Frenet-type frame in Myller configuration for E_3 and three-dimensional Lie groups at the first stage. Namely, we investigate a quite general moving frame, which is called generalized Frenet-type frame in three-dimensional Lie groups with Myller configurations. We determine the moving equations, curvatures, and Frenet-type derivative formulas with the help of the Lie curvature and some special cases of this frame. At the second stage, we construct some special curves such as osculating-type and rectifying-type curves with Frenet-type frame in three-dimensional Lie groups with Myller configurations. We arrive at a generalization of the classical curve theory due to the natural structure of the geometry of the versor fields along a curve with a Frenet-type frame in three-dimensional Lie groups with Myller configuration. Because of this particular relation, osculating-type curves and rectifying-type curves with Frenet-type frame in three-dimensional Lie groups with Myller configurations include some special cases.

2 | BASIC CONCEPTS

In this section, we remind some terminology with respect to Frenet-type frame in Myller configuration for Euclidean 3-space E_3 and three-dimensional Lie groups in the following subsections, respectively.

2.1 | Frenet-type frame in Myller configuration

Let $\left(C, \overline{\zeta}\right)$ be a versor field and $\overline{r}(s)$ be a position vector of the curve C where s is the arc-length parameter on C. For Frenet-type frame $\mathcal{R}_F = \left\{P(s), \overline{\zeta}_1(s), \overline{\zeta}_2(s), \overline{\zeta}_3(s)\right\}$ of the versor field $\left(C, \overline{\zeta}\right)$, one can express the following:

$$\frac{d\overline{r}}{ds} = \rho_1(s)\overline{\zeta}_1(s) + \rho_2(s)\overline{\zeta}_2(s) + \rho_3(s)\overline{\zeta}_3(s),$$

---

[1] See also for generalization [10, 12].

where $\rho_1^2(s)+\rho_2^2(s)+\rho_3^2(s)=1$. Also, the following derivative formulas are satisfied:

$$\begin{cases} \dfrac{d\overline{\zeta}_1(s)}{ds} = \mathcal{K}_1(s)\overline{\zeta}_2(s), \\ \dfrac{d\overline{\zeta}_2(s)}{ds} = -\mathcal{K}_1(s)\overline{\zeta}_1(s) + \mathcal{K}_2(s)\overline{\zeta}_3(s), \\ \dfrac{d\overline{\zeta}_3(s)}{ds} = -\mathcal{K}_2(s)\overline{\zeta}_2(s), \end{cases}$$

where $\mathcal{K}_1(s) > 0$. Then, $\mathcal{K}_1$-curvature and $\mathcal{K}_2$-torsion have the same geometrical interpretation as the curvature and torsion of a curve with Frenet frame in E_3. It can be seen that if the functions $\rho_1(s)=1$, $\rho_2(s)=0$, and $\rho_3(s)=0$, we get Frenet equations of a curve in E_3 [1, 2, 10]. The fundamental theorem of invariants for the versor field $\left(C,\overline{\zeta}\right)$ is given as follows:

Theorem 1 ([10]). *If the invariants* $\mathcal{K}_1(s), \mathcal{K}_2(s), \rho_1(s), \rho_2(s), \rho_3(s)$, *with* $\rho_1^2(s)+\rho_2^2(s)+\rho_3^2(s)=1$ *are smooth functions for* $s \in [a,b]$, *then there exist a curve* $C : [a,b] \to E_3$ *parametrized by arc-length s and a versor field* $\overline{\zeta}(s)$, $s \in [a,b]$, *whose the curvature, torsion, and the functions* $\rho_i(s)$ *are* $\mathcal{K}_1(s)$, $\mathcal{K}_2(s)$, *and* $\rho_i(s)$, $i=1,2,3$. *Any two such versor fields* $(C,\overline{\zeta})$ *differ by a proper Euclidean motion.*

For more detailed information with respect to the Myller configuration, we want to refer to the book [10] by Miron.

2.2 | Three-dimensional Lie groups

Let G be a Lie group with a bi-invariant metric $\langle , \rangle$ and D be the Levi-Civita connection of Lie group G. If notation $\mathfrak{g}$ represents the Lie algebra of Lie group G, then we express that $\mathfrak{g}$ is isomorphic to T_eG where e is the neutral element of Lie group G. If $\langle , \rangle$ is a bi-invariant metric on Lie group G, the following can be written:

$$\langle X,[Y,Z]\rangle = \langle [X,Y],Z\rangle,$$

and

$$D_XY = \frac{1}{2}[X,Y],$$

for every $X,Y,Z \in \mathfrak{g}$. Also, Lie bracket of any two vector fields $\mathcal{W}$ and Z is expressed by

$$[\mathcal{W},Z] = \sum_{i=1}^{n} w_i z_i \left[X_i,X_j\right],$$

where $\mathcal{W} = \sum_{i=1}^{n} w_iX_i$ and $Z = \sum_{i=1}^{n} z_iX_i$ with the orthonormal basis $\{X_1,X_2,\dots,X_n\}$ of $\mathfrak{g}$ and $w_i, z_i : I \to \mathbb{R}$. If $\Upsilon : I \subset \mathbb{R} \to G$ is an arc-length regular curve, then the covariant derivative of the vector field $\mathcal{W}$ along the curve Υ which is represented with the notation $D_{\Upsilon'}\mathcal{W}$ is written:

$$D_{\Upsilon'}\mathcal{W} = D_T\mathcal{W} = \dot{\mathcal{W}} + \frac{1}{2}\left[T,\mathcal{W}\right].$$

3 | FRENET-TYPE FRAME IN THREE-DIMENSIONAL LIE GROUPS WITH MYLLER CONFIGURATION

In this section, we introduce Frenet-type frame in three-dimensional Lie groups with Myller configuration G and some properties of it. Then, we give some characterizations of it by the concept of Frenet-type frame in Myller configuration for Euclidean 3-space E_3 [10].

Definition 1. Let $\left(C,\overline{\zeta}\right)$ be a versor field and $\overline{r}(s) : I \mapsto G$ be a position vector of the curve C in Myller configuration for three-dimensional Lie group G. The covariant derivative of $\mathcal{W}$ along the curve is defined as follows:

$$\begin{aligned} D_{\hat{\zeta}(s)}\mathcal{W}(s) &= \dot{\mathcal{W}}(s) + \frac{1}{2}\left[\hat{\zeta}(s), \mathcal{W}(s)\right] \\ &= \dot{\mathcal{W}}(s) + \frac{1}{2}\rho_1(s)\left[\overline{\zeta}_1(s), \mathcal{W}(s)\right] + \frac{1}{2}\rho_2(s)\left[\overline{\zeta}_2(s), \mathcal{W}(s)\right] + \frac{1}{2}\rho_3(s)\left[\overline{\zeta}_3(s), \mathcal{W}(s)\right], \end{aligned} \tag{1}$$

and

$$\hat{\zeta}(s) = \frac{d\overline{r}}{ds} = \rho_1(s)\overline{\zeta}_1(s) + \rho_2(s)\overline{\zeta}_2(s) + \rho_3(s)\overline{\zeta}_3(s), \tag{2}$$

where $\dot{\mathcal{W}} = \sum_{i=1}^{n} \frac{dw}{ds} X_i$ for the orthonormal basis $\{X_1, X_2, X_3, \dots, X_n\}$ of $\mathfrak{g}$.

Additionally, derivative formulas of the curve $\overline{r}(s)$ in Myller configuration for three-dimensional Lie groups are satisfied:

$$\begin{cases} D_{\hat{\zeta}(s)}\overline{\zeta}_1(s) = \mathcal{K}_1(s)\overline{\zeta}_2(s), \\ D_{\hat{\zeta}(s)}\overline{\zeta}_2(s) = -\mathcal{K}_1(s)\overline{\zeta}_1(s) + \mathcal{K}_2(s)\overline{\zeta}_3(s), \\ D_{\hat{\zeta}(s)}\overline{\zeta}_3(s) = -\mathcal{K}_2(s)\overline{\zeta}_2(s), \end{cases} \tag{3}$$

where D is Levi-Civita connection of Lie group G and $\mathcal{K}_1(s) = \left\|\dot{\overline{\zeta}}_1(s)\right\|$.

Theorem 2. *Let $\overline{r}(s) : I \mapsto G$ be a curve with Frenet-type frame in three-dimensional Lie groups with Myller configuration G. Then, the following equations hold:*

$$\left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right] = \left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_3(s) \right\rangle \overline{\zeta}_3(s) = 2\flat_G \overline{\zeta}_3(s), \tag{4}$$

$$\left[\overline{\zeta}_1(s), \overline{\zeta}_3(s)\right] = \left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_3(s)\right], \overline{\zeta}_2(s) \right\rangle \overline{\zeta}_2(s) = -2\flat_G \overline{\zeta}_2(s), \tag{5}$$

$$\left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right] = \left\langle \left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right], \overline{\zeta}_1(s) \right\rangle \overline{\zeta}_1(s) = 2\flat_G \overline{\zeta}_1(s), \tag{6}$$

where $\flat_G = \frac{1}{2}\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_3(s) \right\rangle$.

Proof. Suppose that $\overline{r}(s) : I \mapsto G$ is a curve with Frenet-type frame in three-dimensional Lie groups with Myller configuration G. Since $\left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right] \in Sp\left\{\overline{\zeta}_1(s), \overline{\zeta}_2(s), \overline{\zeta}_3(s)\right\}$, we can write as follows:

$$\left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right] = \lambda_1\overline{\zeta}_1(s) + \lambda_2\overline{\zeta}_2(s) + \lambda_3\overline{\zeta}_3(s). \tag{7}$$

If we multiply both of sides of Equation (7) by the versors $\overline{\zeta}_1(s), \overline{\zeta}_2(s)$, and $\overline{\zeta}_3(s)$, then we obtain the following, respectively:

$$\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_1(s) \right\rangle = \lambda_1 = 0, \tag{8a}$$

$$\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_2(s) \right\rangle = \lambda_2 = 0, \tag{8b}$$

$$\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_3(s) \right\rangle = \lambda_3. \tag{8c}$$

By using Equations (7) and (8a)-(8c), we get

$$\left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right] = \left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_3(s) \right\rangle \overline{\zeta}_3(s).$$

Assume that $\flat_G = \frac{1}{2}\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_3(s) \right\rangle$. In that case, we have $\left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right] = 2\flat_G\overline{\zeta}_3(s)$. By using the same manner, we can continue to prove. Since $\left[\overline{\zeta}_1(s), \overline{\zeta}_3(s)\right] \in Sp\left\{\overline{\zeta}_1(s), \overline{\zeta}_2(s), \overline{\zeta}_3(s)\right\}$, we have

$$\left[\overline{\zeta}_1(s), \overline{\zeta}_3(s)\right] = \lambda_4\overline{\zeta}_1(s) + \lambda_5\overline{\zeta}_2(s) + \lambda_6\overline{\zeta}_3(s). \tag{9}$$

Then, we obtain

$$\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_3(s)\right], \overline{\zeta}_1(s) \right\rangle = \lambda_4 = 0, \tag{10a}$$

$$\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_3(s)\right], \overline{\zeta}_2(s) \right\rangle = \lambda_5, \tag{10b}$$

$$\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_3(s)\right], \overline{\zeta}_3(s) \right\rangle = \lambda_6 = 0. \tag{10c}$$

Hence, we get

$$\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_3(s)\right], \overline{\zeta}_2(s) \right\rangle = \left\langle \overline{\zeta}_1(s), \left[\overline{\zeta}_3(s), \overline{\zeta}_2(s)\right] \right\rangle = -\left\langle \overline{\zeta}_1(s), \left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right] \right\rangle = -\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_3(s) \right\rangle = -2\flat_G.$$

With the help of Equations (9) and (10a)–(10c), we can express $\left[\overline{\zeta}_1(s), \overline{\zeta}_3(s)\right] = -2\flat_G \overline{\zeta}_2(s)$. With the similar logic, since $\left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right] \in Sp\left\{\overline{\zeta}_1(s), \overline{\zeta}_2(s), \overline{\zeta}_3(s)\right\}$, we can write

$$\left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right] = \lambda_7 \overline{\zeta}_1(s) + \lambda_8 \overline{\zeta}_2(s) + \lambda_9 \overline{\zeta}_3(s). \tag{11}$$

Also, we get

$$\left\langle \left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right], \overline{\zeta}_1(s) \right\rangle = \lambda_7, \tag{12a}$$

$$\left\langle \left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right], \overline{\zeta}_2(s) \right\rangle = \lambda_8 = 0, \tag{12b}$$

$$\left\langle \left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right], \overline{\zeta}_3(s) \right\rangle = \lambda_9 = 0. \tag{12c}$$

Therefore, we have

$$\left\langle \left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right], \overline{\zeta}_1(s) \right\rangle = \left\langle \overline{\zeta}_1(s), \left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right] \right\rangle = \left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_3(s) \right\rangle = 2\flat_G.$$

By Equations (11) and (12a)–(12c), we get $\left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right] = 2\flat_G \overline{\zeta}_1(s)$. Consequently, we have the desired result. ☐

Theorem 3. *Let $\overline{r}(s) : I \mapsto G$ be a curve with Frenet-type frame in three-dimensional Lie groups with Myller configuration G. Then, Frenet-type frame derivative formulas in Myller configuration for three-dimensional Lie groups are written as follows:*

$$\begin{pmatrix} \dot{\overline{\zeta}}_1(s) \\ \dot{\overline{\zeta}}_2(s) \\ \dot{\overline{\zeta}}_3(s) \end{pmatrix} = \begin{pmatrix} 0 & K_1(s) - \rho_3(s)\flat_G & \rho_2(s)\flat_G \\ -K_1(s) + \rho_3(s)\flat_G & 0 & K_2(s) - \rho_1(s)\flat_G \\ -\rho_2(s)\flat_G & -K_2(s) + \rho_1(s)\flat_G & 0 \end{pmatrix} \begin{pmatrix} \overline{\zeta}_1 \\ \overline{\zeta}_2 \\ \overline{\zeta}_3 \end{pmatrix}, \tag{13}$$

where

$$\begin{cases} \mathcal{K}_1(s) - \rho_3(s)\flat_G = \left\langle \dot{\overline{\zeta}}_1(s), \overline{\zeta}_2(s) \right\rangle = -\left\langle \dot{\overline{\zeta}}_2(s), \overline{\zeta}_1(s) \right\rangle, \\ \mathcal{K}_2(s) - \rho_1(s)\flat_G = \left\langle \dot{\overline{\zeta}}_2(s), \overline{\zeta}_3(s) \right\rangle = -\left\langle \dot{\overline{\zeta}}_3(s), \overline{\zeta}_2(s) \right\rangle. \end{cases}$$

Proof. By using Equations (1) and (3), we get the following:

$$\begin{aligned} D_{\hat{\zeta}(s)}\overline{\zeta}_1(s) &= \dot{\overline{\zeta}}_1(s) + \frac{1}{2}\rho_1(s)\left[\overline{\zeta}_1(s), \overline{\zeta}_1(s)\right] + \frac{1}{2}\rho_2(s)\left[\overline{\zeta}_2(s), \overline{\zeta}_1(s)\right] + \frac{1}{2}\rho_3(s)\left[\overline{\zeta}_3(s), \overline{\zeta}_1(s)\right] = \mathcal{K}_1(s)\overline{\zeta}_2(s), \\ D_{\hat{\zeta}(s)}\overline{\zeta}_2(s) &= \dot{\overline{\zeta}}_2(s) + \frac{1}{2}\rho_1(s)\left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right] + \frac{1}{2}\rho_2(s)\left[\overline{\zeta}_2(s), \overline{\zeta}_2(s)\right] + \frac{1}{2}\rho_3(s)\left[\overline{\zeta}_3(s), \overline{\zeta}_2(s)\right] \\ &= -\mathcal{K}_1(s)\overline{\zeta}_1(s) + \mathcal{K}_2(s)\overline{\zeta}_3(s), \\ D_{\hat{\zeta}(s)}\overline{\zeta}_3(s) &= \dot{\overline{\zeta}}_3(s) + \frac{1}{2}\rho_1(s)\left[\overline{\zeta}_1(s), \overline{\zeta}_3(s)\right] + \frac{1}{2}\rho_2(s)\left[\overline{\zeta}_2(s), \overline{\zeta}_3(s)\right] + \frac{1}{2}\rho_3(s)\left[\overline{\zeta}_3(s), \overline{\zeta}_3(s)\right] = -\mathcal{K}_2(s)\overline{\zeta}_2(s). \end{aligned}$$

With the help of Equations (4)–(6), we obtain

$$\begin{aligned}
\dot{\overline{\zeta}}_1(s) &= \left(K_1(s) - \rho_3(s)\flat_G\right)\overline{\zeta}_2(s) + \rho_2(s)\flat_G\overline{\zeta}_3(s),\\
\dot{\overline{\zeta}}_2(s) &= \left(-K_1(s) + \rho_3(s)\flat_G\right)\overline{\zeta}_1(s) + \left(K_2(s) - \rho_1(s)\flat_G\right)\overline{\zeta}_3(s),\\
\dot{\overline{\zeta}}_3(s) &= -\rho_2(s)\flat_G\overline{\zeta}_1(s) + \left(-K_2(s) + \rho_1(s)\flat_G\right)\overline{\zeta}_2(s).
\end{aligned}$$

Also, if we use the inner product in the previous equation by $\overline{\zeta}_1(s), \overline{\zeta}_2(s)$, and $\overline{\zeta}_3(s)$, we get the curvature equations. Therefore, we complete the proof. ☐

The geometry of versor fields along a curve for Frenet-type frame in three-dimensional Lie groups with Myller configurations is a generalization of the usual theory of curves with Frenet frame in classical Euclidean space, Frenet frame in classical three-dimensional Lie groups for Euclidean space, and Frenet-type frame in Myller configuration for Euclidean space. Due to this special generalization, the following special cases can be given:

Corollary 1. *Let $\overline{r}(s) : I \mapsto G$ be a curve with Frenet-type frame in three-dimensional Lie groups with Myller configuration G. Then, the following hold:*

(i) If $\flat_G = 0$, then we get the Frenet-type frame in Myller configuration for three-dimensional Euclidean space [10].
(ii) If $\rho_1(s) = 1, \rho_2(s) = \rho_3(s) = 0$, then we get the Frenet-type frame in three-dimensional Lie groups [21].
(iii) If $\rho_1(s) = 1, \rho_2(s) = \rho_3(s) = 0$, and $\flat_G = 0$, then we get the Frenet frame in Euclidean 3-space [1, 2].

Theorem 4. *Let $\overline{r}(s) : I \mapsto G$ be a curve with Frenet-type frame in three-dimensional Lie groups with Myller configuration G. Then, the Lie group curvature $\flat_G$ is written as follows:*

$$\begin{aligned}
\flat_G ={}& \frac{1}{2\mathcal{K}_1^2(s)\mathcal{K}_2(s) + 2\left(\rho_2(s)\flat_G\right)'\mathcal{K}_1(s)}\left\langle \ddot{\overline{\zeta}}_1(s), \left[\overline{\zeta}_1(s), \dot{\overline{\zeta}}_1(s)\right]\right\rangle\\
&- \frac{\begin{pmatrix} -2\rho_2(s)\mathcal{K}_1'(s) + 2\rho_2(s)\left(\rho_3(s)\flat_G\right)' - 2\rho_3(s)\mathcal{K}_1(s)\mathcal{K}_2(s) - 2\rho_3(s)\left(\rho_2(s)\flat_G\right)' \\ -2\rho_3(s)\mathcal{K}_1(s)\mathcal{K}_2(s) - 2\mathcal{K}_1^2(s)\rho_1(s)\end{pmatrix}\flat_G^2}{2\mathcal{K}_1^2(s)\mathcal{K}_2(s) + 2\left(\rho_2(s)\flat_G\right)'\mathcal{K}_1(s)}\\
&- \frac{\left(2\rho_2^2(s)\mathcal{K}_2(s) + 2\rho_1(s)\rho_3(s)\mathcal{K}_1(s) + 2\rho_3^2(s)\mathcal{K}_1(s)\right)\flat_G^3 - \left(-2\rho_1(s)\rho_2^2(s) - 2\rho_1(s)\rho_3^2(s)\right)\flat_G^4}{2\mathcal{K}_1^2(s)\mathcal{K}_2(s) + 2\left(\rho_2(s)\flat_G\right)'\mathcal{K}_1(s)}.
\end{aligned} \tag{14}$$

Proof. From Equation (1), we can write

$$D_{\hat{\zeta}(s)}\overline{\zeta}_1(s) = \dot{\overline{\zeta}}_1(s) + \frac{1}{2}\rho_1(s)\left[\overline{\zeta}_1(s), \overline{\zeta}_1(s)\right] + \frac{1}{2}\rho_2(s)\left[\overline{\zeta}_2(s), \overline{\zeta}_1(s)\right] + \frac{1}{2}\rho_3(s)\left[\overline{\zeta}_3(s), \overline{\zeta}_1(s)\right].$$

Then, we have $\left[\overline{\zeta}_1(s), \overline{\zeta}_1(s)\right] = 0$, $\left[\overline{\zeta}_2(s), \overline{\zeta}_1(s)\right] = -2\flat_G\overline{\zeta}_3(s)$ and $\left[\overline{\zeta}_3(s), \overline{\zeta}_1(s)\right] = 2\flat_G\overline{\zeta}_2(s)$. Also, we get

$$\begin{aligned}
\dot{\overline{\zeta}}_1(s) &= D_{\hat{\zeta}(s)}\overline{\zeta}_1(s) - \frac{1}{2}\rho_2(s)\left[\overline{\zeta}_2(s), \overline{\zeta}_1(s)\right] + \frac{1}{2}\rho_3(s)\left[\overline{\zeta}_3(s), \overline{\zeta}_1(s)\right]\\
&= \left(\mathcal{K}_1(s) - \rho_3(s)\flat_G\right)\overline{\zeta}_2 + \rho_2(s)\flat_G\overline{\zeta}_3.
\end{aligned}$$

Also, we can write the following:

$$D_{\hat{\zeta}(s)}\dot{\overline{\zeta}}_1(s) = \ddot{\overline{\zeta}}_1(s) + \frac{1}{2}\left[\overline{\zeta}_1(s), \dot{\overline{\zeta}}_1(s)\right] + \frac{1}{2}\left[\overline{\zeta}_2(s), \dot{\overline{\zeta}}_1(s)\right] + \frac{1}{2}\left[\overline{\zeta}_3(s), \dot{\overline{\zeta}}_1(s)\right],$$

and

$$\ddot{\overline{\zeta}}_1(s) = D_{\hat{\zeta}(s)}\dot{\overline{\zeta}}_1(s) - \frac{1}{2}\left[\overline{\zeta}_1(s), \dot{\overline{\zeta}}_1(s)\right] - \frac{1}{2}\left[\overline{\zeta}_2(s), \dot{\overline{\zeta}}_1(s)\right] - \frac{1}{2}\left[\overline{\zeta}_3(s), \dot{\overline{\zeta}}_1(s)\right].$$

If some calculations are completed, we get the desired Equation (14); for the sake of brevity, we omit them. ☐

Corollary 2. *We examine the equations of the Lie curvature* $\flat_G$ *with the help of the orthonormal frame* $\left\{\overline{\zeta}_1(s), \overline{\zeta}_2(s), \overline{\zeta}_3(s)\right\}$ *inspired by the study [21], in which the equations of* τ_G *are presented in some Lie groups. The following three special cases hold:*

(i) If G is Abelian (viz., $\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_3(s)\right\rangle = \left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_3(s)\right], \overline{\zeta}_2(s)\right\rangle$*), then we get* $\flat_G = 0$.

(ii) If G is $SO(3)$ *with the bi-invariant metric which is written as* $K(X, Y) = -\frac{1}{2}$*trace*(XY) *for each* $X, Y \in \mathfrak{so}(3)$*. From* $\mathfrak{so}(3)$ *with the* $\left(\mathbb{R}^3, \wedge\right)$*, we have*

$$\flat_G = \frac{1}{2}\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_3(s)\right\rangle = \frac{1}{2}\left\langle \overline{\zeta}_1(s) \wedge \overline{\zeta}_2(s), \overline{\zeta}_3(s)\right\rangle = \frac{1}{2}\left\langle \overline{\zeta}_3(s), \overline{\zeta}_3(s)\right\rangle = \frac{1}{2},$$

for a curve with Frenet-type frame in Myller configuration in $SO(3)$*.*

(iii) If G is S^3*, let us consider Lie group homeomorphism* $S^3 \cong SU(2)$*, the basis of* $su(2)$ *are* $\vartheta_1, \vartheta_2, \vartheta_3$ *which are written as*

$$\vartheta_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \vartheta_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \vartheta_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Thanks to the study [21], we have $[\vartheta_1, \vartheta_2] = 2\vartheta_3, [\vartheta_2, \vartheta_3] = 2\vartheta_1$ *and* $[\vartheta_3, \vartheta_1] = 2\vartheta_1$*, then we obtain*

$$\flat_G = \frac{1}{2}\left\langle \left[\overline{\zeta}_1(s), \overline{\zeta}_2(s)\right], \overline{\zeta}_3(s)\right\rangle = \frac{1}{2}\left\langle 2\overline{\zeta}_3(s), \overline{\zeta}_3(s)\right\rangle = 1,$$

for a curve with respect to Frenet-type frame in Myller configuration in $SU(2)$*.*

4 | OSCULATING-TYPE CURVES IN THREE-DIMENSIONAL LIE GROUPS WITH MYLLER CONFIGURATIONS

In this section, we investigate the osculating-type curves with the Frenet-type frame in three-dimensional Lie groups with Myller configurations. As is known, since the Frenet-type frame in three-dimensional Lie groups with Myller configurations is a generalized version for both Frenet-type frame in Myller configuration for Euclidean 3-space, Frenet frame in three-dimensional Lie groups for Euclidean 3-space and also Frenet frame in Euclidean 3-space, the osculating-type curves are a generalization for osculating curves with both Frenet-type frame in Myller configuration for Euclidean space, Frenet frame in three-dimensional Lie groups for Euclidean 3-space and Frenet frame in Euclidean 3-space, as well.

Definition 2. Let $\overline{r}(s) : I \to G$ is defined as an osculating-type curve with Frenet-type frame in three-dimensional Lie groups with Myller configuration if

$$\overline{r}(s) = \eta(s)\overline{\zeta}_1(s) + \omega(s)\overline{\zeta}_2(s), \tag{15}$$

where $\eta(s)$ and $\omega(s)$ are smooth functions.

Firstly, let us obtain some preliminary preparations before starting the relevant theorems. Differentiating Equation (15) and by using Equations (2) and (13), we get

$$\begin{aligned} \rho_1(s)\overline{\zeta}_1(s) + \rho_2(s)\overline{\zeta}_2(s) + \rho_3(s)\overline{\zeta}_3(s) = {} & \eta'(s)\overline{\zeta}_1(s) + \eta(s)\left(\mathcal{K}_1(s) - \rho_3(s)\flat_G\right)\overline{\zeta}_2(s) + \eta(s)\rho_2(s)\flat_G\overline{\zeta}_3 + \omega'(s)\overline{\zeta}_2(s) \\ & + \omega(s)\left(-\mathcal{K}_1(s) + \rho_3(s)\flat_G\right)\overline{\zeta}_1(s) + \omega(s)\left(\mathcal{K}_2(s) - \rho_1(s)\flat_G\right)\overline{\zeta}_3(s). \end{aligned} \tag{16}$$

In that case, we have

$$\eta'(s) + \omega(s)\left(-\mathcal{K}_1(s) + \rho_3(s)\flat_G\right) = \rho_1(s), \tag{17a}$$

$$\omega'(s) + \eta(s)\left(\mathcal{K}_1(s) - \rho_3(s)\flat_G\right) = \rho_2(s), \tag{17b}$$

$$\omega(s)\left(\mathcal{K}_2(s) - \rho_1(s)\flat_G\right) + \eta(s)\rho_2(s)\flat_G = \rho_3(s). \tag{17c}$$

Now, we examine the solutions of Equation (17c) according to the cases $\rho_3(s) = 0$ or $\rho_3(s) \neq 0$. First of all, let us study the situation $\rho_3(s) \neq 0$. Since $\rho_1(s) = 1$, $\rho_2(s) = \rho_3(s) = 0$ is accepted for Frenet frame in Euclidean space, the case $\rho_3(s) \neq 0$ will be a new classification that does not correspond to Frenet frame for three-dimensional Lie groups.

Theorem 5. *Let $\bar{r}(s) : I \rightarrow G$ be a curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations. Then, $\bar{r}(s)$ is an osculating-type curve if and only if*

$$\bar{r}(s) = \left(\frac{\rho_2(s)}{\mathcal{K}_1(s)-\rho_3(s)\flat_G} - \frac{1}{\mathcal{K}_1(s)-\rho_3(s)\flat_G}\left(\frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}\right)'\right)\bar{\zeta}_1(s) + \left(\frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}\right)\bar{\zeta}_2(s), \tag{18}$$

where $\rho_3(s) \neq 0$.

Proof. Assume that $\bar{r}(s)$ is an osculating-type curve with $\rho_3(s) \neq 0$ with Frenet-type frame in three-dimensional Lie groups with Myller configurations. By using Equations (17a)–(17c), we have $\omega(s) \neq 0$ and $\mathcal{K}_2(s) - \rho_1(s)\flat_G \neq 0$. Then, we get

$$\omega(s) = \frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}, \tag{19}$$

and since $\mathcal{K}_1(s) - \rho_3(s)\flat_G > 0$, we obtain

$$\eta(s) = \frac{\rho_2(s)}{\mathcal{K}_1(s)-\rho_3(s)\flat_G} - \frac{1}{\mathcal{K}_1(s)-\rho_3(s)\flat_G}\left(\frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}\right)'. \tag{20}$$

Hence, if Equation (15) is considered, what is desired is proved. Conversely, suppose that Equation (18) is satisfied when $\rho_3(s) \neq 0$. Then, we have

$$\left\langle \bar{r}(s), \bar{\zeta}_1(s) \right\rangle = \frac{\rho_2(s)}{\mathcal{K}_1(s)-\rho_3(s)\flat_G} - \frac{1}{\mathcal{K}_1(s)-\rho_3(s)\flat_G}\left(\frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}\right)', \tag{21a}$$

$$\left\langle \bar{r}(s), \bar{\zeta}_2(s) \right\rangle = \frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}. \tag{21b}$$

Differentiating Equation (21b), we obtain

$$\rho_2(s) + \left(-\mathcal{K}_1(s)+\rho_3(s)\flat_G\right)\left\langle \bar{r}(s), \bar{\zeta}_1(s) \right\rangle + \left(\mathcal{K}_2(s)-\rho_1(s)\flat_G\right)\left\langle \bar{r}(s), \bar{\zeta}_3(s) \right\rangle = \left(\frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}\right)'.$$

By using Equation (21a), since $\mathcal{K}_2(s) - \rho_1(s)\flat_G \neq 0$, we get $\left\langle \bar{r}(s), \bar{\zeta}_3(s) \right\rangle = 0$. Consequently, $\bar{r}(s)$ is an osculating-type curve. □

Corollary 3. *Let $\bar{r}(s)$ be an osculating-type curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations.*

- *(i) If $\rho_1(s) = 1, \rho_2(s) = \rho_3(s) = 0$ and $\flat_G = 0$, we have the osculating curves in Euclidean space E_3 which is determined by İlarslan and Nesović [3].*
- *(ii) If $\rho_1(s) = 1, \rho_2(s) = \rho_3(s) = 0$, we get the osculating curve in compact Lie groups which is determined by Bozkurt et al. [22].*
- *(iii) If $\flat_G = 0$, we have the osculating-type curves with Frenet-type frame in Myller configuration for Euclidean space E_3 which is determined by İşbilir and Tosun [15].*

Theorem 6. *Let $\bar{r}(s) : I \rightarrow G$ is a curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations where $\rho_3(s) \neq 0$. Then, $\bar{r}(s)$ is an osculating-type curve if and only if the $\mathcal{K}_1(s)$-curvature, $\mathcal{K}_2(s)$-torsion and the functions $\rho_1(s), \rho_2(s), \rho_3(s)$ hold the following relation:*

$$\begin{gathered}\left(\frac{\rho_2(s)}{\mathcal{K}_1(s)-\rho_3(s)\flat_G} - \frac{1}{\mathcal{K}_1(s)-\rho_3(s)\flat_G}\left(\frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}\right)'\right)' \\ + \left(\frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}\right)\left(\mathcal{K}_1(s)+\rho_3(s)\flat_G\right) - \rho_1(s) = 0.\end{gathered} \tag{22}$$

Proof. Let $\overline{r}(s)$ be an osculating-type curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations where $\rho_3(s) \neq 0$.

By means of Equations (17a), (19), and (20), we get Equation (22). On the other hand, suppose that $\overline{r}(s)$ is a curve satisfying the relation written in Equation (22). According to these, we can write

$$\frac{d}{ds}\left[\begin{array}{l} \overline{r}(s)-\left(\frac{\rho_2(s)}{\mathcal{K}_1(s)-\rho_3(s)\flat_G}-\frac{1}{\mathcal{K}_1(s)-\rho_3(s)\flat_G}\left(\frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}\right)'\right)\overline{\zeta}_1(s)\\ \quad -\left(\frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}\right)\overline{\zeta}_2(s)\end{array}\right]=0.$$

Hence, $\overline{r}(s)$ is an osculating-type curve. □

Since $\rho_3(s) \neq 0$, some special solutions of differential equation (22) where $\rho_1^2(s)+\rho_2^2(s)+\rho_3^2(s)=1$ are as follows:

Corollary 4. *Let* $\overline{r}(s)\ :\ I\ \to\ G$ *is a curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations where* $\rho_3(s) \neq 0$*. If* $\rho_1(s)=0$ *and* $\rho_2^2(s)+\rho_3^2(s)=1$*, we have*

$$\left(\frac{\rho_2(s)}{\mathcal{K}_1(s)-\rho_3(s)\flat_G}-\frac{1}{\mathcal{K}_1(s)-\rho_3(s)\flat_G}\left(\frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)}\right)'\right)'+\left(\frac{\rho_3(s)-\eta(s)\rho_2(s)\flat_G}{\mathcal{K}_2(s)}\right)\left(\mathcal{K}_1(s)+\rho_3(s)\flat_G\right)=0, \tag{23}$$

where Equation (23) is a homogeneous differential equation with variable coefficients.

Corollary 5. *Let* $\overline{r}(s)\ :\ I\ \to\ G$ *is a curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations where* $\rho_3(s) \neq 0$*. If* $\rho_2(s)=0$ *and* $\rho_1^2(s)+\rho_3^2(s)=1$*, we get*

$$\left(-\frac{1}{\mathcal{K}_1(s)-\rho_3(s)\flat_G}\left(\frac{\rho_3(s)}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}\right)'\right)'+\left(\frac{\rho_3(s)}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}\right)\left(\mathcal{K}_1(s)+\rho_3(s)\flat_G\right)-\rho_1(s)=0.$$

Since $\rho_3(s) \neq 0, \mathcal{K}_1(s)-\rho_1(s)\flat_G \neq 0$*, and* $\mathcal{K}_1(s)-\rho_3(s)\flat_G \neq 0$*, assume that* $\frac{\rho_3(s)}{\mathcal{K}_2(s)-\rho_1(s)\flat_G}=y(s)$ *and* $\frac{1}{\mathcal{K}_1(s)-\rho_3(s)\flat_G}=p(s)$*, we have*

$$\left(p(s)y'(s)\right)'+\frac{y(s)}{p(s)}=-\rho_1(s).$$

Hence, an inhomogeneous differential equation is constructed from the second order. This differential equation is first solved with respect to the homogeneous part, then special solutions are generated depending on the function $\rho_1(s)$*.*

Corollary 6. *Let* $\overline{r}(s)\ :\ I\ \to\ G$ *is a curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations where* $\rho_3(s) \neq 0$*. If* $\rho_1(s)=0$*,* $\rho_2(s)=0$*, and* $\rho_3(s)=1$*, we get*

$$\left(-\frac{1}{\mathcal{K}_1(s)-\flat_G}\left(\frac{1}{\mathcal{K}_2(s)}\right)'\right)'+\left(\frac{1}{\mathcal{K}_2(s)}\right)\left(\mathcal{K}_1(s)+\flat_G\right)=0.$$

Suppose that $\frac{1}{\mathcal{K}_2(s)}=y(s)$ *and* $-\frac{1}{\mathcal{K}_1(s)+\flat_G}=p(s)$*, where denominators are not zero, we write*

$$\left(p(s)y'(s)\right)'+\frac{y(s)}{p(s)}=0.$$

If we apply the $t=\int\frac{1}{p}ds$*, we obtain*

$$\frac{d^2y}{dt^2}+y=0. \tag{24}$$

The solution of the differential Equation (24) is $y = c_1 \cos t + c_2 \sin t$ *where* c_1, c_2 *are constants. Since* $\frac{1}{\mathcal{K}_2(s)} = y(s)$ *and* $t = \int \frac{1}{p} ds$*, we have*

$$\frac{1}{\mathcal{K}_2(s)} = c_1 \cos \int \mathcal{K}_1(s)ds + c_2 \sin \int \mathcal{K}_1(s)ds.$$

Therefore, we have as follows:

$$\omega(s) = \frac{1}{\mathcal{K}_2(s)} = c_1 \cos \int \mathcal{K}_1(s)ds + c_2 \sin \int \mathcal{K}_1(s)ds, \tag{25a}$$

$$\eta(s) = -\frac{1}{\mathcal{K}_1(s) - \flat_G}\left(\frac{1}{\mathcal{K}_2(s)}\right)' = c_1 \sin \int \mathcal{K}_1(s)ds - c_2 \cos \int \mathcal{K}_1(s)ds. \tag{25b}$$

From Equations (25a) and (25b), the following is written:

$$\left\langle \overline{r}(s), \overline{r}(s) \right\rangle = \eta^2(s) + \omega^2(s) = c_1^2 + c_2^2,$$

where c_1 *and* c_2 *are constants. That is, the osculating-type curve* $\overline{r}(s)$ *is a spherical curve when* $\rho_1(s) = 0$, $\rho_2(s) = 0$, *and* $\rho_3(s) = 1$.

Now, we express the characterizations for the case of $\rho_3(s) = 0$.

Proposition 1. *Let* $\overline{r}(s)$ *be a curve with Frenet-type frame in Myller configuration for three-dimensional Lie groups. If* $\overline{r}(s)$ *is an osculating-type curve where* $\rho_3(s) = 0$, *the functions* $\eta(s) = \left\langle \overline{r}(s), \overline{\zeta}_1(s) \right\rangle$, $\omega(s) = \left\langle \overline{r}(s), \overline{\zeta}_2(s) \right\rangle$, *and* $\rho_1(s), \rho_2(s), \rho_3(s)$ *satisfy the following equation:*

$$\eta^2(s) + \omega^2(s) = 2 \int \left(\rho_1(s)\eta(s) + \rho_2(s)\omega(s)\right) ds, \tag{26}$$

where $\omega(s)\left(\mathcal{K}_2(s) - \rho_1(s)\flat_G\right) = -\eta(s)\rho_2(s)\flat_G$.

Proof. Let $\overline{r}(s)$ be a curve with $\rho_3(s) = 0$, then from Equation (17c), we have $\omega(s)\left(\mathcal{K}_2(s) - \rho_1(s)\flat_G\right) = -\eta(s)\rho_2(s)\flat_G$. From Equations (17a) and (17b), we have

$$\eta'(s) + \omega(s)\left(-\mathcal{K}_1(s) + \rho_3(s)\flat_G\right) = \rho_1(s), \tag{27a}$$

$$\omega'(s) + \eta(s)\left(\mathcal{K}_1(s) - \rho_3(s)\flat_G\right) = \rho_2(s). \tag{27b}$$

If each side of Equations (27a) and (27b) are multiplied by $\eta(s)$ and $\omega(s)$, respectively, and then summing the two equations obtained, the following equation holds:

$$\eta(s)\eta'(s) + \omega(s)\omega'(s) = \eta(s)\rho_1(s) + \omega(s)\rho_2(s). \tag{28}$$

Integrating both sides of Equation (28), Equation (26) is constructed. □

Proposition 2. *Let* $\overline{r}(s)$ *be a curve with Frenet-type frame in Myller configuration for three-dimensional Lie groups. If*

$$\eta^2(s) + \omega^2(s) = 2 \int \left(\eta(s)\rho_1(s) + \omega(s)\rho_2(s)\right) ds, \tag{29}$$

one of the expressions

(i) if $\mathcal{K}_2(s) - \rho_1(s)\flat_G = 0$, *then* $\rho_3(s) = 0$;
(ii) $\omega(s) = 0$, *that is,* $\rho_3(s) = 0$ *and* $\overline{r}(s)$ *is a rectifying-type curve;*
(iii) $\overline{r}(s)$ *is an osculating-type curves,*

is satisfied.

Proof. Suppose that $\overline{r}(s)$ is a curve with Equation (29). In that case, we get as follows:

$$\eta(s)\eta'(s)+\omega(s)\omega'(s)=\eta(s)\rho_1(s)+\omega(s)\rho_2(s).$$

Also, we have

$$\eta(s)=\left\langle \overline{r}(s),\overline{\zeta}_1(s)\right\rangle \Rightarrow \eta'(s)=\rho_1(s)+\left\langle \overline{r}(s),\left(\mathcal{K}_1(s)-\rho_3(s)\flat_G\right)\overline{\zeta}_2(s)+\rho_2(s)\flat_G\overline{\zeta}_3\right\rangle,$$

and

$$\omega(s)=\left\langle \overline{r}(s),\overline{\zeta}_2(s)\right\rangle \Rightarrow \omega'(s)=\rho_2(s)+\left\langle \overline{r}(s),(-\mathcal{K}_1(s)+\rho_3(s)\flat_G)\overline{\zeta}_1(s)+(\mathcal{K}_2(s)-\rho_1(s)\flat_G)\overline{\zeta}_3(s)\right\rangle.$$

After some geometric interpretations, we get the desired results. □

Corollary 7. *Let $\overline{r}(s)$ be an osculating-type curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations. If $\rho_1(s)=1,\ \rho_2(s)=\rho_3(s)=0$ for Equation (26), we get*

$$\eta^2(s)+\omega^2(s)=2\int \eta(s)ds. \tag{30}$$

The expression (30) is a characterization of osculating curves in Euclidean space according to Frenet frame [3] for $\flat_G=0$, and it is a characterization of osculating-type curves with Frenet frame in Myller configuration for Euclidean space [15].

5 | RECTIFYING-TYPE CURVES IN THREE-DIMENSIONAL LIE GROUPS WITH MYLLER CONFIGURATIONS

In this section, we also examine the rectifying-type curves with Frenet-type frame in three-dimensional Lie groups with Myller configurations. As is known, because of the fact that Frenet-type frame in three-dimensional Lie groups with Myller configurations is a generalized version for both Frenet frame in Myller configuration for Euclidean space and Frenet frame in Euclidean space, the rectifying-type curves are a generalization for rectifying curves with both Frenet frame in Myller configuration for Euclidean space and Frenet frame in Euclidean space, as well.

Definition 3. $\overline{r}(s):I\to G$ is defined as a rectifying-type curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations if

$$\overline{r}(s)=\eta(s)\overline{\zeta}_1(s)+\omega(s)\overline{\zeta}_3(s), \tag{31}$$

where $\eta(s)$ and $\omega(s)$ are smooth functions.

Let us write also some preliminary preparations in order to get some required equations. If we take derivative of Equation (31) and by using Equations (2) and (13), we have

$$\begin{aligned}\rho_1(s)\overline{\zeta}_1(s)+\rho_2(s)\overline{\zeta}_2(s)+\rho_3(s)\overline{\zeta}_3(s)&=\eta'(s)\overline{\zeta}_1+\eta(s)\left[\left(\mathcal{K}_1(s)-\rho_3(s)\flat_G\right)\overline{\zeta}_2+\rho_2(s)\flat_G\overline{\zeta}_3(s)\right]+\omega'(s)\overline{\zeta}_3(s)\\&\quad+\omega(s)\left[-\rho_2(s)\flat_G\overline{\zeta}_1(s)+\left(-\mathcal{K}_2(s)+\rho_1(s)\flat_G\right)\overline{\zeta}_2\right].\end{aligned} \tag{32}$$

Then, we obtain

$$\eta'(s)-\omega(s)\rho_2(s)\flat_G=\rho_1(s), \tag{33a}$$

$$\eta(s)\left(\mathcal{K}_1(s)-\rho_3(s)\flat_G\right)+\omega(s)\left(-\mathcal{K}_2(s)+\rho_1(s)\flat_G\right)=\rho_2(s), \tag{33b}$$

$$\eta(s)\rho_2(s)\flat_G+\omega'(s)=\rho_3(s). \tag{33c}$$

Theorem 7. *Let $\overline{r}(s)\ :\ I\ \rightarrow\ G$ be a curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations. Then, $\overline{r}(s)$ is a rectifying-type curve if and only if the following are satisfied:*

(i) $\frac{d}{ds}\left\langle \overline{r}(s), \overline{\zeta}_1(s) \right\rangle = \rho_1(s) + \omega(s)\rho_2(s)\flat_G,$

(ii) $\frac{d}{ds}\left\langle \overline{r}(s), \overline{\zeta}_3(s) \right\rangle = \rho_3(s) - \eta(s)\rho_2(s)\flat_G$ *where* $-\mathcal{K}_2(s) + \rho_1(s)\flat_G \neq 0.$

Proof. Suppose that (i) is satisfied. We can write the following equation:

$$\left\langle \overline{r}(s), \overline{\zeta}_1(s) \right\rangle = \eta(s). \tag{34}$$

Differentiating Equation (34) with respect to the parameter s, we can write

$$\left\langle \frac{d\overline{r}(s)}{ds}, \overline{\zeta}_1(s) \right\rangle + \left\langle \overline{r}(s), \overline{\zeta}_1'(s) \right\rangle = \eta'(s).$$

Then, we have with the help of Equations (2), (13), and (33a):

$$\left\langle \rho_1(s)\overline{\zeta}_1(s) + \rho_2(s)\overline{\zeta}_2(s) + \rho_3(s)\overline{\zeta}_3(s), \overline{\zeta}_1(s) \right\rangle + \left\langle \overline{r}(s), \left(\mathcal{K}_1(s) - \rho_3(s)\flat_G\right)\overline{\zeta}_2(s) + \rho_2(s)\flat_G\overline{\zeta}_3(s) \right\rangle = \rho_1(s) + \omega(s)\rho_2(s)\flat_G. \tag{35}$$

From Equation (35), we get $\left\langle \overline{r}(s), \overline{\zeta}_3(s) \right\rangle = \omega(s)$ and $\left\langle \overline{r}(s), \overline{\zeta}_2(s) \right\rangle = 0$ where $\mathcal{K}_1(s) - \rho_3(s)\flat_G \neq 0$. Therefore, we obtain the desired result. In the same manner, assume that (ii) holds. We have

$$\left\langle \overline{r}(s), \overline{\zeta}_3(s) \right\rangle = \omega(s). \tag{36}$$

Differentiating Equation (36) with respect to the parameter s, we have

$$\left\langle \frac{d\overline{r}(s)}{ds}, \overline{\zeta}_3(s) \right\rangle + \left\langle \overline{r}(s), \overline{\zeta}_3'(s) \right\rangle = \omega'(s).$$

By using Equations (2), (13), and (33a), the following equation can be written:

$$\left\langle \rho_1(s)\overline{\zeta}_1(s) + \rho_2(s)\overline{\zeta}_2(s) + \rho_3(s)\overline{\zeta}_3(s), \overline{\zeta}_3(s) \right\rangle + \left\langle \overline{r}(s), \rho_2(s)\flat_G\overline{\zeta}_1(s) + (-\mathcal{K}_2(s) + \rho_1(s)\flat_G)\overline{\zeta}_2(s) \right\rangle = \rho_3(s) - \eta(s)\rho_2(s)\flat_G. \tag{37}$$

From Equation (37), we get $\left\langle \overline{r}(s), \overline{\zeta}_2(s) \right\rangle = 0$ where $-\mathcal{K}_2(s) + \rho_1(s)\flat_G \neq 0$ and $\left\langle \overline{r}(s), \overline{\zeta}_1(s) \right\rangle = \eta(s)$. Also, we have in both cases (i) and (ii) imply that $\overline{r}(s)$ is the rectifying-type curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations. Conversely, if the curve $\overline{r}(s)$ is a rectifying-type curve, then we have the equations written in (i) and (ii) of Theorem 7, this is obvious. □

Corollary 8. *Let $\overline{r}(s)\ :\ I\ \rightarrow\ G$ be a curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations.*

(i) If $\rho_1(s) = 1, \rho_2(s) = \rho_3(s) = 0,$ and $\flat_G = 0,$ we have the rectifying curves in Euclidean space E_3 which is determined by Chen [4].

(ii) If $\rho_1(s) = 1, \rho_2(s) = \rho_3(s) = 0,$ we get the rectifying curves in compact Lie groups which is determined by Bozkurt et al. [22].

(iii) If $\flat_G = 0,$ we have the rectifying-type curves with Frenet-type frame in Myller configuration for three-dimensional Euclidean space which is determined by Macsim et al. [11].

Now, let us give a classification with respect to curvatures as follows:

$$\begin{aligned}\frac{d}{ds}\left(\overline{r}(s)-\eta(s)\overline{\zeta}_1(s)-\omega(s)\overline{\zeta}_3(s)\right) &= \rho_1(s)\overline{\zeta}_1(s)+\rho_2(s)\overline{\zeta}_2(s)+\rho_3(s)\overline{\zeta}_3(s)\\ &\quad-\left[\eta'\overline{\zeta}_1(s)+\eta(s)\left(\left(\mathcal{K}_1(s)-\rho_3(s)\flat_G\right)\overline{\zeta}_2(s)+\rho_2(s)\flat_G\overline{\zeta}_3(s)\right)\right]\\ &\quad-\omega'(s)\overline{\zeta}_1(s)-\omega(s)\left(-\rho_2(s)\flat_G\overline{\zeta}_3(s)+\left(-\mathcal{K}_2(s)+\rho_1(s)\flat_G\right)\overline{\zeta}_2(s)\right)\\ &=\left(\rho_1(s)-\eta'(s)+\omega(s)\rho_2(s)\flat_G\right)\overline{\zeta}_1\\ &\quad+\left(\rho_2(s)-\eta(s)\left(\mathcal{K}_1(s)-\rho_3(s)\flat_G\right)-\omega(s)\left(-\mathcal{K}_2(s)+\rho_1(s)\flat_G\right)\right)\overline{\zeta}_2\\ &\quad+\left(\rho_3(s)-\eta(s)\rho_2(s)\flat_G-\omega'(s)\right)\overline{\zeta}_3\\ &=0.\end{aligned} \tag{38}$$

If $\rho_2(s)=0$, $\mathcal{K}_2(s)-\rho_1(s)\flat_G\neq 0$, and $\mathcal{K}_1(s)-\rho_3(s)\flat_G\neq 0$, we get the following classification with respect to the curvatures:

$$\frac{\eta(s)}{\omega(s)}=\frac{\mathcal{K}_2(s)-\rho_1(s)\flat_G}{\mathcal{K}_1(s)-\rho_3(s)\flat_G}.$$

Theorem 8. *Let $\overline{r}(s)\ :\ I\ \rightarrow\ G$ be a curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations. Then, $\overline{r}(s)$ is congruent to a rectifying-type curve if and only if the following equation holds:*

$$\begin{aligned}\left[\rho_1(s)-\eta'(s)+\omega(s)\rho_2(s)\flat_G\right]\overline{\zeta}_1+\left[\rho_2(s)-\eta(s)\left(\mathcal{K}_1(s)-\rho_3(s)\flat_G\right)-\omega(s)\left(-\mathcal{K}_2(s)+\rho_1(s)\flat_G\right)\right]\overline{\zeta}_2\\ +\left[\rho_3(s)-\eta(s)\rho_2(s)\flat_G-\omega'(s)\right]\overline{\zeta}_3=0.\end{aligned}$$

Proof. By using Equation (38), we can complete the proof easily. ☐

Theorem 9. *Let $\overline{r}(s)\ :\ I\ \rightarrow\ G$ be a rectifying-type curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations. Then, the following holds:*

$$\overline{r}(s)=\delta\sec(t)\psi(t),$$

where $\delta\in\mathbb{R}^+$ and $\psi=\psi(t)$ is a constant speed curve on the sphere $\mathbb{S}^2$.

Proof. Let us denote the distance function of the rectifying-type curve $\overline{r}(s)$ by $\varrho(s)$. That is, $\varrho(s)=||\overline{r}(s)||$. Also, let us determine a unit speed curve as follows:

$$\begin{aligned}\psi(s)&=\frac{\overline{r}(s)}{||\overline{r}(s)||}\\ &=\frac{\overline{r}(s)}{\varrho(s)}.\end{aligned} \tag{39}$$

Since the curve $\overline{r}(s)$ is a rectifying-type curve and the curve $\psi(s)$ is a unit speed curve, we get

$$\varrho(s)=\sqrt{\eta^2(s)+\omega^2(s)}\quad\text{and}\quad ||\psi(s)||=1.$$

In that case, we can write the following by using Equations (33a) and (33c):

$$\begin{aligned}\varrho'(s)&=\frac{\eta'(s)\eta(s)+\omega'(s)\omega(s)}{\sqrt{\eta^2(s)+\omega^2(s)}}\\ &=\frac{\eta(s)\left(\rho_1(s)+\omega(s)\rho_2(s)\flat_G\right)+\omega(s)\left(\rho_3(s)-\eta(s)\rho_2(s)\flat_G\right)}{\sqrt{\eta^2(s)+\omega^2(s)}}.\end{aligned}$$

Since the curve $\psi(s)$ is a unit speed curve, we have $\langle \psi'(s), \psi(s) \rangle = 0$. By using Equation (39), we can write $\overline{r}(s) = \psi(s)\varrho(s)$. Then, by taking the derivative of the equation $\overline{r}(s) = \psi(s)\varrho(s)$, we get

$$\frac{d\overline{r}(s)}{ds} = \psi'(s)\varrho(s) + \psi(s)\varrho'(s).$$

Now, let us denote the speed of the curve $\psi(s)$ by $v(s) = ||\psi'(s)||$, and since $\left|\left|\frac{d\overline{r}(s)}{ds}\right|\right| = 1$, we obtain

$$\left(\varrho'(s)\right)^2 + \varrho^2(s)v^2 = 1.$$

Also, we can write

$$v^2 = \frac{1 - \left(\varrho'(s)\right)^2}{\varrho^2(s)}.$$

Then, by using Equations (33a) and (33c) if $\rho_2(s) = 0$, we have

$$v = \frac{|\eta(s)\rho_3(s) + \omega(s)\rho_1(s)|}{\eta^2(s) + \omega^2(s)}.$$

Additionally, if $\rho_1(s)$ and $\rho_3(s)$ are constants and $\rho_2(s) = 0$, we can express the $\eta(s)$ and $\omega(s)$ as follows:

$$\eta(s) = \phi_1 s + \gamma_1 \quad \text{and} \quad \omega(s) = \phi_2 s + \gamma_2,$$

where $\rho_1(s) = \phi_1$ and $\rho_3(s) = \phi_2$. Then, we have

$$v = \frac{|\left(\phi_1 s + \gamma_1\right)\rho_3(s) + \left(\phi_2 s + \gamma_2\right)\rho_1(s)|}{\left(\phi_1 s + \gamma_1\right)^2 + \left(\phi_2 s + \gamma_2\right)^2}.$$

Also, we express:

$$v = \frac{\Omega}{s^2 + 2\left(\phi_1\gamma_1 + \phi_2\gamma_2\right)s + \gamma_1^2 + \gamma_2^2}.$$

Therefore, we can write the following equation:

$$v = \frac{\Omega}{\left(s + \phi_1\gamma_1 + \phi_2\gamma_2\right)^2 + \gamma_1^2 + \gamma_2^2 - \left(\phi_1\gamma_1 + \phi_2\gamma_2\right)^2}.$$

Now, we want to use a translation as $v = \frac{\Omega}{s^2+\delta^2}$ where $\delta > 0$. Then, let us denote:

$$t = \frac{\delta}{\Omega}\int_0^s v(u)du = \frac{\delta}{\Omega}\int_0^s \frac{\Omega}{u^2+\delta^2}du = \frac{\delta}{\Omega}\frac{\Omega}{\delta}\arctan\left(\frac{s}{d}\right) = \arctan\left(\frac{s}{d}\right).$$

Moreover, we can write

$$s(t) = \delta\arctan(t),$$

and

$$\begin{aligned}
\overline{r}(t) &= \overline{r}(s(t)) \\
&= \varrho(s(t))\psi(s(t)) \\
&= \sqrt{s^2(t) + \delta^2}\psi(s(t)) \\
&= \sqrt{\delta^2\tan^2(t) + \delta^2\psi(s(t))} \\
&= \delta\sec(t)\psi(s(t)) \\
&= \delta\sec(t)\psi(t).
\end{aligned}$$

In addition to these, since $\psi'(t) = \psi'(s)\frac{ds}{dt}$, then the following holds:

$$||\psi'(t)|| = ||\psi'(s)||\frac{ds}{dt} = v\frac{ds}{dt} = \frac{\Omega}{\delta}.$$

□

6 | CONCLUSION

In this paper, we introduced the generalized Frenet-type frame in three-dimensional Lie groups with Myller configurations and some special cases. Then, we constructed derivative formulas by Lie curvature. Also, we determined some special type curves such as osculating-type and rectifying-type curves with Frenet-type frame in three-dimensional Lie groups with Myller configurations. From the natural structure of Myller configuration, we get a new generalization for the moving frames. The geometrical theory of versor fields along a curve with Frenet-type frame in three-dimensional Lie groups with Myller configurations is a generalization of the geometry of the usual theory of curves in classical Euclidean space, in classical three-dimensional Lie groups for Euclidean space, and in Myller configuration for Euclidean space. Also, we obtained some special curves such as osculating-type and rectifying-type curves in three-dimensional Lie groups with Myller configuration. Due to this particular relation, the osculating-type and rectifying-type curves with Frenet-type frame in three-dimensional Lie groups with Myller configurations include some special cases for osculating and rectifying curves in different spaces.

AUTHOR CONTRIBUTIONS

Zehra İşbilir: Writing—review and editing; resources; methodology; investigation; writing—original draft; conceptualization; validation; visualization; software; formal analysis; data curation. **Bahar Doğan Yazıcı:** Conceptualization; investigation; writing—original draft; methodology; validation; visualization; writing—review and editing; resources; formal analysis; software; data curation. **Murat Tosun:** Supervision; resources; methodology; validation; visualization; writing—review and editing; writing—original draft; investigation; conceptualization; project administration; formal analysis; data curation; software.

ACKNOWLEDGEMENTS

We would like to thank editors and anonymous referees for their valuable comments and careful reading.

CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interest.

ORCID

Zehra İşbilir https://orcid.org/0000-0001-5414-5887
Bahar Doğan Yazıcı https://orcid.org/0000-0001-5690-4840
Murat Tosun https://orcid.org/0000-0002-4888-1412

REFERENCES

1. F. Frenet, *Sur les courbes à double courbure*, J. Math. Pures Appl. (1852), 437–447.
2. J. A. Serret, *Sur quelques formules relatives à la théorie des courbes à double courbure*, J. Math. Pures Appl. (1851), 193–207.
3. K. İlarslan and E. Nesović, *Some characterizations of osculating curves in the Euclidean spaces*, Demonstr. Math. **41** (2008), no. 4, 931–940.
4. B. Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane?* Amer. Math. Monthly **110** (2003), no. 2, 147–152.
5. K. İlarslan and E. Nesović, *Some characterizations of rectifying curves in Euclidean 4-spaces E^4*, Turk. J Math. **32** (2008), 21–30.
6. K. İlarslan and E. Nesović, *The first kind and the second kind osculating curves in Minkowski space-time*, Compt. Rend Acad. Bulg. Sci. **62** (2009), no. 6, 677–686.
7. K. İlarslan, *Spacelike normal curves in Minkowski space E_1^3*, Turk. J. Math. **29** (2005), no. 2, 53–63.
8. K. İlarslan and E. Nesović, *Spacelike and timelike normal curves in Minkowski space-time*, Publ. de l'Institut Math. **85** (2009), no. 99, 111–118.
9. R. Miron, *Geometria unor configuraţii Myller*, Analele Sţ Univ. Iaşi. **VI** (1960), no. 3.

10. R. Miron, *The geometry of Myller configurations. Applications to theory of surfaces and nonholonomic manifolds*, Romanian Academy, Bucharest, 2010.
11. G. Macsim, A. Mihai, and A. Olteanu, *On rectifying-type curves in a Myller configuration*, Bull. Korean Math. Soc. **56** (2019), no. 2, 383–390.
12. B. Heroiu, *Versor fields along a curve in a four dimensional Lorentz space*, J. Adv. Math. Stud. **4** (2011), no. 1, 49–57.
13. G. Macsim, A. Mihai, and A. Olteanu, *Curves in a Myller configuration*, Int. Conf. Appl. Pure Math. (ICAPM), Iaşi, Institute of Mathematics, Romanian Academy, 2017.
14. G. F. Macsim, A. Mihai, and A. Olteanu, Special curves in a Myller configuration, *Proc. 16th Workshop Math. Comput. Sci. Tech. Educ. Department Math. Comput. Sci.*, Vol. **2**, Technical University of Civil Engineering Bucharest, Bucharest, 2019.
15. Z. İşbilir and M. Tosun, *On generalized osculating-type curves in Myller configuration*, An. St. Univ. Ovidius Constanta, Ser. Mat. **2** (2024).
16. B. Doğan Yazıcı and M. Tosun, *Quasi-type osculating curves in Myller configuration*, Filomat **38** (2024), no. 8, 2813–2820.
17. V. I. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier (Grenoble) **16** (1966), 319–361.
18. P. Crouch and L. F. Silva, *The dynamic interpolation problem: on Riemannian manifolds. Lie groups, and symmetric spaces*, J. Dyn. Control Syst. **1** (1995), 177–202.
19. B. Kolev, *Lie groups and mechanics: an introduction*, J. Nonlinear Math. Phys. **11** (2004), 480–498.
20. A. C. Çöken and Ü. Çiftçi, *A note on the geometry of Lie groups*, Nonlinear Anal. **68** (2008), 2013–2016.
21. Ü. Çiftçi, *A generalization of Lancret's theorem*, J. Geom. Phys. **59** (2009), 1597–1603.
22. Z. Bozkurt, İ. Gök, O. Z. Okuyucu, and N. Ekmekçi, *Characterizations of rectifying, normal and osculating curves in the three-dimensional compact Lie groups*, Life Sci. J. **10** (2013), 819–823.
23. O. Z. Okuyucu, Ö. G. Yıldız, and M. Tosun, *Spinor Frenet equations in the three-dimensional Lie groups*, Adv. Appl. Clifford Algebras **26** (2016), 1341–1348.
24. M. Mak, *Natural and conjugate mates of Frenet curves in three-dimensional Lie group*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **70** (2021), no. 1, 522–540.
25. B. Doğan Yazıcı, O. Z. Okuyucu, and M. Tosun, *Framed curves in three-dimensional Lie groups and a Berry phase model*, J. Geom. Phys. **182** (2022), 104682.
26. S. Karaku and F. K. Aksoyak, *Generalized bicomplex numbers and Lie groups*, Adv. Appl. Clifford Algebras **25** (2015), no. 4, 943–963.
27. O. Z. Okuyucu, İ. Gök, Y. Yayli, and N. Ekmekci, *Slant helices in three dimensional Lie groups*, Appl. Math. Comput. **221** (2013), 672–683.
28. O. Z. Okuyucu and M. Canbirdi, *Framed slant helices in Euclidean 3-space*, Adv. Differ. Equ. (2021), 504.
29. S. Özkaldi Karaku and Y. Yayli, *Tensor product surfaces in $\mathbb{R}^4$ and Lie groups*, Bull. Malaysian Math. Sci. Soc. **33** (2010), no. 1, 69–77.
30. G. Yildiz and O. Z. Okuyucu, *Inextensible flows of curves in Lie groups*, Caspian J. Math. Sci. **2** (2013), no. 1, 23–32.

How to cite this article: Z. İşbilir, B. Doğan Yazıcı, and M. Tosun, *On special curves in Lie groups with Myller configuration*, Math. Meth. Appl. Sci. **47** (2024), 11693–11708, DOI 10.1002/mma.10149.