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BICOMPLEX NUMBER AND TENSOR PRODUCT SURFACES IN $\mathbb{R}^4_2$

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We show that a hyperquadric M in $\mathbb{R}^4_2$ is a Lie group by using the bicomplex number product. For our purpose, we change the definition of tensor product. We define a new tensor product by considering the tensor product surface in the hyperquadric M. By using this new tensor product, we classify totally real tensor product surfaces and complex tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve. By means of the tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve, we determine a special subgroup of the Lie group M. Thus, we obtain the Lie group structure of tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve. Moreover, we obtain left invariant vector fields of these Lie groups. We consider the left invariant vector fields on these groups, which constitute a pseudo-Hermitian structure. By using this, we characterize these Lie groups as totally real or slant in $\mathbb{R}^4_2$.

1. Introduction

In the Euclidean space $\mathbb{E}^n$, the tensor product immersion of two immersions of a given Riemannian manifold was first defined and studied by Chen in [3]. In particular, the direct sum and tensor product of two immersions of two different Riemannian manifolds were defined by Decruyenaere et al. in [4] in the following way:

Let M and N be two differentiable manifolds and let $f\colon M \to \mathbb{E}^m$ and $h\colon N \to \mathbb{E}^n$ be two immersions. The direct sum and the tensor product are defined, respectively, as follows:

$$f \oplus h\colon M \times N \to \mathbb{E}^{m+n}, \qquad (f \oplus h)(p,q) = \big(f_1(p), \ldots, f_m(p), h_1(q), \ldots, h_n(q)\big),$$

and

$$f \otimes h\colon M \times N \to \mathbb{E}^{mn}, \qquad (f \otimes h)(p,q) = \big(f_1(p)h_1(q), \ldots, f_1(p)h_n(q), \ldots, f_m(p)h_n(q)\big).$$

Under certain conditions obtained in [4], the tensor product $f \otimes h$ is an immersion in the space $\mathbb{E}^{mn}$.

The simplest examples of tensor product immersions are tensor product surfaces. In the Euclidean space $\mathbb{E}^n$, the tensor product surfaces of two Euclidean plane curves and the tensor product surfaces of a Euclidean space curve and a Euclidean plane curve were investigated in [8] and [1], respectively. Moreover, in the semi-Euclidean space $\mathbb{E}^n_v$, the tensor product surfaces of two Lorentzian plane curves and the tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve were studied in [9] and [10], respectively. Furthermore, the tensor product surfaces of a Lorentzian space curve and a Euclidean plane curve and the tensor product surfaces of a Euclidean space curve and a Lorentzian plane curve were studied in [5] and [6], respectively.

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It is often quite difficult to decide if a manifold is parallelizable. S^n is parallelizable if and only if $n = 1, 3, 7$. Is it possible to make a parallelization of any surface? The answer is positive. If M is a Lie group, then M is parallelizable.

In [12], it was shown that a hyperquadric M in $\mathbb{R}^4$ is a Lie group by using the bicomplex number product. In the same paper, the Lie group structure of tensor product surfaces of Euclidean plane curves was also obtained.

In the present paper, we obtain the Lie group structure of a special hypersurface in $\mathbb{R}^4_2$. By changing the rule of tensor multiplication in $\mathbb{R}^4_2$ given in [3], we give a new definition of tensor product. As a result, the tensor product surface is obtained as a subset of the hyperquadric M. Hence, we investigate the tensor product surface as a Lie group. If we change the definition of tensor product given above as

$$(\alpha \otimes \beta)(t,s) = \big(\alpha_1(t)\beta_1(s), \alpha_1(t)\beta_2(s), -\alpha_2(t)\beta_2(s), \alpha_2(t)\beta_1(s)\big),$$

then we can easily obtain the same results as in [10]. By using the new tensor product, we classify totally real tensor product surfaces and complex tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve. Furthermore, we give some theorems for tensor product surfaces to be Lie groups and one-parameter Lie subgroups. Finally, we give necessary conditions for the Lie group structures of tensor product surfaces to be totally real or slant in $\mathbb{R}^4_2$.

At the beginning, we recall the notion of bicomplex numbers.

2. Preliminaries

A bicomplex number is defined by a basis $\{1, i, j, ij\}$, where i, j, and ij satisfy the conditions $i^2 = -1$, $j^2 = -1$, and $ij = ji$. Thus, any bicomplex number x can be expressed as $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ $\forall x_1, x_2, x_3, x_4 \in \mathbb{R}$. We denote the set of bicomplex numbers by C_2. For any $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ and $y = y_1 1 + y_2 i + y_3 j + y_4 ij$ in C_2, the addition of bicomplex numbers is defined as follows:

$$x + y = (x_1 + y_1)\,1 + (x_2 + y_2)\,i + (x_3 + y_3)\,j + (x_4 + y_4)\,ij.$$

The multiplication of a bicomplex number $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ by a real scalar λ is defined as

$$\lambda x = \lambda x_1 1 + \lambda x_2 i + \lambda x_3 j + \lambda x_4 ij.$$

With this addition and scalar multiplication operations, C_2 is a real vector space.

The bicomplex number product, denoted by $\times$, over the set of bicomplex numbers C_2 is given by the following table:

$\times$	1	i	j	ij
1	1	i	j	ij
i	i	-1	ij	$-j$
j	j	ij	-1	$-i$
ij	ij	$-j$	$-i$	1

The vector space C_2 equipped with bicomplex product $\times$ is a real algebra [13].

Since the bicomplex number algebra is associative, it can be considered in terms of matrices. Consider the set of matrices

$$Q = \left\{ \begin{bmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix} : x_i \in \mathbb{R},\ 1 \le i \le 4 \right\}.$$

The set Q equipped with matrix addition and scalar matrix multiplication is a real vector space. Furthermore, the vector space equipped with matrix product is an algebra.

The transformation

$$h\colon C_2 \to Q$$

given by

$$h\left(x = x_1 1 + x_2 i + x_3 j + x_4 ij\right) = \begin{bmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix}$$

is one-to-one and onto. Moreover, for any $x, y \in C_2$ and $\lambda \in \mathbb{R}$, we have

$$h\left(x + y\right) = h\left(x\right) \oplus h\left(y\right),$$

$$h\left(\lambda x\right) = \lambda h\left(x\right),$$

$$h\left(x \times y\right) = h\left(x\right) h\left(y\right).$$

Thus, the algebras C_2 and Q are isomorphic.

For further bicomplex number concepts, see [13].

3. Tensor Product Surfaces of a Lorentzian Plane Curve and a Euclidean Plane Curve

In this section, we change the definition of tensor product as follows:

Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1(+-)$ and $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be a Lorentzian plane curve and a Euclidean plane curve, respectively. We set $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ and $\beta(s) = (\beta_1(s), \beta_2(s))$. Let us define their tensor product as

$$\begin{aligned} f &= \alpha \otimes \beta\colon \mathbb{R}^2 \to \mathbb{R}^4_2(++--), \\ f(t,s) &= \big(\alpha_1(t)\beta_1(s), \alpha_1(t)\beta_2(s), -\alpha_2(t)\beta_2(s), \alpha_2(t)\beta_1(s)\big). \end{aligned} \tag{3.1}$$

By using Eq. (3.1), one can easily compute the canonical tangent vectors of $f(t,s)$ as follows:

$$\frac{\partial f}{\partial t} = \left(\alpha_1'(t)\beta_1(s), \alpha_1'(t)\beta_2(s), -\alpha_2'(t)\beta_2(s), \alpha_2'(t)\beta_1(s)\right),$$
$$\frac{\partial f}{\partial s} = \left(\alpha_1(t)\beta_1'(s), \alpha_1(t)\beta_2'(s), -\alpha_2(t)\beta_2'(s), \alpha_2(t)\beta_1'(s)\right), \tag{3.2}$$

where α' is the derivative of α.

In what follows, we assume that α is a spacelike or a timelike curve with a spacelike or a timelike position vector and that β is a regular curve. We also assume that the tensor product surface $f(t,s)$ is a regular surface, i.e., $g_{11}g_{22} - g_{12}^2 \neq 0$.

Hence, relations (3.1) and (3.2) imply that the coefficients of the pseudo-Riemannian metric induced on $f(t,s)$ by the pseudo-Euclidean metric g of $\mathbb{R}^4_2$ given by $g = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$ have the form

$$g_{11} = g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) = g_1\left(\alpha', \alpha'\right) g_2\left(\beta, \beta\right),$$

$$g_{12} = g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) = g_1\left(\alpha, \alpha'\right) g_2\left(\beta, \beta'\right),$$

$$g_{22} = g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right) = g_1\left(\alpha, \alpha\right) g_2\left(\beta', \beta'\right),$$

where $g_1 = dx_1^2 - dx_2^2$ and $g_2 = dx_1^2 + dx_2^2$ are the metrics of $\mathbb{R}^2_1$ and $\mathbb{R}^2$, respectively. Consequently, an orthonormal basis for the tangent space of $f(t,s)$ is given by

$$e_1 = \frac{1}{\sqrt{|g_{11}|}}\frac{\partial f}{\partial t},$$

$$e_2 = \frac{1}{\sqrt{\left|g_{11}\left(g_{11}g_{22} - g_{12}^2\right)\right|}}\left(g_{11}\frac{\partial f}{\partial s} - g_{12}\frac{\partial f}{\partial t}\right).$$

Recall that the mean curvature vector field H is defined as follows:

$$H = \frac{1}{2}\left(\varepsilon_1 h(e_1, e_1) + \varepsilon_2 h(e_2, e_2)\right),$$

where h is the second fundamental form of $\alpha \otimes \beta$ and $\varepsilon_i = g(e_i, e_i)$, $i = 1, 2$. In particular, by the Beltrami formula, we have

$$H = -\frac{1}{2}\Delta f.$$

Next, recall that a surface M in $\mathbb{R}^4_2$ is said to be minimal if its mean curvature vector field H vanishes identically.

A basis of the normal space of $f(t,s)$ can be calculated as follows: Let $J_1\colon \mathbb{R}^2_1 \to \mathbb{R}^2_1$ and $J_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the following maps:

$$J_1(x,y) = (y,x),$$

$$J_2(x,y) = (-y,x).$$

Note that $g_1\left(X, J_1(X)\right) = 0$ for $X \in \mathbb{R}^2_1$ and $g_2\left(Y, J_2(Y)\right) = 0$ for $Y \in \mathbb{R}^2$.

Then the normal space is spanned by

$$n_1(t,s) = J_1(\alpha(t)) \otimes J_2\left(\beta(s)\right) = (\alpha_2(t), \alpha_1(t)) \otimes (-\beta_2(s), \beta_1(s))$$

$$= \Big(-\alpha_2(t)\beta_2(s), \alpha_2(t)\beta_1(s), -\alpha_1(t)\beta_1(s), -\alpha_1(t)\beta_2(s)\Big),$$

$$n_2(t,s) = J_1(\alpha'(t)) \otimes J_2(\beta'(s)) = (\alpha_2'(t), \alpha_1'(t)) \otimes (-\beta_2'(s), \beta_1'(s))$$

$$= \Big(-\alpha_2'(t)\beta_2'(s), \alpha_2'(t)\beta_1'(s), -\alpha_1'(t)\beta_1'(s), -\alpha_1'(t)\beta_2'(s)\Big).$$

4. Totally Real and Complex Lorentzian Immersion and Slant Tensor Product Surface

Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1(+-)$ and $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be a Lorentzian plane curve and a Euclidean plane curve, respectively, and let $f = \alpha \otimes \beta$ be their tensor product. We consider the pseudo-Hermitian structure J given by

$$J\left(u,v,z,w\right) = \left(-v,u,-w,z\right), \quad u,v,z,w \in \mathbb{R}.$$

In the next theorem, by using the new product, we classify totally real tensor product surfaces in the semi-Euclidean space $\mathbb{R}^4_2$, i.e., at each point, the pseudo-Hermitian structure J transforms the tangent space to the surface into the normal space.

Theorem 1. *The tensor product immersion $f = \alpha \otimes \beta$ of a Lorentzian plane curve and a Euclidean plane curve is a totally real Lorentzian immersion with respect to the pseudo-Hermitian structure J on $\mathbb{R}^4_2$ if and only if α is a Lorentzian circle centered at 0 or β is a straight line passing through the origin.*

Proof. Im f is a totally real surface if and only if $J\left(\frac{\partial f}{\partial t}\right)$ is orthogonal to $\frac{\partial f}{\partial s}$ and $J\left(\frac{\partial f}{\partial s}\right)$ is orthogonal to $\frac{\partial f}{\partial t}$.

We have

$$J\left(\frac{\partial f}{\partial t}\right) = \Big(-\alpha_1'(t)\beta_2(s), \alpha_1'(t)\beta_1(s), -\alpha_2'(t)\beta_1(s), -\alpha_2'(t)\beta_2(s)\Big).$$

By a straightforward calculation, we establish that

$$g\left(J\left(\frac{\partial f}{\partial t}\right), \frac{\partial f}{\partial s}\right) = -g\left(J\left(\frac{\partial f}{\partial s}\right), \frac{\partial f}{\partial t}\right),$$

$$g\left(J\left(\frac{\partial f}{\partial t}\right), \frac{\partial f}{\partial s}\right) = 0$$

if and only if $\left(\alpha_1\alpha_1' - \alpha_2\alpha_2'\right) = 0$ or $\left(\beta_1\beta_2' - \beta_1'\beta_2\right) = 0$. Integrating these equations, we find that either β is a straight line passing through the origin or α is a Lorentzian circle centered at 0.

Theorem 1 is proved.

Theorem 2. *The tensor product immersion $f = \alpha \otimes \beta$ of a Lorentzian plane curve and a Euclidean plane curve is a complex Lorentzian immersion with respect to the pseudo-Hermitian structure J on $\mathbb{R}^4_2$ if and only if α is a straight line passing through the origin and β is a Euclidean plane curve.*

Proof. By definition, the following equations are satisfied:

$$g\left(J\left(\frac{\partial f}{\partial t}\right), n_i\right) = 0, \qquad g\left(J\left(\frac{\partial f}{\partial s}\right), n_i\right) = 0, \quad i = 1, 2. \tag{4.1}$$

We have

$$J\left(\frac{\partial f}{\partial t}\right) = \left(-\alpha_1'(t)\beta_2(s), \alpha_1'(t)\beta_1(s), -\alpha_2'(t)\beta_1(s), -\alpha_2'(t)\beta_2(s)\right),$$

$$J\left(\frac{\partial f}{\partial s}\right) = \left(-\alpha_1(t)\beta_2'(s), \alpha_1(t)\beta_1'(s), -\alpha_2(t)\beta_1'(s), -\alpha_2(t)\beta_2'(s)\right).$$

By a straightforward calculation, we obtain

$$g\left(J\left(\frac{\partial f}{\partial t}\right), n_2\right) = g\left(J\left(\frac{\partial f}{\partial s}\right), n_1\right) = 0,$$

$$g\left(J\left(\frac{\partial f}{\partial t}\right), n_1\right) = \left(\alpha_1(t)\alpha_2'(t) - \alpha_1'(t)\alpha_2(t)\right)\left(\beta_1^2(s) + \beta_2^2(s)\right), \tag{4.2}$$

$$g\left(J\left(\frac{\partial f}{\partial s}\right), n_2\right) = \left(\alpha_1'(t)\alpha_2(t) - \alpha_1(t)\alpha_2'(t)\right)\left(\beta_1'^2(s) + \beta_2'^2(s)\right).$$

Using Eqs. (4.1) and (4.2), we get

$$\alpha_1'(t)\alpha_2(t) - \alpha_1(t)\alpha_2'(t) = 0.$$

This implies that α is straight line passing through the origin.

Theorem 2 is proved.

Recall the definition of a slant surface with respect to the pseudo-Hermitian structure J on $\mathbb{R}^4_2$. Let M be a surface with respect to the pseudo-Hermitian structure J on $\mathbb{R}^4_2$. One says that M is a proper slant surface if [3]

$$g(J(e_1), e_2) = \lambda, \quad \lambda \in \mathbb{R},$$

along M for a given orthonormal basis $\{e_1, e_2\}$ of T_pM $(p \in M)$ that is independent of the choice of $\{e_1, e_2\}$.

Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1(+-)$ and $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be a Lorentzian plane curve and a Euclidean plane curve, respectively. We consider polar coordinates on α and β. Then

$$\alpha(t) = \rho_1(t)\,(\cosh t, \sinh t)\,,$$

$$\beta(s) = \rho_2(s)\,(\cos s, \sin s)\,.$$

A straightforward computation leads to

$$g\left(J\left(e_1\right), e_2\right) = \frac{\rho_1'\rho_2}{\sqrt{\left|\left(\rho_1'^2 - \rho_1^2\right)\left(\rho_2'^2 + \rho_2^2\right) - \rho_1'^2\rho_2'^2\right|}}.$$

If $\rho_2 = \text{const},$ then $\rho_1 = a_1 e^{b_1 t},$ $a_1 \in \mathbb{R},$ $b_1 \in \mathbb{R}.$ Hence, α is a hyperbolic spiral and β is a circle centered at the origin. If $\rho_2 \neq \text{const},$ then we set

$$\frac{\rho_k}{\rho_k'} = c_k, \quad k = 1, 2.$$

Then

$$g\left(J\left(e_1\right), e_2\right) = \frac{c_2}{\sqrt{\left|\left(c_2^2 + 1\right)\left(1 - c_1^2\right) - 1\right|}}.$$

Therefore, $\operatorname{Im} f$ is a proper slant surface if and only if

$$\frac{c_2^2}{\left(c_2^2 + 1\right)\left(1 - c_1^2\right) - 1} = \lambda^2.$$

Hence,

$$\frac{c_2^2 + \lambda^2}{c_2^2 + 1} = \lambda^2\left(1 - c_1^2\right).$$

This means that $c_1(t)$ and $c_2(s)$ must be constant functions, which implies that $\rho_1(t) = a_1 e^{b_1 t}$ and $\rho_2(s) = a_2 e^{b_2 s},$ $a_1, b_1, a_2, b_2 \in \mathbb{R}.$ Consequently, α is a hyperbolic spiral and β is a logarithmic spiral. Thus, we have proved the following theorem:

Theorem 3. *The tensor product immersion $f = \alpha \otimes \beta$ of a Lorentzian plane curve and a Euclidean plane curve is a slant surface with respect to the pseudo-Hermitian structure J on $\mathbb{R}^4_2$ if and only if α is a hyperbolic spiral and β is either a circle centered at O or a spiral curve.*

5. Lie Groups and a Special Subgroup

In this section, we deal with the hyperquadric

$$M = \left\{x = \left(x_1, x_2, x_3, x_4\right) :\ x_1x_3 + x_2x_4 = 0,\ g\left(x, x\right) \neq 0\right\},$$

$$M = \{x = (x_1, x_2, x_3, x_4) : x_1 x_3 + x_2 x_4 = 0,\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \neq 0\}.$$

We consider M as a set of bicomplex numbers:

$$M = \{x = x_1 1 + x_2 i + x_3 j + x_4 ij \in C_2 \colon x_1 x_3 + x_2 x_4 = 0,\ \ g(x, x) \neq 0\}.$$

The components of M are easily obtained by representing the bicomplex number multiplication in the matrix form:

$$\widetilde{M} = \left\{ x = \begin{bmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix} : x_1 x_3 + x_2 x_4 = 0,\ \ g(x, x) \neq 0 \right\}.$$

Theorem 4. *The set M equipped with bicomplex number product is a Lie group.*

Proof. $\widetilde{M}$ is a differentiable manifold and, at the same time, a group with group operation given by the matrix multiplication. The group function

$$.\colon \widetilde{M} \times \widetilde{M} \to \widetilde{M}$$

defined by $(x, y) \to x.y$ is differentiable. Therefore, $(M, \times)$ can be made a Lie group so that h is a isomorphism.

Theorem 4 is proved.

Consider the group M_1 of all unit bicomplex numbers $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ on M with the group operation of bicomplex multiplication, namely,

$$M_1 = \{x \in M \colon g(x, x) = 1\},$$

$$M_1 = \{x \in M \colon x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1\}.$$

Lemma 1. *M_1 is a 2-dimensional Lie subgroup of M.*

6. Lie Algebra of the Lie Group M and M_1

M is a Lie group of dimension three. Let us find its Lie algebra. Thus, let

$$\alpha(t) = \alpha_1(t)1 + \alpha_2(t)i + \alpha_3(t)j + \alpha_4(t)ij$$

be a curve on M such that $\alpha(0) = 1$, i.e., $\alpha_1(0) = 1$ and $\alpha_m(0) = 0$ for $m = 2, 3, 4$. The differentiation of the equation

$$\alpha_1(t)\alpha_3(t) + \alpha_2(t)\alpha_4(t) = 0$$

yields the equation

$$\alpha_1'(t)\alpha_3(t) + \alpha_1(t)\alpha_3'(t) + \alpha_2'(t)\alpha_4(t) + \alpha_2(t)\alpha_4'(t) = 0.$$

Substituting $t = 0$, we obtain $\alpha_3'(0) = 0$. The Lie algebra thus consists of vectors of the form

$$\zeta = \zeta_m \left(\frac{\partial}{\partial \alpha_m} \right) \Big|_{\alpha=1},$$

where $m = 1, 2, 4$. The vector ζ can formally be written as $\zeta = \zeta_1 + \zeta_2 i + \zeta_4 ij$. Let us find the left invariant vector field X on M for which $X\big|_{\alpha=1} = \zeta$. Let $\beta(t)$ be a curve on M such that $\beta(0) = 1$ and $\beta'(0) = \zeta$. Then $L_x(\beta(t)) = x\beta(t)$ is the left translation of the curve $\beta(t)$ by a bicomplex number x, and its tangent vector is $x\beta'(0) = x\zeta$. In particular, let X_m denote the left invariant vector fields on M for which

$$X_m \,|_{\alpha=1} = \frac{\partial}{\partial \alpha_m} \Big|_{\alpha=1},$$

where $m = 1, 2, 4$. These three vector fields are represented in the bicomplex notation at the point $\alpha = 1$ by the bicomplex units 1, i, and ij. For the components of these vector fields at the point $x = x_1 1 + x_2 i + x_3 j + x_4 ij$, we have $(X_1)_x = x1$, $(X_2)_x = xi$, $(X_3)_x = xij$, and

$$X_1 = (x_1, x_2, x_3, x_4),$$

$$X_2 = (-x_2, x_1, -x_4, x_3),$$

$$X_4 = (x_4, -x_3, -x_2, x_1),$$

where all partial derivatives are taken at the point x.

M_1 is a Lie group of dimension two. Its Lie algebra can easily be found, and

$$X_2 = (-x_2, x_1, -x_4, x_3),$$

$$X_4 = (x_4, -x_3, -x_2, x_1).$$

Theorem 5. *M is parallelizable.*

Proof. If we set

$$x_1 = \rho_1 \cos\phi,$$

$$x_2 = \rho_1 \sin\phi,$$

$$x_3 = \rho_2 \cos\theta,$$

$$x_4 = \rho_2 \sin\theta,$$

then the relation $x_1x_3 + x_2x_4 = 0$ yields

$$\cos\phi = \sin\theta, \qquad \sin\phi = -\cos\theta$$

or

$$\cos\phi = -\sin\theta, \qquad \sin\phi = \cos\theta.$$

Furthermore, it follows from the condition $x_1^2 + x_2^2 - x_3^2 - x_4^2 \neq 0$ that $\rho_1^2 - \rho_2^2 \neq 0$. We have the parametric representation of one component of M with vector position $r(\rho_1, \rho_2, \phi)$, where

$$r = (\rho_1 \cos\phi, \rho_1 \sin\phi, -\rho_2 \sin\phi, \rho_2 \cos\phi).$$

Hence, we have three vectors tangent to coordinate curves:

$$r_{\rho_1} = (\cos\phi, \sin\phi, 0, 0),$$

$$r_{\rho_2} = (0, 0, -\sin\phi, \cos\phi),$$

$$r_\phi = (-\rho_1 \sin\phi, \rho_1 \cos\phi, -\rho_2 \cos\phi, -\rho_2 \sin\phi).$$

These vectors are obviously orthogonal to each other and form a parallelization of M.

Theorem 5 is proved.

7. Tensor Product Surfaces and Lie Groups

In this section, by using the tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve, we define a special subgroup of the Lie group M. Thus, we obtain the Lie group structure of tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve. We also obtain the left invariant vector fields of these Lie groups.

Theorem 6. *Let $\alpha\colon \mathbb{R} \to \mathbb{R}^2_1$ be a hyperbolic spiral and let $\beta\colon \mathbb{R} \to \mathbb{R}^2$ be a spiral with the same parameter, i.e., $\alpha(t) = e^{at}(\cosh t, \sinh t)$ and $\beta(t) = e^{bt}(\cos t, \sin t)$, $a, b \in \mathbb{R}$. Then their tensor product is a one-parameter subgroup in the Lie group M.*

Proof. We obtain

$$\gamma(t) = \alpha(t) \otimes \beta(t) = e^{(a+b)t}(\cosh t \cos t, \cosh t \sin t, -\sinh t \sin t, \sinh t \cos t).$$

It is easy to see that

$$\gamma(t_1) \times \gamma(t_2) = \gamma(t_1 + t_2)$$

for all t_1 and t_2. Hence, $(\gamma(t), \times)$ is a one-parameter Lie subgroup of $(M, \times)$.

Theorem 6 is proved.

Corollary 1. *Let* $\alpha: \mathbb{R} \to \mathbb{R}_1^2$ *be a hyperbolic spiral and* $\beta: \mathbb{R} \to \mathbb{R}^2$ *be a circle centered at* O *with the same parameter, i.e.,* $\alpha(t) = e^{at}(\cosh t, \sinh t)$, $a \in \mathbb{R}$, *and* $\beta(t) = (\cos t, \sin t)$. *Then their tensor product is a one-parameter subgroup in the Lie group* M.

Proof. Taking $b = 0$ in Theorem 6, we establish that β is a circle centered at O. Then their tensor product is a one-parameter subgroup in the Lie group M.

Corollary 2. *Let* $\alpha: \mathbb{R} \to \mathbb{R}_1^2$ *be a Lorentzian circle centered at* O *and* $\beta: \mathbb{R} \to \mathbb{R}^2$ *be a circle centered at* O *with the same parameter, i.e.,* $\alpha(t) = (\cosh t, \sinh t)$ *and* $\beta(t) = (\cos t, \sin t)$. *Then their tensor product is a one-parameter subgroup in the Lie group* M_1.

Proof. Since $\|\alpha(t) \otimes \beta(t)\|_L = 1$, we have $\alpha(t) \otimes \beta(t) \subset M_1$. Taking $a = b = 0$ in Theorem 6, we establish that α is a Lorentzian circle centered at O and β is a circle centered at O. Then their tensor product is a one-parameter subgroup in the Lie group M_1.

Theorem 7. *Let* $\alpha: \mathbb{R} \to \mathbb{R}_1^2$ *be a Lorentzian circle centered at* O, *let* $\beta: \mathbb{R} \to \mathbb{R}^2$ *be a circle centered at* O *with the same parameter, i.e.,* $\alpha(t) = (\cosh t, \sinh t)$ *and* $\beta(t) = (\cos t, \sin t)$, *and let* $\gamma(t) = \alpha(t) \otimes \beta(t)$ *be their tensor product. Then the left invariant vector field on* $\gamma(t)$ *is* $X = X_2 + X_4$, *where* X_2 *and* X_4 *are the left invariant vector fields on* M_1.

Proof. Let us find the left invariant vector field on $\gamma(t)$ for the vector

$$u = \left.\frac{d}{dt}\right|_{e=0}.$$

$\eta(t) = (1, t, 0, t)$ is a curve with tangent vector u. Its image under L_g is the curve

$$L_g(\eta(t)) = g\eta(t) = (x_1 1 + x_2 i + x_3 j + x_4 ij) \times (1 + ti + tij)$$

$$= (x_1 - x_2 t + x_4 t) + i\,(x_1 t + x_2 - x_3 t) + j\,(-x_2 t + x_3 - x_4 t) + ij\,(x_1 t + x_3 t + x_4).$$

Its tangent vector is

$$L_g(\eta(t))(t) = (-x_2 + x_4) + i\,(x_1 - x_3) + j\,(-x_2 - x_4) + ij\,(x_1 + x_3).$$

For the left invariant vector X, we have

$$X = (-x_2 + x_4)\frac{\partial}{\partial x_1} + (x_1 - x_3)\frac{\partial}{\partial x_2} + (-x_2 - x_4)\frac{\partial}{\partial x_3} + (x_1 + x_3)\frac{\partial}{\partial x_4}.$$

Theorem 7 is proved.

Conclusion 1. *Let* $\alpha: \mathbb{R} \to \mathbb{R}_1^2$ *be a hyperbolic spiral (or a Lorentzian circle centered at* O*) and let* $\beta: \mathbb{R} \to \mathbb{R}^2$ *be a spiral (or a circle centered at* O*) with the same parameter. Then their tensor product is the maximal integral curve.*

We now want to classify these Lie groups as totally real or slant in the semi-Euclidean space $\mathbb{R}_2^4$. For this purpose, we consider the left invariant vector field on these groups, which constitute the pseudo-Hermitian structure given by $J = X_2$.

Corollary 3. *Let* $\alpha : \mathbb{R} \to \mathbb{R}^2_1$ *be a Lorentzian circle centered at* O, *let* $\beta : \mathbb{R} \to \mathbb{R}^2$ *be either a spiral or a circle centered at* O, *and let* $f = \alpha \otimes \beta$ *be their tensor product immersion. Then the Lie group* $f(t, s)$ *is a totally real Lorentzian immersion with respect to the pseudo-Hermitian structure* J.

Proof. We know from Theorem 1 that if α is a Lorentzian circle centered at O, then $f = \alpha \otimes \beta$ is a totally real surface with respect to the pseudo-Hermitian structure J.

Corollary 4. *Let* $\alpha : \mathbb{R} \to \mathbb{R}^2_1$ *be a hyperbolic spiral, let* $\beta : \mathbb{R} \to \mathbb{R}^2$ *be either a circle centered at* O *or a spiral, and let* $f = \alpha \otimes \beta$ *be their tensor product. Then the Lie group* $f(t, s)$ *is a proper slant surface with respect to the pseudo-Hermitian structure* J *on* $\mathbb{R}^4_2$.

Proof. We know from Theorem 3 that if α is a hyperbolic spiral and β is either a circle centered at O or a spiral curve, then $f = \alpha \otimes \beta$ is a proper slant surface with respect to the pseudo-Hermitian structure J.

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