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On the Semisimilarity and Consemisimilarity of Split Quaternions

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Abstract. In this study, we introduce the concept of semisimilarity and consemisimilarity of split quaternions. Moreover, we examine the solvability conditions and general solutions of systems $xay = b, ybx = a$ and $\widetilde{x}ay = b, \widetilde{y}bx = a$ in split quaternions. If there exist x and y that satisfy first equations system, then a and b are said to be semisimilar, if there exist x and y that satisfy second equations system, then a and b are said to be consemisimilar.

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1. Introduction

In 1843, Hamilton introduced real quaternions that can be represented as [8]

$$\mathbb{H} = \{a = a_0 + a_1 i + a_2 j + a_3 k\colon a_0, a_1, a_2, a_3 \in \mathbb{R}, i, j, k \notin \mathbb{R}\}$$

where

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

From these rules it follows immediately that multiplication of real quaternions is not commutative. There are many studies on geometric, algebraic, physical e.t.c. meaning of the real quaternions. These applications of real quaternions require solving quaternionic equations. So, Niven [11,12] and Brand [3] gave nth roots of a real quaternion a and calculate them explicity. In [4], Cho generalized Euler's formula and De Moivre's formula for real quaternions. Also, he showed that there are uncountably many unit quaternions satisfying $x^n = 1$ for $n \geq 3$. Huang derived explicit formulas for computing the roots of a quaternionic quadratic polynomial [9]. In [15], Tian established the solvability conditions and general solutions of systems $xay = b, ybx = a$ and $\overline{x}ay = b, \overline{y}bx = a$ in real quaternions. In [14], Shpakivskyi solved the

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general linear quaternionic equation with one unknown and systems of linear quaternionic equations with two unknowns.

After Hamilton had discovered the real quaternions, James Cockle defined the set of split quaternions $\mathbb{H}_S$ in 1849 [5]. The split quaternions are not commutative like real quaternions. But contain zero divisors, nilpotent and non-trivial idempotent elements [1]. The split quaternions are a recently developing topic, since the split quaternions are used to express Lorentizian relation. Also, there are many studies on geometric, algebraic and physical meaning of the split quaternions [10,13]. Kula and Yaylı showed that the algebra of split quaternions of $\mathbb{H}_S$ has a scalar product that allows us to identify it with semi-Euclidean space $\mathbb{R}^4_2$. They expressed that a pair q and p of unit split quaternions in $\mathbb{H}_S$ determines a rotation $R_{qp}\colon \mathbb{H}_S \to \mathbb{H}_S$. Moreover, they examined the general solution of the linear equation $qx - xp = 0$ over $\mathbb{H}_S$ [10]. In [6], Erdogdu and Ozdemir investigated linear split quaternionic equations with the term of the form axb. They gave a new method of solving general linear split quaternionic equations with one, two and n unknowns.

2. Algebraic Properties of Split Quaternions

Let $\mathbb{R}$ be the real number field, $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$ be complex numbers, and $\mathbb{H}_S = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the split quaternions, where

$$\begin{aligned} &i^2 = -1, \quad j^2 = k^2 = 1 \\ &ij = -ji = k, jk = -kj = -i, ki = -ik = j. \end{aligned} \tag{2.1}$$

The real part and the imaginary part of $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}_S$ are defined as $\operatorname{Re} a = a_0$ and $\operatorname{Im} a = a_1 i + a_2 j + a_3 k$, respectively.

The multiplication of $a = a_0 + a_1 i + a_2 j + a_3 k$ and $b = b_0 + b_1 i + b_2 j + b_3 k$ is defined as

$$ab = \operatorname{Re} a \operatorname{Re} b + g(\operatorname{Im} a, \operatorname{Im} b) + \operatorname{Re} a \operatorname{Im} b + \operatorname{Re} b \operatorname{Im} a + \operatorname{Im} a \times \operatorname{Im} b$$

where

$$\begin{aligned} g(\operatorname{Im} a, \operatorname{Im} b) &= -a_1 b_1 + a_2 b_2 + a_3 b_3, \\ \operatorname{Im} a \times \operatorname{Im} b &= (a_3 b_2 - a_2 b_3) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k. \end{aligned}$$

The conjugate of a split quaternion is denoted by $\overline{a}$ and it is

$$\bar{a} = a_0 - a_1 i - a_2 j - a_3 k = \operatorname{Re} a - \operatorname{Im} a.$$

The norm of a split quaternion is defined as

$$\|a\| = \sqrt{|a\overline{a}|} = \sqrt{|a_0^2 + a_1^2 - a_2^2 - a_3^2|}.$$

Also, a split quaternion a is said to be spacelike, timelike or lightlike (null), if $a\overline{a} < 0$, $a\overline{a} > 0$ or $a\overline{a} = 0$, respectively.

The linear transformation $\phi, \tau\colon \mathbb{H}_S \to End(\mathbb{H}_S)$, given by

$$\phi(a) : \mathbb{H}_S \to \mathbb{H}_S, \quad \phi(a)(x) = ax$$

and

$$\tau(a) : \mathbb{H}_S \to \mathbb{H}_S, \quad \tau(a)(x) = xa,$$

are called the left representation and the right representation of the algebra $\mathbb{H}_S$, respectively. We know that every associative finite-dimensional algebra A over an arbitrary K is isomorphic with a subalgebra of the algebra $M_n(K)$. So we could find a faithful representation for the algebra A in the algebra $M_n(K)$, [7]. For the split quaternion algebra $\mathbb{H}_S$, the mapping:

$$\phi\colon \mathbb{H}_S \to M_4(\mathbb{R}), \quad \phi(a) = \begin{pmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \tag{2.2}$$

is an isomorphism between $\mathbb{H}_S$ and the algebra of matrices with the above form. The matrix $\phi(a)$ is called the left matrix representation for split quaternion $a \in \mathbb{H}_S$. In the same manner, we introduce the right matrix representation for the split quaternion a as;

$$\tau\colon H_S \to M_4(R), \quad \tau(a) = \begin{pmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix}, \tag{2.3}$$

where $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}_S$ [10].

It is nearby to identify a split quaternion $a \in \mathbb{H}_S$ with a vector $\overrightarrow{a} \in \mathbb{R}_2^4$ [13]. We will denote such identification by the symbol i.e.

$$a = a_0 + a_1 i + a_2 j + a_3 k \cong \overrightarrow{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Also, we show that conjugate, real part and imaginary part of a

$$\overline{a} \cong \begin{pmatrix} a_0 \\ -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} = Ca, \quad \mathrm{Re}a = a_0 \cong a_0 e_1 \quad \text{and} \quad \mathrm{Im}a \cong \overrightarrow{\mathrm{Im}a} = \begin{pmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

respectively, where

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 2.1. [10] *If* $a, b, x \in \mathbb{H}_S$ *and* $c \in \mathbb{R}$, *then we have*:

1. $a = b \Leftrightarrow \phi(a) = \phi(b) \Leftrightarrow \tau(a) = \tau(b)$,
2. $\phi(a+b) = \phi(a) + \phi(b), \tau(a+b) = \tau(a) + \tau(b)$,
3. $\phi(ca) = c\phi(a), \tau(ca) = c\tau(a)$,
4. $ab = \phi(a)\overrightarrow{b}, ab = \tau(b)\overrightarrow{a}, \phi(a)\tau(b) = \tau(b)\phi(a)$,
5. $axb = \phi(a)\tau(b)\overrightarrow{x} = \tau(b)\phi(a)\overrightarrow{x}$,

6. $\phi(ab) = \phi(a)\phi(b), \tau(ab) = \tau(a)\tau(b)$,
7. $\phi(\overline{a}) = \varepsilon(\phi(a))^T\varepsilon, \tau(\overline{a}) = \varepsilon(\tau(a))^T\varepsilon, \varepsilon = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}$,
8. $\phi(a^{-1}) = \phi(a)^{-1}, \tau(a^{-1}) = \tau(a)^{-1}, \|a\| \neq 0$,
9. $\phi(a^{-1}) = \begin{cases} -\frac{1}{\|a\|^2}\varepsilon(\phi(a))^T\varepsilon & if\ a\ is\ spacelike \\ \frac{1}{\|a\|^2}\varepsilon(\phi(a))^T\varepsilon & if\ a\ is\ timelike \\ There\,is\,no\,invers & if\ a\ is\ lightlike \end{cases}$
10. $|\phi(a)| = |\tau(a)| = \|a\|^2$.

From Theorem (2.1)-(4), we get

$$ax - xb = (\phi(a) - \tau(b))\overrightarrow{x}. \tag{2.4}$$

The authors showed the following result about the difference $\phi(a) - \tau(b)$ in the last equation [10].

Theorem 2.2. [10] *Let* $a = a_0 + a_1 i + a_2 j + a_3 k,\ \ b = b_0 + b_1 i + b_2 j + b_3 k \in \mathbb{H}_S$ *be given, and denote* $\delta(a,b) = \phi(a) - \tau(b)$. *Then*

(i) *If a and b are two split quaternions with* $g(\mathrm{Im}a, \mathrm{Im}a) < 0,\ \ g(\mathrm{Im}b, \mathrm{Im}b) < 0$ [*or* $g(\mathrm{Im}a, \mathrm{Im}a) > 0,\ \ g(\mathrm{Im}b, \mathrm{Im}b) > 0$] *then, the determinant of* $\delta(a,b)$ *is*

$$|\delta(a,b)| = s^4 - 2s^2((\mathrm{Im}a)^2 + (\mathrm{Im}b)^2) + ((\mathrm{Im}a)^2 - (\mathrm{Im}b)^2)^2$$

where $s = a_0 - b_0$. *Thus* $|\delta(a,b)| = 0$ *if and only if* $\mathrm{Re}a = \mathrm{Re}b$ *and* $g(\mathrm{Im}a, \mathrm{Im}a) = g(\mathrm{Im}b, \mathrm{Im}b)$.

(ii) *If* $a_0 \neq b_0$, *or* $g(\mathrm{Im}\,a, \mathrm{Im}\,a) \neq g(\mathrm{Im}\,b, \mathrm{Im}\,b)$, *then* $\delta(a,b)$ *is non-singular and its inverse can be written as*

$$\begin{aligned}\delta^{-1}(a,b) &= \phi^{-1}(a^2 - 2b_0 a + \|b\|)(\phi(a) - \tau(\bar{b})) \\ &= \phi^{-1}(2(a_0 - b_0)a + \|b\| - \|a\|)(\phi(a) - \tau(\bar{b}))\end{aligned}$$

and

$$\begin{aligned}\delta(a,b)^{-1} &= \tau^{-1}(b^2 - 2a_0 b + \|a\|)(\phi(\bar{a}) - \tau(b)) \\ &= \tau^{-1}(2(b_0 - a_0)b + \|a\| - \|b\|)(\phi(\bar{a}) - \tau(b))\end{aligned}$$

(iii) *If* $a_0 = b_0$ *and* $g(\mathrm{Im}(a), \mathrm{Im}(a)) = g(\mathrm{Im}(b), \mathrm{Im}(b))$, *then* $\delta(a,b)$ *is singular and has a generalized inverse as follows*

$$\delta(a,b)^{-} = \frac{1}{4(\mathrm{Im}a)^2}\delta(a,b) = \frac{1}{4(\mathrm{Im}a)^2}(\phi(\mathrm{Im}a) - \tau(\mathrm{Im}b)).$$

Theorem 2.3. *If* $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}_S$ *is spacelike, timelike or lightlike, then the eigenvalues of* $\phi(a)$ [*or* $\tau(a)$] *can be given by* $a_0 \pm \|\mathrm{Im}\,a\|, a_0 \pm \|\mathrm{Im}\,a\|\,i$ *and* a_0, *respectively. In case* $\mathrm{Im}(a)$ *is spacelike or timelike, each eigenvalue occurs with algebraic multiplicity* 2, *and otherwise the eigenvalues* a_0 *has algebraic multiplicity* 4.

Proof. For $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}_S$, we consider the eigenvalue-eigenvector equation

$$\phi(a)\overrightarrow{x} = \lambda\overrightarrow{x}$$

where $\lambda \in \mathbb{C}$ is an eigenvalue and $0 \neq x \in \mathbb{C}^4$ is a corresponding eigenvector of $\phi(a)$. The matrix $\phi(a)$ can be written as $\phi(a) = a_0 I_4 + \phi(\operatorname{Im} a)$. Consequently, the eigenvalues of $\phi(a)$ are obtained by adding a_0 to the eigenvalues of $\phi(\operatorname{Im} a)$. If μ is an eigenvalue of $\phi(\operatorname{Im} a)$, then μ^2 is an eigenvalue of $\phi(\operatorname{Im} a)^2$.

From

$$\phi(\operatorname{Im}(a))^2 = \begin{cases} \|\mathrm{Im}a\|^2 I_4 & \text{if Im}a \text{ is spacelike} \\ -\|\mathrm{Im}a\|^2 I_4 & \text{if Im}a \text{ is timelike} \\ 0 & \text{if Im}a \text{ is lightlike} \end{cases}$$

we conclude that

$$\mu^2 = \begin{cases} \|\mathrm{Im}a\|^2 & \text{if Im}a \text{ is spacelike} \\ -\|\mathrm{Im}a\|^2 & \text{if Im}a \text{ is timelike} \\ 0 & \text{if Im}a \text{ is lightlike} \end{cases}.$$

Hence, the eigenvalues of $\phi(\operatorname{Im}(a))$ can only be

$$\mu = \begin{cases} \pm\|\mathrm{Im}a\| & \text{if Im}a \text{ is spacelike} \\ \pm\|\mathrm{Im}a\|\, i & \text{if Im}a \text{ is timelike} \\ 0 & \text{if Im}a \text{ is lightlike} \end{cases}.$$

If $\operatorname{Im}(a)$ is spacelike, timelike or lightlike, the eigenvalues of fundamental matrix $\phi(a)$ are given by $a_0 \pm \|\operatorname{Im} a\|, a_0 \pm \|\operatorname{Im} a\|\, i$ and a_0, respectively. In case $\operatorname{Im}(a)$ is spacelike or timelike, each eigenvalue occurs with algebraic multiplicity 2, and otherwise the eigenvalues a_0 has algebraic multiplicity 4. □

Theorem 2.4. *Let* $\|\operatorname{Im} a\| \neq 0$ *for* $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}_S$. *If* $\operatorname{Im} a$ *is spacelike, then the eigenspaces of* $\phi(a)$ *corresponding to* $a_0 + \|\operatorname{Im} a\|$ *and* $a_0 - \|\operatorname{Im} a\|$ *are*

$$\left\{\phi(g_1)\overrightarrow{y} : \overrightarrow{y} \in \mathbb{C}^4\right\} \quad \text{and} \quad \left\{\phi(h_1)\overrightarrow{y} : \overrightarrow{y} \in \mathbb{C}^4\right\}$$

respectively, where $g_1 = \|\operatorname{Im} a\| e_1 + \overrightarrow{\operatorname{Im} a}$ *and* $h_1 = -\|\operatorname{Im} a\| e_1 + \overrightarrow{\operatorname{Im} a}$.
If $\operatorname{Im}(a)$ *is timelike, then the eigenspaces of* $\phi(a)$ *corresponding to* $a_0 + \|\operatorname{Im} a\|\, i$ *and* $a_0 - \|\operatorname{Im} a\|\, i$ *are*

$$\left\{\phi(g_2)\, y : y \in \mathbb{C}^4\right\} \quad \text{and} \quad \left\{\phi(h_2)\, y : y \in \mathbb{C}^4\right\}$$

respectively, where $g_2 = i\|\operatorname{Im} a\| e_1 + \overrightarrow{\operatorname{Im} a}$ *and* $h_1 = -i\|\operatorname{Im} a\| e_1 + \overrightarrow{\operatorname{Im} a}$.

Theorem 2.4 can be verified by calculating

$$\phi(a)\phi(g_1)\overrightarrow{y} - (a_0 + \|\operatorname{Im} a\|)\phi(g_1)\overrightarrow{y},$$
$$\phi(a)\phi(h_1)\overrightarrow{y} - (a_0 - \|\operatorname{Im} a\|)\phi(h_1)\overrightarrow{y},$$
$$\phi(a)\phi(g_2)\overrightarrow{y} - (a_0 + \|\operatorname{Im} a\|\, i)\phi(g_2)\overrightarrow{y}$$

and

$$\phi(a)\,\phi(h_2)\,\overrightarrow{y}-(a_0-\|\operatorname{Im}a\|\,i)\,\phi(h_2)\,\overrightarrow{y}$$

which all of them equal zero vector for any $y\in\mathbb{C}^4$.

3. Semisimilarity and Consemisimilarity of Elements in $\mathbb{H}_S$

3.1. Semisimilarity of Elements in $\mathbb{H}_S$

Recall that if there exist $p\in\mathbb{H}_S$ ($\|p\|\neq 0$) such that $p^{-1}ap=b$ then split quaternions a and b are called similar, this is written as $a\sim b$. The relation $\sim$ is an equivalence relation on the split quaternions [10].

Definition 3.1. Two split quaternion a and b are said to be semisimilar if there exist a split quaternions x and y that satisfying the equation

$$xay=b,\quad ybx=a, \tag{3.1}$$

this is written as $a\approx b$. Also, $\approx$ is an equivalence relation on $\mathbb{H}_S$.

Theorem 3.2. *If two split quaternions a and b are semisimilar* ($\|a\|\neq 0,\ \|b\|\neq 0$) *then we have $\|a\|=\|b\|$, $a^2\sim b^2$ and* $\operatorname{Re}a^2=\operatorname{Re}b^2$.

Proof. If $\|a\|$ and $\|b\|$ are nonzero, the norms of split quaternions x and y satisfy (3.1) are nonzero. Let $a\approx b$. Then there exist x and y with $xay=b,\ ybx=a$. The ratio of the norms of these two equalities gives us

$$\frac{\|a\|}{\|b\|}=\frac{\|b\|}{\|a\|}$$

that is $\|a\|=\|b\|$.

If the right side of first equation in (3.1) multiplied by $y^{-1}a^{-1}$, then we get $x=by^{-1}a^{-1}$. Substitution $x=by^{-1}a^{-1}$ into the second equation in (3.1) gives us $yb^2y^{-1}=a^2$ that is $a^2\sim b^2$. Taking real parts of both sides of equation $yb^2y^{-1}=a^2$, we have $\operatorname{Re}a^2=\operatorname{Re}b^2$. □

Theorem 3.3. *Let $a,b\in\mathbb{H}_S$ and $\|a\|\neq 0,\|b\|\neq 0$,*

(a) *Let a^2 and b^2 are real numbers. Then there exist $x,y\in\mathbb{H}_S$ satisfying (3.1) if and only if $a^2=b^2$. In the present case, the general solution (3.1) is*

$$x=p,\quad y=ap^{-1}b^{-1} \tag{3.2}$$

or

$$x=bq^{-1}a^{-1},\quad y=q \tag{3.3}$$

where $p,q\in\mathbb{H}_S$ are arbitrary ($\|p\|\neq 0,\ \|q\|\neq 0$).

(b) *In case $a^2\in\mathbb{R}$ and $b^2\notin\mathbb{R}$ or $a^2\notin\mathbb{R}$ and $b^2\in\mathbb{R}$, Eq. (3.1) has no solution.*

(c) *Let a^2 and b^2 are not real numbers. Then there exist $x,y\in\mathbb{H}_S$ that satisfy (3.1) if and only if*

$$\operatorname{Re}a^2=\operatorname{Re}b^2\quad\text{and}\quad g\left(\operatorname{Im}a^2,\operatorname{Im}a^2\right)=g\left(\operatorname{Im}b^2,\operatorname{Im}b^2\right). \tag{3.4}$$

In the present case, the general solution to (3.1) *is*

$$x = b\left[q + \frac{1}{(\operatorname{Im} a^2)^2} \operatorname{Im} a^2 q \operatorname{Im} b^2\right]^{-1} a^{-1} \tag{3.5}$$

$$y = \left[q + \frac{1}{(\operatorname{Im} a^2)^2} \operatorname{Im} a^2 q \operatorname{Im} b^2\right] \tag{3.6}$$

where $q \in \mathbb{H}_S$ *is arbitrary such that* $\|y\| \neq 0$ *or equivalently*

$$x = \left[p + \frac{1}{(\operatorname{Im} a^2)^2} \operatorname{Im} b^2 p \operatorname{Im} a^2\right] \tag{3.7}$$

$$y = a\left[p + \frac{1}{(\operatorname{Im} a^2)^2} \operatorname{Im} b^2 p \operatorname{Im} a^2\right]^{-1} b^{-1} \tag{3.8}$$

where $p \in \mathbb{H}_S$ *is arbitrary such that* $\|x\| \neq 0$. *In particular, if* $b^2 \neq (\overline{a})^2$, *i.e.,* $\operatorname{Im} a^2 + \operatorname{Im} b^2 \neq 0$ *then the general solution of* (3.1) *can be written as*

$$x = \lambda_1(\operatorname{Im} a^2 + \operatorname{Im} b^2) + \lambda_2((\operatorname{Im} a^2)^2 + \operatorname{Im} b^2 \operatorname{Im} a^2) \tag{3.9}$$

$$y = a[\lambda_1(\operatorname{Im} a^2 + \operatorname{Im} b^2) + \lambda_2((\operatorname{Im} a^2)^2 + \operatorname{Im} b^2 \operatorname{Im} a^2)]^{-1} b^{-1} \tag{3.10}$$

or equivalently,

$$x = b[\lambda_3(\mathrm{Im}a^2 + \mathrm{Im}b^2) + \lambda_4((\mathrm{Im}a^2)^2 + \mathrm{Im}a^2\mathrm{Im}b^2)]^{-1}a^{-1} \tag{3.11}$$

$$y = \lambda_3(\mathrm{Im}a^2 + \mathrm{Im}b^2) + \lambda_4((\mathrm{Im}a^2)^2 + \mathrm{Im}a^2\mathrm{Im}b^2) \tag{3.12}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ *are chosen arbitrary such that* $\|x\| \neq 0$ *and* $\|y\| \neq 0$.

Proof. If $\|a\| \neq 0, \|b\| \neq 0$, then the norms of a and b satisfying (3.1) are nonzero. If the right side of first equation in (3.1) multiplied by $y^{-1}a^{-1}$, then we get $x = by^{-1}a^{-1}$. Substitution $x = by^{-1}a^{-1}$ into the second equation in (3.1) gives us

$$a^2 y = y b^2. \tag{3.13}$$

In a similar way, getting solution for y from the second equation in (3.1) and substituting it into the first equation in (3.1) gives

$$b^2 x = x a^2. \tag{3.14}$$

For $a^2, b^2 \in \mathbb{R}$, the equation $a^2 y = y b^2$ has solution for y if and only if a^2 is equal to b^2. In the present case, y that satisfies (3.13) is arbitrary. Substituting this y into the first equation in (3.1) gives (3.3). Solving for x in (3.14) and substituting this x into the second equation in (3.1) gives (3.2).

For $a^2 \in \mathbb{R}$ and $b^2 \notin \mathbb{R}$ or $a^2 \notin \mathbb{R}$ and $b^2 \in \mathbb{R}$, we have $y(b^2 - a^2) = 0$ or $x(b^2 - a^2) = 0$. Since $b^2 - a^2 \neq 0$, (3.1) has no solution ($y \neq 0$).

Under the given conditions $a^2, b^2 \notin \mathbb{R}$, the Eq. (3.13) can be expressed as

$$[\phi(a^2) - \tau(b^2)]\overrightarrow{y} = \delta(a^2, b^2)\overrightarrow{y} = 0.$$

This equation has a nonzero solution for y if and only if $\left|\delta\left(a^2, b^2\right)\right| = 0$, which is equivalent to (3.4). In the present case, the general solution can be written as

$$y = 2[I_4 - \delta^-(a^2, b^2)\delta(a^2, b^2)]\overrightarrow{q}$$

where $\overrightarrow{q}$ is an arbitrary vector. An expression of $\delta^-\left(a^2, b^2\right)$ can be derived from theorem (2.2). Thus

$$\begin{aligned} y &= 2\left[I_4 - \frac{1}{4(\operatorname{Im} a^2)^2}\delta(a^2, b^2)\delta(a^2, b^2)\right]\overrightarrow{q} \\ &= 2\left[I_4 - \frac{1}{(\operatorname{Im} a^2)^2}(2(\operatorname{Im} a^2)^2 I_4 - 2\phi(\operatorname{Im} a^2)\tau(\operatorname{Im} b^2))\right]\overrightarrow{q} \\ &= \left[I_4 + \frac{1}{(\operatorname{Im} a^2)^2}\phi(\operatorname{Im} a^2)\tau(\operatorname{Im} b^2)\right]\overrightarrow{q} \end{aligned}$$

Applying it to split quaternion form by Theorem (2.2)-(v) yields (3.6).

$$y = \left[q + \frac{1}{(\operatorname{Im} a^2)^2}\operatorname{Im} a^2 q \operatorname{Im} b^2\right].$$

Substituting y into the equation $xay = b$ gives (3.5). If $b^2 \neq \bar{a}^2$, then set $q = \operatorname{Im} a^2$ and $q = (\operatorname{Im} a^2)^2$ in (3.6). In the present case, we get two special solutions to (3.13) as

$$\begin{aligned} y_1 &= \text{Im}a^2 + \frac{1}{(\text{Im}a^2)^2}\text{Im}a^2\text{Im}a^2\text{Im}b^2 \\ &= \text{Im}a^2 + \text{Im}b^2 \\ y_2 &= (\operatorname{Im} a^2)^2 + \frac{1}{(\text{Im}a^2)^2}\text{Im}a^2\left(\text{Im}a^2\right)^2\text{Im}b^2 \\ &= (\operatorname{Im} a^2)^2 + \text{Im}a^2\text{Im}b^2. \end{aligned}$$

Therefore, the solution of the Eq. (3.13) is a solution for (3.12). Since $\operatorname{Re} y_1 = 0$ and $\operatorname{Re} y_2 \neq 0$, y_1 and y_2 are linearly independent. Thus, the general solution of (3.13) is (3.12), since $rank(\delta(a^2, b^2)) = 2$ under $a^2, b^2 \notin \mathbb{R}$ and (3.4). Note that under the given conditions $a^2, b^2 \notin \mathbb{R}$ and (3.4) the solution to (3.1) is nonzero. Thus it is required that $q \in \mathbb{H}_S$ in (3.5) and (3.6) and $\lambda_3, \lambda_4 \in \mathbb{R}$ in (3.11) and (3.12) are chosen such that $\|x\| \neq 0$ and $\|y\| \neq 0$. In a similar way, solving the equation in (3.14) under the given condition yields (3.7) and (3.8), as well as (3.9) and (3.10). □

Theorem 3.4. *$x, y \in \mathbb{H}_S$, $a^2 \notin \mathbb{R}$ and $\|a\| \neq 0$. Then the general solution of*

$$xay = a \quad \textit{and} \quad yax = a \tag{3.15}$$

is

$$x = a[\lambda_1 + \lambda_2 a^2]^{-1} a^{-1}, \quad y = \lambda_1 + \lambda_2 a^2 \tag{3.16}$$

or

$$x = \lambda_3 + \lambda_4 a^2, \quad y = a\big[\lambda_3 + \lambda_4 a^2\big]^{-1} a^{-1}. \tag{3.17}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{H}_S$ *are arbitrary such that* $\|x\|, \|y\| \neq 0$.

Proof. According to Theorem (3.3), we have

$$y = q + \frac{1}{(\operatorname{Im} a^2)^2} \operatorname{Im} a^2 q \operatorname{Im} a^2.$$

Next let $q = (\operatorname{Im} a^2)p$ where p is arbitrary. Then

$$\begin{aligned} y &= \operatorname{Im} a^2 p + p \operatorname{Im} a^2 \\ &= \operatorname{Im} a^2 (\operatorname{Re} p + \operatorname{Im} p) + (\operatorname{Re} p + \operatorname{Im} p) \operatorname{Im} a^2 \\ &= 2 \operatorname{Im} a^2 \operatorname{Re} p + \operatorname{Im} a^2 \operatorname{Im} p + \operatorname{Im} p \operatorname{Im} a^2 \\ &= 2 \operatorname{Re} p \operatorname{Im} a^2 + t_0 \\ &= t_1 \operatorname{Im} a^2 + t_0 \end{aligned}$$

where $t_0, t_1 \in \mathbb{R}$, which is equivalent

$$y = \lambda_1 + \lambda_2 a^2.$$

Substituting this y into the first equation gives (3.16). Similar way, we have equaitions in (3.17) from (3.7). □

Theorem 3.5. *Let* $a = a_0 + a_1 i + a_2 j + a_3 k$ *with* $a \notin \mathbb{C}$, $\|a\| \neq 0$ *and* $g(\operatorname{Im} a, \operatorname{Im} a) < 0$. *Then* a *is always semisimilar to the complex number* $\operatorname{Re} a + \|\operatorname{Im} a\|\, i$, *that is the pair of the equation*

$$xay = (\operatorname{Re} a + \|\operatorname{Im} a\|\, i), \quad y(\operatorname{Re} a + \|\operatorname{Im} a\|\, i)x = a$$

always have nonzero solution. The general solutions to this pair of equation can be written as

$$x = \lambda_1[\mathrm{Im}a^2 + 2\mathrm{Re}a\|\mathrm{Im}a\|i] + \lambda_2[-\|\mathrm{Im}a^2\|^2 + 2i\mathrm{Re}a\|\mathrm{Im}a\|\mathrm{Im}a^2]$$
$$y = a[\lambda_1[\mathrm{Im}a^2 + 2\mathrm{Re}a\|\mathrm{Im}a\|i] + \lambda_2[-\|\mathrm{Im}a^2\|^2 + 2i\mathrm{Re}a\|\mathrm{Im}a\|\mathrm{Im}a^2]]^{-1} b^{-1}$$

or equivalently,

$$x = b[\lambda_3[\mathrm{Im}a^2 + 2\mathrm{Re}a\|\mathrm{Im}a\|i] + \lambda_4[-\|\mathrm{Im}a^2\|^2 + 2\mathrm{Re}a\mathrm{Im}a^2\|\mathrm{Im}a\|i]]^{-1} a^{-1}$$
$$y = \lambda_3[\mathrm{Im}a^2 + 2\mathrm{Re}a\|\mathrm{Im}a\|i] + \lambda_4[-\|\mathrm{Im}a^2\|^2 + 2\mathrm{Re}a\mathrm{Im}a^2\|\mathrm{Im}a\|i]$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ *are chosen arbitrarily such that* $\|x\| \neq 0$ *and* $\|y\| \neq 0$.

Theorem 3.6. *Let* $a = a_0 + a_1 i + a_2 j + a_3 k$ *with* $a \notin C, a^2 \notin R$ *and* $g(\operatorname{Im} a, \operatorname{Im} a) > 0$. *Then* a *is always semi-similar to the hyperbolic number* $\operatorname{Re}(a) + \|\operatorname{Im} a\|\, j$, *that is the pair of equation*

$$xay = (\operatorname{Re} a + \|\operatorname{Im} a\|\, j), \quad y(\operatorname{Re} a + \|\operatorname{Im} a\|\, j)x = a$$

always have nonzero solution. The general solutions to this pair of equation can be written as

$$x = \lambda_1[\mathrm{Im}a^2 + 2\mathrm{Re}a\|\mathrm{Im}a\|j] + \lambda_2[-\|\mathrm{Im}a^2\|^2 + 2j\mathrm{Re}a\|\mathrm{Im}a\|\mathrm{Im}a^2]$$
$$y = a[\lambda_1[\mathrm{Im}a^2 + 2\mathrm{Re}a\|\mathrm{Im}a\|j] + \lambda_2[-\|\mathrm{Im}a^2\|^2 + 2j\mathrm{Re}a\|\mathrm{Im}a\|\mathrm{Im}a^2]]^{-1} b^{-1}$$

or equivalently,

$$x = b[\lambda_3[\mathrm{Im}a^2 + 2\mathrm{Re}a\|\mathrm{Im}a\|j] + \lambda_4[-\|\mathrm{Im}a^2\|^2 + 2\mathrm{Re}a\,\mathrm{Im}a^2\|\mathrm{Im}a^2\|j]]^{-1}a^{-1}$$
$$y = \lambda_3[\mathrm{Im}a^2 + 2\mathrm{Re}a\|\mathrm{Im}a\|j] + \lambda_4[-\|\mathrm{Im}a^2\|^2 + 2\mathrm{Re}a\mathrm{Im}a^2\|\mathrm{Im}a\|j]$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ *are chosen arbitrarily such that* $\|x\| \neq 0$ *and* $\|y\| \neq 0$.

3.2. Consemisimilarity of Elements in $\mathbb{H}_S$

A complex matrix $A \in \mathbb{C}^{n\times n}$ is said to be consimilar $B \in \mathbb{C}^{n\times n}$ if there exists nonsingular matrix P such that $\overline{P}AP^{-1} = B$. Complex consimilairty is an equivalence relation on $\mathbb{C}^{n\times n}$ and has been extensively studied in [14]. The another equivalence relation on $\mathbb{C}^{n\times n}$ is consemisimilarity. $A \in \mathbb{C}^{n\times n}$is said to be consimilar $B \in \mathbb{C}^{n\times n}$ if there exist X and Y such that $\overline{X}AY = B, \overline{Y}BX = A$ [2].

In a similar way, we can construct consimilarity and consemisimilarity relation on $\mathbb{H}_S$. Let $a, b \in \mathbb{H}_S$. Generally $\overline{(ab)} \neq \overline{a}\,\overline{b}$. Thus the mappings $a \to \overline{p}ap^{-1}$ or $a \to \overline{x}ay,\ b \to \overline{y}by$ are not an equivalence relation on $\mathbb{H}_S$. Thus we need to give a new definition of consimilarity and consemisimilarity of split quaternions.

Definition 3.7. Let $a = a_0 + a_1i + a_2j + a_3k \in \mathbb{H}_S$, then we define $\widetilde{a} = a_0 - a_1i + a_2j - a_3k$. We say that $\widetilde{a}$ is the j-conjugate of a.

For any $a, b \in \mathbb{H}_S$, the following equalities are easy to confirm

1. $\widetilde{(\widetilde{a})} = a$,
2. $\widetilde{(a + b)} = \widetilde{a} + \widetilde{b}$
3. $\widetilde{(ab)} = \widetilde{a}\,\widetilde{b}$
4. $\overline{(\widetilde{a})} = \widetilde{(\overline{a})}$
5. $\|a\| = \|\widetilde{a}\|$.

Definition 3.8. Two split quaternions a and b are consimilar if there exists a split quaternion $p \in \mathbb{H}_S$ $(\|p\| \neq 0)$ that satisfying $\widetilde{p}\,ap^{-1} = b$, this is written as $a\overset{c}{\sim} b$. Obviously, the consimilarity is an equivalence relation on the split quaternions.

Definition 3.9. Two split quaternions a and b are consemisimilar if there exist a split quaternions x and y satisfying

$$\widetilde{x}\,ay = b, \quad \widetilde{y}\,bx = a, \tag{3.18}$$

this is written as $a\overset{c}{\approx} b$. Obviously, the consemisimilarity is an equivalence relation on the split quaternions.

Theorem 3.10. *If two split quaternions* a *and* b *are consemisimilar* $(\|a\| \neq 0, \|b\| \neq 0)$ *then we have* $\|a\| = \|b\|$, $\widetilde{a}a \sim \widetilde{b}b$ *and* $\mathrm{Re}\,\widetilde{a}a = \mathrm{Re}\,\widetilde{b}b$.

Proof. If $\|a\|$ and $\|b\|$ are nonzero, the norms of split quaternions $\widetilde{x}$ and $\widetilde{y}$ satisfy (3.18) are nonzero. Let $a\overset{c}{\approx} b$. Then there exist x and y with $\widetilde{x}\,ay = b,\ \ \widetilde{y}\,bx = a$. The ratio of the norms of these two equalities gives us

$$\frac{\|a\|}{\|b\|} = \frac{\|b\|}{\|a\|}$$

that is $\|a\| = \|b\|$. If the left side of first equation in (3.18) multiplied by $a^{-1}\widetilde{x}^{-1}$, then we have $y = a^{-1}\widetilde{x}^{-1}b$. Substitution $y = a^{-1}\widetilde{x}^{-1}b$. into the second equation in (3.18) and simplifying gives us $x^{-1}\widetilde{b}bx = \widetilde{a}a$ that is $\widetilde{a}a \sim \widetilde{b}b$. By taking real parts of both sides of equation $x^{-1}\widetilde{b}bx = \widetilde{a}a$, we have $\mathrm{Re}\,\widetilde{a}a = \mathrm{Re}\,\widetilde{b}b$. □

The following theorem gives the solvability conditions and general solutions of equations system $\widetilde{x}\,ay = b, \widetilde{y}\,bx = a$.

Theorem 3.11. *Let $a, b \in \mathbb{H}$ and $\|a\| \neq 0,\ \|b\| \neq 0$,*

1. *Let $\widetilde{a}a$ and $\widetilde{b}b$ are reals. Then, there exist $x, y \in \mathbb{H}_S$ that satisfy (3.18) if and only if $\widetilde{a}a = \widetilde{b}b$. In the present case, the general solution to (3.18) is*

$$x = p, \quad y = a^{-1}\widetilde{p}^{-1}b$$

or equivalently

$$x = b^{-1}\widetilde{q}^{-1}a, \quad y = q$$

where p, q $(\|p\|, \|q\| \neq 0)$ are arbitrary.

2. *Let $\widetilde{a}a$ and $\widetilde{b}b$ are not reals. Then, there exist $x, y \in \mathbb{H}_S$ that satisfy (3.18) if and only if $\mathrm{Re}\widetilde{a}a = \mathrm{Re}\widetilde{b}b$ and $g(\mathrm{Im}\widetilde{a}a, \mathrm{Im}\widetilde{a}a\,) = g(\mathrm{Im}\widetilde{b}b, \mathrm{Im}\widetilde{b}b\,)$. In the present case, the general solution to (3.18) is*

$$x = p + \frac{1}{(\mathrm{Im}\,\widetilde{a}a)^2}(\mathrm{Im}\,\widetilde{b}b)p(\mathrm{Im}\,\widetilde{a}a), \;\; y = a^{-1}\left[\widetilde{p} + \frac{1}{(\mathrm{Im}\,\widetilde{a}a)^2}(\mathrm{Im}\,b\widetilde{b})\widetilde{p}(\mathrm{Im}\,a\widetilde{a})\right]^{-1} b$$

or equivalently,

$$x = b^{-1}\left[\widetilde{q} + \frac{1}{(\mathrm{Im}\,\widetilde{a}a)^2}(\mathrm{Im}\,a\widetilde{a})\widetilde{q}(\mathrm{Im}\,b\widetilde{b})\right]^{-1} a, \quad y = q + \frac{1}{(\mathrm{Im}\,\widetilde{a}a)^2}(\mathrm{Im}\,\widetilde{a}a)q(\mathrm{Im}\,\widetilde{b}b)$$

where $p, q \in H_S$ are arbitrary such that $\|x\|, \|y\| \neq 0$.

The proof is a routine process as was performed in the Theorem 3.3.

4. Conclusions

Similarity and consimilarity are special forms of transformation $f(a) = (x^{-1})\hat{}\,a\,x$ where $()\hat{}$ is any involutory automorphism on $\mathbb{H}_S$ that satisfies $(a\hat{})\hat{} = a$, $(a+b)\hat{} = a\hat{} + b\hat{}$ and $(ab)\hat{} = a\hat{}\,b\hat{}$. If we choose $a\hat{} = a$ (the identity), then we have similarity relation on H_S which was introduced in [14]. If we choose $a\hat{} = \widetilde{a}$ (j-conjugation), then we have consimilarity relation in $\mathbb{H}_S$. Semisimilarity and consemisimilarity are closely related to similarity and consimilarity. Semisimilarity is defined as $b = f(a) = y\hat{}ax,\ a = f^{-1}(b) = x\hat{}ay$ where $x\hat{} = x$. Consemisimilarity is defined as $b = f(a) = y\hat{}ax$, $a = f^{-1}(b) = x\hat{}ay$ where $x\hat{} = \widetilde{x}$. If a and b are similar, then a and b are semisimilar. Thus semisimilarity is weaker than similarity in $\mathbb{H}_S$. In a similar way, If a and b are consimilar, then a and b are consemisimilar. Thus, consemisimilarity is weaker than consimilarity.

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