Czechoslovak Mathematical Journal, 63 (138) (2013), 783–797

ON THE DIOPHANTINE EQUATION $x^2 - kxy + y^2 - 2^n = 0$

Refik Keskin, Esentepe, Zafer Şiar, Gülümbe, Olcay Karaatli, Esentepe

(Received May 22, 2012)

Abstract. In this study, we determine when the Diophantine equation $x^2-kxy+y^2-2^n=0$ has an infinite number of positive integer solutions x and y for $0 \leqslant n \leqslant 10$. Moreover, we give all positive integer solutions of the same equation for $0 \leqslant n \leqslant 10$ in terms of generalized Fibonacci sequence. Lastly, we formulate a conjecture related to the Diophantine equation $x^2 - kxy + y^2 - 2^n = 0$.

Keywords: Diophantine equation; Pell equation; generalized Fibonacci number; generalized Lucas number

MSC 2010: 11B37, 11B39, 11B50, 11B99

1. Introduction

In [8], the authors dealt with the equation

$$x^2 - kxy + y^2 + x = 0 \tag{1.1}$$

and they showed that it has no positive integer solutions x and y for $k > 3$ but it has an infinite number of positive integer solutions for $k = 3$. In [5], Keskin considered the Diophantine equations

$$x^2 - kxy + y^2 \pm x = 0, \tag{1.2}$$

$$x^2 - kxy - y^2 \pm y = 0, \tag{1.3}$$

and showed that when $k > 3$, $x^2 - kxy + y^2 + x = 0$ has no positive integer solutions but the equation $x^2 - kxy + y^2 - x = 0$ has positive integer solutions and equation (1.3) has positive integer solutions for $k > 1$.

In [15], Yuan and Hu determined when the following two equations

$$x^2 - kxy + y^2 + 2x = 0 \tag{1.4}$$

and

$$x^2 - kxy + y^2 + 4x = 0 \tag{1.5}$$

have an infinite number of positive integer solutions x and y. They showed that equation (1.4) has an infinite number of positive integer solutions if and only if $k = 3, 4$ and equation (1.5) has an infinite number of positive integer solutions if and only if $k = 3, 4, 6$.

In [6], the authors considered the equation

$$x^2 - kxy + y^2 + 2^r x = 0, \tag{1.6}$$

where k is a positive integer and r is a nonnegative integer. In order to determine when equation (1.6) has an infinite number of positive integer solutions, the authors investigated when the equation

$$x^2 - kxy + y^2 + 2^n = 0 \tag{1.7}$$

has an infinite number of positive integer solutions for nonnegative integer n. Also they found all positive integer solutions of equation (1.7) for $0 \leqslant n \leqslant 10$.

Let us consider the equation

$$x^2 - kxy + y^2 - 2^r x = 0, \tag{1.8}$$

where k is a positive integer and r is a nonnegative integer. In order to determine when equation (1.8) has an infinite number of positive integer solutions, it is sufficient to determine when the equation

$$x^2 - kxy + y^2 - 2^n = 0 \tag{1.9}$$

has an infinite number of positive integer solutions for nonnegative integer n. Now assume that equation (1.8) has positive integer solutions x and y. Then it follows that $x \mid y^2$ and thus $y^2 = xz$ for some positive integer z. A simple computation shows that $\gcd(x, z) = 2^j$ for some nonnegative integer j. Thus $x = 2^j a^2$ and $z = 2^j b^2$ for some positive integers a and b with $\gcd(a, b) = 1$. Then it follows that $y = 2^j ab$. Substituting these values of x and y into equation (1.8), we obtain

$$a^2 - kab + b^2 - 2^{r-j} = 0.$$

Therefore it is sufficient to know when $x^2 - kxy + y^2 - 2^{r-j} = 0$ has an infinite number of positive integer solutions for $0 \leqslant j \leqslant r$.

Now, we begin with some well known elementary properties of Pell equations. Let d be a positive integer which is not a perfect square and N be any nonzero fixed integer. Then the equation $x^2 - dy^2 = N$ is known as Pell equation. For $N = \pm 1$, the equation $x^2 - dy^2 = \pm 1$ is known as classical Pell equation. We use the notations (x, y) and $x + y\sqrt{d}$ interchangeably to denote solutions of the equation $x^2 - dy^2 = N$. If x and y are both positive, we say that $x + y\sqrt{d}$ is a positive solution to the equation $x^2 - dy^2 = N$. The least positive integer solution $x_1 + y_1\sqrt{d}$ to the equation $x^2 - dy^2 = N$ is called the fundamental solution to this equation. If $x_1 + y_1\sqrt{d}$ is the fundamental solution to the equation $x^2 - dy^2 = -1$, it is well known that $(x_1 + y_1\sqrt{d})^2$ is the fundamental solution to the equation $x^2 - dy^2 = 1$. Moreover, if $x_1 + y_1\sqrt{d}$ is the fundamental solution to the equation $x^2 - dy^2 = 1$, then all positive integer solutions to the equation $x^2 - dy^2 = 1$ are given by

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n \tag{1.10}$$

with $n \geqslant 1$. Also, the solutions (x_n, y_n) satisfy the following recurrence relations

$$x_{n+1} = 2x_1x_n - x_{n-1}, \tag{1.11}$$

and

$$y_{n+1} = 2x_1y_n - y_{n-1}. \tag{1.12}$$

For more information about Pell equations, one can consult [11], [13], and [2].

In section 2, we determine when equation (1.9) has an infinite number of positive integer solutions x and y for $0 \leqslant n \leqslant 10$. Then in section 3, we give all positive integer solutions to equation (1.9) for $0 \leqslant n \leqslant 10$.

2. Main theorems

In this section, we determine when equation (1.9) has an infinite number of positive integer solutions x and y for $0 \leqslant n \leqslant 10$. First, we give the following lemma and theorem without proof, which will be needed in the proof of the main theorems. The proof of the lemma can be found in [6], the theorem is given in [11].

Lemma 2.1. *Let $d > 2$. If $u_1 + v_1\sqrt{d}$ is the fundamental solution of the equation $u^2 - dv^2 = \pm 2$, then $\frac{1}{2}(u_1^2 + dv_1^2) + u_1v_1\sqrt{d}$ is the fundamental solution of the equation $x^2 - dy^2 = 1$.*

Theorem 2.1. *If x_1 and y_1 are natural numbers satisfying the inequality*

$$x_1 > \frac{y_1^2}{2} - 1 \tag{2.1}$$

and if $\alpha = x_1 + y_1\sqrt{d}$ is a solution of the equation $x^2 - dy^2 = 1$, then α is the fundamental solution of this equation.

Theorem 2.2. *The equation $x^2 - kxy + y^2 - 1 = 0$ has an infinite number of positive integer solutions x and y if and only if $k > 1$.*

Proof. By Theorem 3.10 given in [5], $x^2 - kxy + y^2 - 1 = 0$ has an infinite number of positive integer solutions x and y when $k > 3$. It is clear that the equation $x^2 - kxy + y^2 - 1 = 0$ has an infinite number of positive integer solutions x and y for $k = 2$. For $k = 3$, the equation $x^2 - 3xy + y^2 - 1 = 0$ has an infinite number of positive integer solutions $(x, y) = (F_{2n+2}, F_{2n})$ with $n \geqslant 1$ (see [5], Theorem 1.5). If $k = 1$, the equation becomes $x^2 - xy + y^2 - 1 = 0$, which implies that $(2x - y)^2 + 3y^2 = 4$. Therefore $x = 1$ and $y = 1$. This means that the equation $x^2 - kxy + y^2 - 1 = 0$ has not an infinite number of positive integer solutions x and y for $k = 1$. □

Theorem 2.3. *The equation $x^2 - kxy + y^2 - 2 = 0$ has no positive integer solutions x and y.*

Proof. Assume that $x^2 - kxy + y^2 - 2 = 0$ for some positive integers x and y. It is clear that x and y must be odd. Then it follows that k is even. Let $k = 2t$ for some positive integer t. Then $x^2 - kxy + y^2 - 2 = 0$ implies that $(x - ty)^2 - (t^2 - 1)y^2 = 2$. Let $u_1 + v_1\sqrt{t^2 - 1}$ be the fundamental solution of the equation $u^2 - (t^2 - 1)v^2 = 2$. Then from Lemma 2.1, it follows that $\frac{1}{2}(u_1^2 + (t^2 - 1)v_1^2) + u_1v_1\sqrt{t^2 - 1}$ is the fundamental solution of the equation $x^2 - (t^2 - 1)y^2 = 1$. For $t > 1$, since $(t, 1)$ is the fundamental solution of the equation $x^2 - (t^2 - 1)y^2 = 1$ by Theorem 2.1, we get $\frac{1}{2}(u_1^2 + (t^2 - 1)v_1^2) = t$ and $u_1v_1 = 1$. From this, it follows that $t = 2$. Substituting this value of t into $(x - ty)^2 - (t^2 - 1)y^2 = 2$, we get $(x - 3y)^2 - 3y^2 = 2$. But this equation has no positive integer solutions since $(x - 3y)^2 \equiv 2 \pmod 3$ which is impossible. □

Theorem 2.4. *The equation $x^2 - kxy + y^2 - 4 = 0$ has an infinite number of positive integer solutions x and y if and only if $k > 1$. Also, x and y may be odd only when $k = 2$.*

Proof. Assume that $x^2 - kxy + y^2 - 4 = 0$ for some positive integers x and y. If x is even, then y is even and thus $x = 2a$ and $y = 2b$ for some positive integers a and b. Then it follows that $a^2 - kab + b^2 - 1 = 0$, which implies that $k > 1$ by Theorem 2.2.

Now assume that x and y are odd. Then k is even and $4 \nmid k$. Therefore $k = 2t$ for some odd positive integer t. Completing the square gives $(x - ty)^2 - (t^2 - 1)y^2 = 4$. Since $4 \mid t^2 - 1$, it follows that $x - ty = 2m$ and thus $m^2 - \frac{1}{4}(t^2 - 1)y^2 = 1$. For $t > 1$, $(t, 2)$ is the fundamental solution of the equation $x^2 - \frac{1}{4}(t^2 - 1)y^2 = 1$ by Theorem 2.1. From equation (1.12), if (x_n, y_n) is the solution of the equation $x^2 - \frac{1}{4}(t^2 - 1)y^2 = 1$, then y_n is even. But this is impossible since y is odd. When $t = 1$, i.e., $k = 2$, it is clear that the equation $x^2 - kxy + y^2 - 4 = 0$ has an infinite number of positive integer solutions x and y. □

Theorem 2.5. *The equation $x^2 - kxy + y^2 - 8 = 0$ has an infinite number of positive integer solutions x and y if and only if $k = 6$.*

Proof. Assume that $x^2 - kxy + y^2 - 8 = 0$ for some positive integers x and y. If x is even, then y is even and thus $x = 2a$ and $y = 2b$ for some positive integers a and b. Thus we get $a^2 - kab + b^2 - 2 = 0$. By Theorem 2.3, this equation has no positive integer solutions. Now assume that x and y are odd. Then k is even and $4 \nmid k$. Thus $k = 2t$ for some odd positive integer t. Completing the square gives $(x - ty)^2 - (t^2 - 1)y^2 = 8$, which implies that $x - ty = 2m$ for some positive integer m. Thus we get $m^2 - \frac{1}{4}(t^2 - 1)y^2 = 2$. Let $d = \frac{1}{4}(t^2 - 1)$ and assume that $u_1 + v_1\sqrt{d}$ is the fundamental solution of the equation $u^2 - dv^2 = 2$. Then by Lemma 2.1, $\frac{1}{2}(u_1^2 + dv_1^2) + u_1v_1\sqrt{d}$ is the fundamental solution of the equation $x^2 - dy^2 = 1$. For $t > 1$, since $(t, 2)$ is the fundamental solution of the equation $x^2 - dy^2 = 1$ by Theorem 2.1, we get $\frac{1}{2}(u_1^2 + dv_1^2) + u_1v_1\sqrt{d} = t + 2\sqrt{d}$. From this, it follows that $u_1v_1 = 2$ and $u_1^2 + \frac{1}{4}(t^2 - 1)v_1^2 = 2t$. Solving these equations, we see that $t = 3$ and $t = 5$, and thus we get $k = 6$ and $k = 10$. But it can be seen that the equation $x^2 - kxy + y^2 - 8 = 0$ has no positive integer solutions for $k = 10$. □

The proofs of the following theorems are similar to those of the above theorems and therefore we omit them.

Theorem 2.6. *The equation $x^2 - kxy + y^2 - 16 = 0$ has an infinite number of positive integer solutions x and y if and only if $k > 1$. Also, x and y may be odd only when $k = 2, 14$.*

Theorem 2.7. *The equation $x^2 - kxy + y^2 - 32 = 0$ has an infinite number of positive integer solutions x and y if and only if $k = 6, 30$.*

Theorem 2.8. *The equation $x^2 - kxy + y^2 - 64 = 0$ has an infinite number of positive integer solutions x and y if and only if $k > 1$. Also, x and y may be odd only when $k = 2, 18, 62$.*

Now, we consider the equation

$$x^2 - dy^2 = N, \tag{2.2}$$

where $N \neq 0$ and d is a positive integer which is not a perfect square. If $u^2 - dv^2 = N$, then we say that $\alpha = u + v\sqrt{d}$ is a solution to equation (2.2). Let α_1 and α_2 be any two solutions to equation (2.2). Then α_1 and α_2 are called associated solutions if there exists a solution α to $x^2 - dy^2 = 1$ such that

$$\alpha_1 = \alpha\alpha_2.$$

The set of all solutions associated with each other forms a class of solutions to equation (2.2). If K is a class, then $\overline{K} = \{u - v\sqrt{d};\ u + v\sqrt{d} \in K\}$ is also a class. We say that the class K is ambiguous if $K = \overline{K}$.

Now, we give the following definition and theorem from [1].

Definition 2.1. Assume that $N > 0$. Let $u_0 + v_0\sqrt{d}$ be a solution to equation (2.2) given in a class K such that u_0 is the least positive value of u which occurs in K. If K is not ambiguous, then the number v_0 is uniquely determined. If K is ambiguous we get an uniquely determined v_0 by prescribing that $v_0 \geqslant 0$. Then $u_0 + v_0\sqrt{d}$ is called the fundamental solution in its class K.

Theorem 2.9. *If $u+v\sqrt{d}$ is a solution in nonnegative integers to the Diophantine equation $u^2 - dv^2 = N$, where $N > 1$, then there exists a nonnegative integer m such that*

$$u + v\sqrt{d} = (u_1 + v_1\sqrt{d})(x_1 + y_1\sqrt{d})^m$$

where $u_1 + v_1\sqrt{d}$ is the fundamental solution to the class of solutions of the equation $u^2 - dv^2 = N$ to which $u + v\sqrt{d}$ belongs and $x_1 + y_1\sqrt{d}$ is the fundamental solution to the equation $x^2 - dy^2 = 1$.

In [11], Nagell gives the fundamental solution in a given class K in a different way. We give the following theorem from [11].

Theorem 2.10. *Let $N > 0$ and $x_1 + y_1\sqrt{d}$ be the fundamental solution to $x^2 - dy^2 = 1$. If $u_0 + v_0\sqrt{d}$ is the fundamental solution to the equation $u^2 - dv^2 = N$ in its class, then*

$$0 \leqslant v_0 \leqslant \frac{y_1\sqrt{N}}{\sqrt{2(x_1+1)}} \quad \textit{and} \quad 0 < |u_0| \leqslant \sqrt{\frac{1}{2}(x_1+1)N}.$$

Now we can give the following theorems.

Theorem 2.11. *The equation* $x^2 - kxy + y^2 - 128 = 0$ *has an infinite number of positive integer solutions* x *and* y *if and only if* $k = 6, 30, 126$.

P r o o f. Assume that $x^2 - kxy + y^2 - 128 = 0$ for some positive integers x and y. If x is even, then y is even and thus $x = 2a$ and $y = 2b$ for some positive integers a and b. Substituting these values of x and y into the equation $x^2 - kxy + y^2 - 128 = 0$, we get $a^2 - kab + b^2 - 32 = 0$, which implies that $k = 6, 30$ by Theorem 2.7. Now assume that x and y are odd. Then k is even and $4 \nmid k$. Thus $k = 2t$ for some positive odd integer t. Completing the square gives $(x - ty)^2 - (t^2 - 1)y^2 = 128$. Since $8 \mid t^2 - 1$, we see that $x - ty = 4m$ and thus we get $m^2 - \frac{1}{16}(t^2 - 1)y^2 = 8$. Now we consider the equation

$$u^2 - \frac{t^2 - 1}{16} v^2 = 8. \tag{2.3}$$

Let $u_0 + v_0\sqrt{d}$ be the fundamental solution to equation (2.3) in a given class K. Since $(t, 4)$ is the fundamental solution to the equation $x^2 - \frac{1}{16}(t^2 - 1)y^2 = 1$ for $t > 7$ by Theorem 2.1, we get

$$0 \leqslant v_0 \leqslant \frac{4\sqrt{8}}{\sqrt{2(t+1)}} \leqslant \frac{4\sqrt{8}}{\sqrt{2(9+1)}} < 3$$

by Theorem 2.10. If (m, y) is a solution in the class K, then since y is odd, also v_0 is odd. This implies that $v_0 = 1$. Substituting this value of v_0 into equation (2.3), we get $(4u - t)(4u + t) = 127$. A simple computation shows that $t = 63$ and thus $k = 126$. Now assume that $1 \leqslant t \leqslant 7$. Since $\frac{1}{16}(t^2 - 1)$ is not an integer for $1 < t < 7$, we get $t = 7$. But when $t = 7$, it follows that $m^2 - 3y^2 = 8$, which is impossible since y is odd. Also when $t = 1$, it can be easily seen that equation (2.3) has no positive integer solutions. This completes the proof. $\square$

Theorem 2.12. *The equation* $x^2 - kxy + y^2 - 256 = 0$ *has an infinite number of positive integer solutions* x *and* y *if and only if* $k > 1$. *Also,* x *and* y *may be odd only when* $k = 2, 46, 82, 254$.

P r o o f. Assume that $x^2 - kxy + y^2 - 256 = 0$ for some positive integers x and y. If x is even, then y is even and thus $x = 2a$ and $y = 2b$ for some positive integers a and b. Substituting these values of x and y into the equation $x^2 - kxy + y^2 - 256 = 0$, we get $a^2 - kab + b^2 - 64 = 0$, which implies that $k > 1$ by Theorem 2.8. Now assume that x and y are odd. Then k is even and $4 \nmid k$. Thus $k = 2t$ for some positive odd integer t. Completing the square gives $(x - ty)^2 - (t^2 - 1)y^2 = 256$. Since $8 \mid t^2 - 1$, we see that $x - ty = 4m$ and thus we get $m^2 - \frac{1}{16}(t^2-1)y^2 = 16$. Now we consider the equation

$$u^2 - \frac{t^2-1}{16}v^2 = 16. \tag{2.4}$$

Let $u_0 + v_0\sqrt{d}$ be the fundamental solution to equation (2.4) in a given class K. Since $(t, 4)$ is the fundamental solution to the equation $x^2 - \frac{1}{16}(t^2-1)y^2 = 1$ for $t > 7$ by Theorem 2.1, we get

$$0 \leqslant v_0 \leqslant \frac{4\sqrt{16}}{\sqrt{2(t+1)}} \leqslant \frac{4\sqrt{16}}{\sqrt{2(9+1)}} < 4$$

by Theorem 2.10. If (m, y) is a solution in the class K, then, since y is odd, also v_0 is odd. This implies that $v_0 = 1$ or $v_0 = 3$. Substituting the value of $v_0 = 1$ into equation (2.4), we get $(4u-t)(4u+t) = 255$. A simple computation shows that $t = 23, t = 41$ or $t = 127$. Thus we get $k = 46, k = 82$, or $k = 254$. Now assume that $1 \leqslant t \leqslant 7$. But $\frac{1}{16}(t^2-1)$ is not an integer for $1 < t < 7$. When $t = 7$, we get

$$u^2 - 3v^2 = 16$$

from equation (2.4). Thus it follows that $u^2 - 3v^2 \equiv 0 \pmod 4$, which is impossible since v is odd. When $t = 1$, i.e., $k = 2$, it can be easily seen that equation (2.4) has an infinite number of positive integer solution x and y. Substituting the value of $v_0 = 3$ into equation (2.4), we get $(4u-3t)(4u+3t) = 247$. A simple computation shows that $t = 41$, or $t = 1$ and thus $k = 82$, or $k = 2$. This completes the proof. □

Since the proofs of the following theorems are similar to those of the above theorems, we omit them.

Theorem 2.13. *The equation $x^2 - kxy + y^2 - 512 = 0$ has an infinite number of positive integer solutions x and y if and only if* $k = 6, 30, 66, 126, 510$.

Theorem 2.14. *The equation $x^2 - kxy + y^2 - 1024 = 0$ has an infinite number of positive integer solutions x and y if and only if $k > 1$. Also x and y may be odd only when* $k = 2,\ 46, 66, 82, 338,\ 1022$.

Now, if $r \geqslant 0$ is an even integer, then we see that the equation $x^2 - kxy + y^2 - 2^r = 0$ has an infinite number of positive integer solutions x and y for $k = 2$. Also, if $r \geqslant 1$ is an odd integer, then we see that the equation $x^2 - kxy + y^2 - 2^r = 0$ has no positive integer solutions x and y for $k = 2$. Therefore, from now on, we will assume that $k \neq 2$.

3. Solutions of some of the equations $x^2 - kxy + y^2 - 2^n = 0$

In this section, we will give solutions of the equation $x^2 - kxy + y^2 - 2^n = 0$ for $0 \leqslant n \leqslant 10$. Solutions of the equation $x^2 - kxy + y^2 - 2^n = 0$ are related to the generalized Fibonacci and Lucas numbers. We briefly introduce the generalized Fibonacci and Lucas sequences $(U_n(k,s))$ and $(V_n(k,s))$. Let k and s be two integers with $k^2 + 4s > 0$. Generalized Fibonacci sequence is defined by $U_0(k,s) = 0$, $U_1(k,s) = 1$ and $U_{n+1}(k,s) = kU_n(k,s) + sU_{n-1}(k,s)$ for $n \geqslant 1$ and generalized Lucas sequence is defined by $V_0(k,s) = 2$, $V_1(k,s) = k$ and $V_{n+1}(k,s) = kV_n(k,s) + sV_{n-1}(k,s)$ for $n \geqslant 1$. Generalized Fibonacci and Lucas numbers for negative subscript are defined as

$$U_{-n}(k,s) = \frac{-U_n(k,s)}{(-s)^n} \quad \text{and} \quad V_{-n}(k,s) = \frac{V_n(k,s)}{(-s)^n} \tag{3.1}$$

for $n \geqslant 1$. We will use U_n and V_n instead of $U_n(k,s)$ and $V_n(k,s)$, respectively. For $s = -1$, we represent (U_n) and (V_n) by $(u_n(k,-1))$ and $(v_n(k,-1))$ or briefly by (u_n) and (v_n), respectively. We see from equation (3.1) that

$$u_{-n} = -u_n(k,-1) \text{ and } v_{-n} = v_n(k,-1)$$

for all $n \in \mathbb{Z}$. For $k = s = 1$, the sequences (U_n) and (V_n) are called Fibonacci and Lucas sequences and they are denoted as (F_n) and (L_n), respectively. For $k = 2$ and $s = 1$, the sequences (U_n) and (V_n) are called Pell and Pell Lucas sequences and they are denoted as (P_n) and (Q_n), respectively. Let α and β be the roots of the equation $x^2 - kx - s = 0$. Then it is well known that

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n \tag{3.2}$$

where $\alpha = \frac{1}{2}(k + \sqrt{k^2 + 4s})$ and $\beta = \frac{1}{2}(k - \sqrt{k^2 + 4s})$. The above identities are known as Binet's formulae. Clearly $\alpha + \beta = k$, $\alpha - \beta = \sqrt{k^2 + 4s}$, and $\alpha\beta = -s$. Moreover, it is well known that

$$U_n^2 - kU_nU_{n-1} - U_{n-1}^2 = (-1)^{n-1}, \tag{3.3}$$

$$v_n = u_{n+1} - u_{n-1},$$

and

$$u_n^2 - ku_nu_{n-1} + u_{n-1}^2 = 1, \tag{3.4}$$

where $U_n = U_n(k,1)$ and $u_n = u_n(k,-1)$. For more information about generalized Fibonacci and Lucas sequences, one can consult [14], [4], [12], [7], [9], and [10].

Now, we give the following theorem that help us to find solutions of some of the equations $x^2 - kxy + y^2 - 2^n = 0$. Since the proof of this theorem can be found in [5], [7], [9], [10], and [3], we omit it here. Before giving the theorem, we have to state that some of the following equations have positive odd integer solutions or positive even integer solutions. By the positive odd integer solutions x and y of the equation $x^2 - kxy + y^2 - 2^n = 0$ we mean that x and y are solutions of this equation and x and y are both odd. The positive even integer solutions x and y of the equation $x^2 - kxy + y^2 - 2^n = 0$ are defined similarly.

Theorem 3.1. *Let $k \geqslant 3$. Then all positive integer solutions of the equation $x^2 - kxy + y^2 - 1 = 0$ are given by $(x, y) = (u_n, u_{n-1})$ with $n > 1$, where $u_n = u_n(k, -1)$.*

The proof of the following corollary can be done by induction on r.

Corollary 3.1. *Let $r \geqslant 1$ be an integer. Then all positive even integer solutions of the equation $x^2 - kxy + y^2 - 2^{2r} = 0$ are given by $(x, y) = (2^r u_n(k, -1), 2^r u_{n-1}(k, -1))$ with $n > 1$.*

Theorem 3.2. *Let $r \geqslant 2$ be an integer. Then all positive integer solutions of the equation $x^2 - (2^{2r} - 2)xy + y^2 - 2^{2r} = 0$ are given by $(x, y) = (u_{n+1}(2^r, -1), u_{n-1}(2^r, -1))$ with $n > 1$.*

Proof. Assume that $x^2 - (2^{2r} - 2)xy + y^2 - 2^{2r} = 0$ for some positive integers x and y. It is easily seen that $2^r \mid x + y$. Let $u = (x + y)/2^r$ and $v = y$. Then we get $x = 2^r u - v$ and $y = v$. Substituting these values of x and y into the equation $x^2 - (2^{2r} - 2)xy + y^2 - 2^{2r} = 0$, we obtain

$$(2^r u - v)^2 - (2^{2r} - 2)(2^r u - v)v + v^2 - 2^{2r} = 0$$

and this implies that $u^2 - 2^r uv + v^2 - 1 = 0$. Therefore, by Theorem 3.1, we get $u = u_n(2^r, -1)$ and $v = u_{n-1}(2^r, -1)$ with $n > 1$. Thus $x = 2^r u_n - u_{n-1} = u_{n+1}$ and $y = u_{n-1}$. That is, $(x, y) = (u_{n+1}(2^r, -1), u_{n-1}(2^r, -1))$ with $n > 1$. Conversely, if $(x, y) = (u_{n+1}, u_{n-1})$, then from identity (3.4), it follows that $x^2 - (2^{2r} - 2)xy + y^2 - 2^{2r} = 0$. □

Corollary 3.2. *All positive integer solutions of the equation $x^2 - 14xy + y^2 - 16 = 0$ are given by $(x, y) = (u_{n+1}, u_{n-1})$ with $n > 1$, where $u_n = u_n(4, -1)$.*

Corollary 3.3. *All positive integer solutions of the equation $x^2 - 62xy + y^2 - 64 = 0$ are given by $(x, y) = (u_{n+1}, u_{n-1})$ with $n > 1$, where $u_n = u_n(8, -1)$.*

Corollary 3.4. *All positive integer solutions of the equation* $x^2 - 254xy + y^2 - 256 = 0$ *are given by* $(x, y) = (u_{n+1}, u_{n-1})$ *with* $n > 1$*, where* $u_n = u_n(16, -1)$.

Corollary 3.5. *All positive integer solutions of the equation* $x^2 - 1022xy + y^2 - 1024 = 0$ *are given by* $(x, y) = (u_{n+1}, u_{n-1})$ *with* $n > 1$*, where* $u_n = u_n(32, -1)$.

Theorem 3.3. *Let* $r > 1$ *be an odd integer. Then all positive integer solutions of the equation* $x^2 - (2^r - 2)xy + y^2 - 2^r = 0$ *are given by* $(x, y) = (u_{n+1} + u_n, u_n + u_{n-1})$ *with* $n \geqslant 1$*, where* $u_n = u_n(2^r - 2, -1)$.

Proof. Assume that $x^2 - (2^r - 2)xy + y^2 - 2^r = 0$ for some positive integers x and y. We see that x and y have the same parity. Without loss of generality, we may suppose $x \geqslant y$. It can be easily shown that $2^{(r+1)/2} \mid x + y$. Moreover, it can be seen that

$$\frac{2^r}{4}(x - y)^2 - \Big(\frac{2^r}{4} - 1\Big)(x + y)^2 = 2^r.$$

This implies that

$$\Big(\frac{x - y}{2}\Big)^2 - (2^{r-1} - 2)\Big(\frac{x + y}{2^{(r+1)/2}}\Big)^2 = 1.$$

Since $\alpha = \big(2^{r-1} - 1 + 2^{(r-1)/2}\sqrt{2^{r-1} - 2}\big)$ is the fundamental solution to the equation $x^2 - (2^{r-1} - 2)y^2 = 1$ by Theorem 2.1, it follows from (1.10) that

$$\frac{x - y}{2} = x_n \quad \text{and} \quad \frac{x + y}{2^{(r+1)/2}} = y_n$$

for some $n \geqslant 1$, where

$$x_n + y_n\sqrt{2^{r-1} - 2} = \alpha^n.$$

It is easily seen that $x_n = \frac{1}{2}v_n(2^r - 2, -1)$ and $y_n = 2^{(r-1)/2}u_n(2^r - 2, -1)$. Then we get $x = \frac{1}{2}(v_n + 2^r u_n)$ and $y = \frac{1}{2}(2^r u_n - v_n)$. Since $v_n = u_{n+1} - u_{n-1}$, it follows that $x = \frac{1}{2}(u_{n+1} - u_{n-1} + 2^r u_n) = \frac{1}{2}(u_{n+1} + u_{n+1} + 2u_n) = u_{n+1} + u_n$. In a similar way, we see that $y = u_n + u_{n-1}$. This shows that $(x, y) = (u_{n+1} + u_n, u_n + u_{n-1})$ with $n \geqslant 1$. Conversely, if $(x, y) = (u_{n+1} + u_n, u_n + u_{n-1})$, then from identity (3.4), it follows that $x^2 - (2^r - 2)xy + y^2 - 2^r = 0$. □

Corollary 3.6. *All positive integer solutions of the equation* $x^2 - 6xy + y^2 - 8 = 0$ *are given by* $(x, y) = (u_{n+1} + u_n, u_n + u_{n-1})$ *with* $n \geqslant 1$*, where* $u_n = u_n(6, -1)$.

Corollary 3.7. *All positive integer solutions of the equation* $x^2 - 30xy + y^2 - 32 = 0$ *are given by* $(x, y) = (u_{n+1} + u_n, u_n + u_{n-1})$ *with* $n \geqslant 1$*, where* $u_n = u_n(30, -1)$.

Corollary 3.8. *All positive integer solutions of the equation* $x^2 - 126xy + y^2 - 128 = 0$ *are given by* $(x, y) = (u_{n+1} + u_n, u_n + u_{n-1})$ *with* $n \geqslant 1$*, where* $u_n = u_n(126, -1)$.

Corollary 3.9. *All positive integer solutions of the equation* $x^2 - 510xy + y^2 - 512 = 0$ *are given by* $(x, y) = (u_{n+1} + u_n, u_n + u_{n-1})$ *with* $n \geqslant 1$*, where* $u_n = u_n(510, -1)$.

In order to find all positive odd integer solutions of the equations

$$x^2 - \tfrac{1}{3}(2^r - 10)xy + y^2 - 2^r = 0,$$
$$x^2 - \tfrac{1}{5}(2^r - 26)xy + y^2 - 2^r = 0,$$
$$x^2 - \tfrac{1}{7}(2^r - 50)xy + y^2 - 2^r = 0,$$

and

$$x^2 - \tfrac{1}{11}(2^r - 122)xy + y^2 - 2^r = 0,$$

we need Theorem 2.9.

Theorem 3.4. *Let* $r > 4$ *be an even integer. Then all positive odd integer solutions of the equation* $x^2 - \frac{1}{3}(2^r - 10)xy + y^2 - 2^r = 0$ *are given by* $(x, y) = (u_{n+2} + 3u_{n+1}, u_{n+1} + 3u_n)$ *with* $n \geqslant 0$ *or* $(x, y) = (3u_{n+1} + u_n, 3u_n + u_{n-1})$ *with* $n > 0$*, where* $u_n = u_n(\frac{1}{3}(2^r - 10), -1)$.

Proof. Assume that $x^2 - \frac{1}{3}(2^r - 10)xy + y^2 - 2^r = 0$ for some positive integers x and y. Completing the square gives

$$(x - \tfrac{1}{3}(2^{r-1} - 5)y)^2 - \tfrac{1}{3}(2^{r-1} - 8) \cdot \tfrac{1}{3}(2^{r-1} - 2)y^2 = 2^r.$$

Without loss of generality, we may suppose $x \geqslant \frac{1}{3}(2^{r-1} - 5)y$. Since $r > 4$, we get $x - \frac{1}{3}(2^{r-1} - 5)y = 4m$ for some positive integer m. Rearranging the equation gives

$$m^2 - \frac{(2^{r-1} - 8)(2^{r-1} - 2)}{144}y^2 = 2^{r-4}. \tag{3.5}$$

Let $d = (2^{r-1} - 8)(2^{r-1} - 2)/144$. Then we get

$$m^2 - dy^2 = 2^{r-4} \tag{3.6}$$

It can be seen from Theorem 2.10 that equation (3.6) has two solution classes. The fundamental solutions of these classes are

$$\frac{2^{r-1} + 4}{12} + \sqrt{d} \quad \text{and} \quad \frac{2^{r-1} + 4}{12} - \sqrt{d}.$$

By Theorem 2.9, all positive integer solutions of equation (3.6) are given by

$$a_n + b_n\sqrt{d} = \Big(\frac{2^{r-1}+4}{12} + \sqrt{d}\Big)(x_n + y_n\sqrt{d})$$

with $n \geqslant 0$ or

$$c_n + d_n\sqrt{d} = \Big(\frac{2^{r-1}+4}{12} - \sqrt{d}\Big)(x_n + y_n\sqrt{d})$$

with $n \geqslant 1$, where $x_n^2 - dy_n^2 = 1$. Since the fundamental solution of this equation is $\alpha = \frac{1}{3}(2^{r-1}-5) + 4\sqrt{d}$, we get $x_n + y_n\sqrt{d} = \alpha^n$ and therefore $x_n = \frac{1}{2}(\alpha^n + \beta^n)$ and $y_n = \frac{1}{2}(\alpha^n - \beta^n)\sqrt{d}$, where $\beta = \frac{1}{3}(2^{r-1}-5) - 4\sqrt{d}$. Thus we get $b_n = \frac{1}{12}(2^{r-1}+4)y_n + x_n$ and $d_n = \frac{1}{12}(2^{r-1}+4)y_n - x_n$. We see that $x_n = \frac{1}{2}v_n(\frac{1}{3}(2^r - 10), -1)$ and $y_n = 4u_n(\frac{1}{3}(2^r-10), -1)$. This shows that $b_n = \frac{1}{3}(2^{r-1}+4)u_n + \frac{1}{2}v_n = u_{n+1} + 3u_n$ with $n \geqslant 0$ and $d_n = \frac{1}{3}(2^{r-1}+4)u_n - \frac{1}{2}v_n = 3u_n + u_{n-1}$ with $n \geqslant 1$. Substituting the values of y into equation (3.5), we get $(x,y) = (u_{n+2} + 3u_{n+1}, u_{n+1} + 3u_n)$ with $n \geqslant 0$ or $(x,y) = (3u_{n+1} + u_n, 3u_n + u_{n-1})$ with $n > 0$. Conversely, if $(x,y) = (u_{n+2} + 3u_{n+1}, u_{n+1} + 3u_n)$ with $n \geqslant 0$ or $(x,y) = (3u_{n+1} + u_n, 3u_n + u_{n-1})$ with $n > 0$, then from identity (3.4), it follows that $x^2 - \frac{1}{3}(2^r - 10)xy + y^2 - 2^r = 0$. $\square$

Corollary 3.10. *All positive odd integer solutions of the equation* $x^2 - 18xy + y^2 - 64 = 0$ *are given by* $(x,y) = (u_{n+2} + 3u_{n+1}, u_{n+1} + 3u_n)$ *with* $n \geqslant 0$ *or* $(x,y) = (3u_{n+1} + u_n, 3u_n + u_{n-1})$ *with* $n > 0$, *where* $u_n = u_n(18, -1)$.

Corollary 3.11. *All positive odd integer solutions of the equation* $x^2 - 82xy + y^2 - 256 = 0$ *are given by* $(x,y) = (u_{n+2} + 3u_{n+1}, u_{n+1} + 3u_n)$ *with* $n \geqslant 0$ *or* $(x,y) = (3u_{n+1} + u_n, 3u_n + u_{n-1})$ *with* $n > 0$, *where* $u_n = u_n(82, -1)$.

Corollary 3.12. *All positive odd integer solutions of the equation* $x^2 - 338xy + y^2 - 1024 = 0$ *are given by* $(x,y) = (u_{n+2} + 3u_{n+1}, u_{n+1} + 3u_n)$ *with* $n \geqslant 0$ *or* $(x,y) = (3u_{n+1} + u_n, 3u_n + u_{n-1})$ *with* $n > 0$, *where* $u_n = u_n(338, -1)$.

We will give the following theorems without proof since their proofs are similar to that of Theorem 3.4.

Theorem 3.5. *Let* $r \geqslant 8$ *be an integer and* $4 \mid r$. *Then all positive odd integer solutions of the equation* $x^2 - \frac{1}{5}(2^r - 26)xy + y^2 - 2^r = 0$ *are given by* $(x,y) = (u_{n+2} + 5u_{n+1}, u_{n+1} + 5u_n)$ *with* $n \geqslant 0$ *or* $(x,y) = (5u_{n+1} + u_n, 5u_n + u_{n-1})$ *with* $n > 0$, *where* $u_n = u_n(\frac{1}{5}(2^r - 26), -1)$.

Corollary 3.13. *All positive odd integer solutions of the equation* $x^2 - 46xy + y^2 - 256 = 0$ *are given by* $(x,y) = (u_{n+2} + 5u_{n+1}, u_{n+1} + 5u_n)$ *with* $n \geqslant 0$ *or* $(x,y) = (5u_{n+1} + u_n, 5u_n + u_{n-1})$ *with* $n > 0$, *where* $u_n = u_n(46, -1)$.

Theorem 3.6. *Let* $r \geqslant 9$ *be an integer and* $3 \mid r$. *Then all positive odd integer solutions of the equation* $x^2 - \frac{1}{73}(2^r - 50)xy + y^2 - 2^r = 0$ *are given by* $(x,y) = (u_{n+2} + 7u_{n+1}, u_{n+1} + 7u_n)$ *with* $n \geqslant 0$ *or* $(x,y) = (7u_{n+1} + u_n, 7u_n + u_{n-1})$ *with* $n > 0$, *where* $u_n = u_n(\frac{1}{7}(2^r - 50), -1)$.

Corollary 3.14. *All positive odd integer solutions of the equation* $x^2 - 66xy + y^2 - 512 = 0$ *are given by* $(x,y) = (u_{n+2} + 7u_{n+1}, u_{n+1} + 7u_n)$ *with* $n \geqslant 0$ *or* $(x,y) = (7u_{n+1} + u_n, 7u_n + u_{n-1})$ *with* $n > 0$, *where* $u_n = u_n(66, -1)$.

Theorem 3.7. *Let* $r \geqslant 10$ *be an integer and* $10 \mid r$. *Then all positive odd integer solutions of the equation* $x^2 - \frac{1}{11}(2^r - 122)xy + y^2 - 2^r = 0$ *are given by* $(x,y) = (u_{n+2} + 11u_{n+1}, u_{n+1} + 11u_n)$ *with* $n \geqslant 0$ *or* $(x,y) = (11u_{n+1} + u_n, 11u_n + u_{n-1})$ *with* $n > 0$, *where* $u_n = u_n(\frac{1}{11}(2^r - 122), -1)$.

Corollary 3.15. *All positive odd integer solutions of the equation* $x^2 - 82xy + y^2 - 1024 = 0$ *are given by* $(x,y) = (u_{n+2} + 11u_{n+1}, u_{n+1} + 11u_n)$ *with* $n \geqslant 0$ *or* $(x,y) = (11u_{n+1} + u_n, 11u_n + u_{n-1})$ *with* $n > 0$, *where* $u_n = u_n(82, -1)$.

Theorem 3.8. *All positive odd integer solutions of the equation* $x^2 - 46xy + y^2 - 1024 = 0$ *are given by* $(x,y) = (3u_{n+2} + 7u_{n+1}, 3u_{n+1} + 7u_n)$ *with* $n \geqslant 0$ *or* $(x,y) = (7u_{n+1} + 3u_n, 7u_n + 3u_{n-1})$ *with* $n > 0$, *where* $u_n = u_n(46, -1)$.

Theorem 3.9. *All positive odd integer solutions of the equation* $x^2 - 66xy + y^2 - 1024 = 0$ *are given by* $(x,y) = (3u_{n+2} + 5u_{n+1}, 3u_{n+1} + 5u_n)$ *with* $n \geqslant 0$ *or* $(x,y) = (5u_{n+1} + 3u_n, 5u_n + 3u_{n-1})$ *with* $n > 0$, *where* $u_n = u_n(66, -1)$.

Since all positive integer solutions of the equations

$$x^2 - kxy + y^2 - 32 = 0, \quad k \in \{6\},$$
$$x^2 - kxy + y^2 - 128 = 0, \quad k \in \{6, 30\},$$

and

$$x^2 - kxy + y^2 - 512 = 0, \quad k \in \{6, 30, 126\}$$

can be given easily by using the previous theorems, we do not give their solutions.

From the above theorems, we see that when r is odd (r is even) and $k > 2^r - 2$, the equation $x^2 - kxy + y^2 = 2^r$ has no positive integer (odd positive integer) solutions. At this point, we can give the following conjecture.

Conjecture 3.1. (i) *Let r be an odd integer and $r > 2$. If $k > 2^r - 2$, then the equation $x^2 - kxy + y^2 = 2^r$ has no positive integer solutions. If $k \leqslant 2^r - 2$ and the equation $x^2 - kxy + y^2 = 2^r$ has a solution, then k is even.*

(ii) *Let r be an even integer. If $k > 2^r - 2$, then the equation $x^2 - kxy + y^2 = 2^r$ has no positive odd integer solutions. If $k \leqslant 2^r - 2$ and the equation $x^2 - kxy + y^2 = 2^r$ has a positive odd integer solution, then k is even.*

References

[1] *M. J. De Leon*: Pell's equations and Pell number triples. Fibonacci Q. *14* (1976), 456–460.

[2] *M. J. Jacobson, H. C. Williams*: Solving the Pell Equation. CMS Books in Mathematics. Springer, New York, 2009.

[3] *J. P. Jones*: Representation of solutions of Pell equations using Lucas sequences. Acta Acad. Paedagog. Agriensis, Sect. Mat. (N.S.) *30* (2003), 75–86.

[4] *D. Kalman, R. Mena*: The Fibonacci numbers—exposed. Math. Mag. *76* (2003), 167–181.

[5] *R. Keskin*: Solutions of some quadratic Diophantine equations. Comput. Math. Appl. *60* (2010), 2225–2230.

[6] *R. Keskin, O. Karaatli, Z. Şiar*: On the Diophantine equation $x^2 - kxy + y^2 + 2^n = 0$. Miskolc Math. Notes *13* (2012), 375–388.

[7] *R. Keskin, B. Demirtürk*: Solutions of some Diophantine equations using generalized Fibonacci and Lucas sequences. Ars Comb. *111* (2013), 161–179.

[8] *A. Marlewski, P. Zarzycki*: Infinitely many solutions of the Diophantine equation $x^2 - kxy + y^2 + x = 0$. Comput. Math. Appl. *47* (2004), 115–121.

[9] *W. L. McDaniel*: Diophantine representation of Lucas sequences. Fibonacci Q. *33* (1995), 59–63.

[10] *R. Melham*: Conics which characterize certain Lucas sequences. Fibonacci Q. *35* (1997), 248–251.

[11] *T. Nagell*: Introduction to Number Theory. John Wiley & Sons, Inc., New York; Almqvist & Wiksell, Stockholm, 1951.

[12] *P. Ribenboim*: My Numbers, My Friends. Popular Lectures on Number Theory. Springer, New York, 2000.

[13] *J. P. Robertson*: Solving the generalized Pell equation $x^2 - Dy^2 = N$. `http://hometown.aol.com/jpr2718/pell.pdf` (2003).

[14] *S. Robinowitz*: Algorithmic manipulation of Fibonacci identities. Applications of Fibonacci Numbers. (G. E. Bergum, et al., eds.). vol. 6, Kluwer Acadademic Publishers, Dordrecht, 1996, pp. 389–408.

[15] *P. Yuan, Y. Hu*: On the Diophantine equation $x^2 - kxy + y^2 + lx = 0$, $l \in \{1, 2, 4\}$. Comput. Math. Appl. *61* (2011), 573–577.

Authors' addresses: Refik Keskin, Sakarya University, Sakarya, Turkey, e-mail: `rkeskin@sakarya.edu.tr`; Zafer Şiar, Bilecik Şeyh Edebali University, Bilecik, Turkey, e-mail: `zafer.siar@bilecik.edu.tr`; Olcay Karaatli, Sakarya University, Sakarya, Turkey, e-mail: okaraatli@sakarya.edu.tr.