Research Article

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Convergence of generalized sampling series in weighted spaces

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Abstract: The present paper deals with an extension of approximation properties of generalized sampling series to weighted spaces of functions. A pointwise and uniform convergence theorem for the series is proved for functions belonging to weighted spaces. A rate of convergence by means of weighted moduli of continuity is presented and a quantitative Voronovskaja-type theorem is obtained.

Keywords: generalized sampling operators, weighted spaces, weighted modulus of continuity, Voronovskaja-type theorem, quantitative order of approximation

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1 Introduction

The sampling theory deals with the reconstruction of a function f with its sampled valued at some discrete points if the corresponding function satisfies certain conditions. In this frame, the pioneer result is the Whittaker-Kotel'nikov-Shannon (WKS) sampling theorem. An approximate version of WKS sampling theorem was developed at RWTH Aachen by P. L. Butzer and his school in the late 1970s. This version mainly depends on generalized sampling series defined by (see [1–3])

$$(G_w^\chi f)(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi(wx-k), \quad x\in\mathbb{R}, w>0, \tag{1.1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is any function for which the series (1.1) is convergent for every $x\in\mathbb{R}$, and $\chi:\mathbb{R}\to\mathbb{R}$ (called the *kernel* of the operator) denotes a continuous, discrete approximate identity which satisfies suitable assumptions. The operators G_w^χ represent a method in order to study in continuous functions spaces [1,2,4,5], simultaneous approximation and linear prediction [6], which means to predict the behavior of a given function f at time t by considering only sample values taken from the past with respect to t.

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Let $I = [a, b]$, $a, b \in \mathbb{R}$, $P_n(I) = \text{Span}\{1, x, x^2, \dots, x^n\}$ and $C(I)$ be the space of all continuous functions defined on I. The most elegant proof of Weierstrass approximation theorem by algebraic method is based on Bernstein polynomials which map $C(I)$ into $P_n(I)$ ([7]). On the other hand, G_w^χ can be seen as the counterpart of Bernstein polynomials in case of continuous functions on the whole real axis.

In the context of approximation theory, Bernstein polynomials are considered as a pioneer of the theory of linear positive operators. After construction of Bernstein polynomials, many other polynomials and operators were introduced such as Szász-Mirakyan operators, Baskakov operators, Chlodovsky polynomials [8,9]. The mentioned operators act on non-negative semi real axis. Since Weierstrass approximation theorem is given for functions belonging to $C(I)$, the studies on approximation of functions by such operators were restricted on compact intervals of $\mathbb{R}$.

The similar problem occurs in case of Bohman-Korovkin theorem, which is a systematic method to determine whether a family of linear positive operators $L_n : C(I) \to C(I)$ is an approximation process [10,11]. To overcome this problem, Gadjiev [12,13] introduced weighted spaces of functions and Korovkin-type theorem for functions belonging to these spaces. For the recent studies on weighted approximation of linear positive operators, we refer the readers to [14–16] and references therein.

In the present paper, we investigate approximation behaviors of generalized sampling operators (1.1) for functions belonging to weighted space of functions. After some preliminaries and the proof of well-definiteness of the operators (1.1) between weighted spaces of functions, we provide an estimate of the rate of convergence by means of weighted moduli of continuity. As a corollary of this estimate, we present a norm convergence of the operators (1.1). In order to present a rate of pointwise convergence and an upper bound for the approximation error in a unique theorem, we obtain a quantitative Voronovskaja-type theorem via weighted moduli of continuity.

One of the main advantages in considering functions belonging to a weighted space is that, any function that is bounded with respect to the corresponding norm of the space, can be unbounded with respect to the usual sup-norm; this allows us to enlarge the class of functions for which we consider the above approximation problems. Indeed, in the literature, approximation results by means of the generalized sampling series have been mainly considered in the space $C(\mathbb{R})$, or in other functional spaces, such as in Orlicz and modular spaces [17], but never in weighted-type spaces.

From the applications point of view, the possibility to approximate functions belonging to suitable weighted spaces that are in fact not necessarily bounded can be useful in order to approximate signal having polynomial grow at $\pm\infty$, as happens, e.g., for nonstationary (quadratic) power signals [18]. In the latter application, since one deals with quadratic signals, the choice of the weighted space generated by a quadratic-like weight function seems to be the more appropriate.

2 Preliminaries

Throughout the paper, a function $\chi : \mathbb{R} \to \mathbb{R}$ will be called a kernel function if it satisfies the following assumptions:

(χ1) χ is continuous on $\mathbb{R}$,

(χ2) the discrete algebraic moment of order 0:

$$m_0(\chi, u) := \sum_{k \in \mathbb{Z}} \chi(u - k) = 1,$$

for every $u \in \mathbb{R}$,

(χ3) there exists $\beta > 0$, such that the discrete absolute moments of order β:

$$M_\beta(\chi) = \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\chi(u - k)||u - k|^\beta,$$

is finite.

The following lemma holds (see [19]).

Lemma 1. *Let χ be a function satisfying* $(\chi 1)$ *and* $(\chi 3)$. *For every* $\delta > 0$ *there holds*:

$$\lim_{w\to\infty} \sum_{|k-wx|>w\delta} |\chi(wx-k)| = 0,$$

uniformly with respect to $x \in \mathbb{R}$.

It follows from [20, Lemma 2.1. (i)] that

$$M_\gamma(\chi) = \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\chi(u-k)||u-k|^\gamma < +\infty,$$

for every $0 \le \gamma \le \beta$ if χ satisfies the assumptions $(\chi 1)$ and $(\chi 3)$.

A function $\widetilde{w}$ is called a weight function if it is a positive continuous function on the whole real axis $\mathbb{R}$. In this study, we consider the weight function

$$\widetilde{w}(x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}.$$

By $B_{\widetilde{w}}(\mathbb{R})$, we shall denote the space of real functions whose product with the weight function $\widetilde{w}$ on $\mathbb{R}$ is bounded. Namely, we consider the set

$$B_{\widetilde{w}}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \sup_{x\in\mathbb{R}} \widetilde{w}(x)|f(x)| \in \mathbb{R} \right\}.$$

Now, we denote by $C^0(\mathbb{R})$ the space of continuous functions on the whole $\mathbb{R}$. We can also consider the following natural subspaces of $B_{\widetilde{w}}(\mathbb{R})$:

$$\begin{aligned}
C_{\widetilde{w}}(\mathbb{R}) &:= C^0(\mathbb{R}) \cap B_{\widetilde{w}}(\mathbb{R}),\\
C^*_{\widetilde{w}}(\mathbb{R}) &:= \left\{ f \in C_{\widetilde{w}}(\mathbb{R}) : \exists \lim_{x\to\mp\infty} \widetilde{w}(x)f(x) \in \mathbb{R} \right\},\\
U_{\widetilde{w}}(\mathbb{R}) &:= \{ f \in C_{\widetilde{w}}(\mathbb{R}) : \widetilde{w}f \text{ is uniformly continuous} \}.
\end{aligned}$$

The linear space of functions $B_{\widetilde{w}}(\mathbb{R})$, and its above subspaces are normed spaces with the norm

$$\|f\|_{\widetilde{w}} := \sup_{x\in\mathbb{R}} \widetilde{w}(x)|f(x)|$$

see [12–15,21].

The weighted modulus of continuity for functions $f \in C_{\widetilde{w}}(\mathbb{R})$ is denoted by $\Omega(f;\ \cdot)$ and given for $\delta > 0$ by

$$\Omega(f;\ \delta) := \sup_{|h|<\delta, x\in\mathbb{R}} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}. \tag{2.1}$$

For details related to this general modulus of continuity, one can see [22], in which the following elementary properties of $\Omega(f;\ \delta)$ can be found.

Lemma 2. [22] *Let* $\delta > 0$, $x \in \mathbb{R}$. *Then,*

(1) $\Omega(f;\ \delta)$ *is a monotonically increasing function of* δ;
(2) $\Omega(f;\ \delta) \to 0$ *as* $\delta \to 0$ *for functions* $f \in C^*_{\widetilde{w}}(\mathbb{R})$;
(3) *For each* $\lambda > 0$ *and* $f \in C_{\widetilde{w}}(\mathbb{R})$,

$$\Omega(f;\ \lambda\delta) \le 2(1+\lambda)(1+\delta^2)\Omega(f;\ \delta). \tag{2.2}$$

Remark 1. In inequality (2.2) if we replace $\lambda = \frac{|y-x|}{\delta}$, $x, y \in \mathbb{R}$, $\delta > 0$ and consider the definition of the weighted modulus of continuity, it turns out that

$$|f(y) - f(x)| \le 2\left(1 + \frac{|y-x|}{\delta}\right)(1+\delta^2)(1+x^2)(1+(y-x)^2)\Omega(f;\ \delta).$$

On the other hand, quantity of $|y-x|$ with respect to δ directs us to

$$|f(y)-f(x)| \leq \begin{cases} 4(1+\delta^2)^2(1+x^2)\Omega(f;\delta), & |y-x| \leq \delta, \\ 4(1+\delta^2)^2(1+x^2)\Omega(f;\delta)\dfrac{|y-x|^3}{\delta^3}, & |y-x| > \delta. \end{cases}$$

Hence, combining two cases of $|y-x|$ we obtain

$$|f(y)-f(x)| \leq 4(1+\delta^2)^2(1+x^2)\Omega(f;\delta)\left(1+\frac{|y-x|^3}{\delta^3}\right).$$

Finally, we obtain

$$|f(y)-f(x)| \leq 16(1+x^2)\Omega(f;\delta)\left(1+\frac{|y-x|^3}{\delta^3}\right), \tag{2.3}$$

with the choice of $\delta \leq 1$.

3 Main results

The first main result is to show that the operators G_w^χ are well-defined on weighted spaces of functions. First, we need the following preliminary proposition.

Proposition 1. *Let χ be a kernel satisfying the assumptions* $(\chi 1)$, $(\chi 2)$ *and* $(\chi 3)$ *for* $\beta = 2$. *Furthermore, we denote by* $\psi(x) := 1/\widetilde{w}(x) = 1+x^2$, $x \in \mathbb{R}$. *Then,*

$$|(G_w^\chi \psi)(x)| \leq (1+x^2)\left(M_0(\chi)+\frac{1}{w^2}M_2(\chi)+\frac{2}{w}M_1(\chi)\right), \quad x \in \mathbb{R},\ w>0, \tag{3.1}$$

holds.

Proof. For $w>0$, $x \in \mathbb{R}$ and $k \in \mathbb{Z}$, since

$$\psi\left(\frac{k}{w}\right) = 1+\left(\frac{k}{w}-x\right)^2+2x\left(\frac{k}{w}-x\right)+x^2,$$

the linearity of the operators G_w^χ allows us to write

$$\begin{aligned} |(G_w^\chi \psi)(x)| &\leq \sum_{k\in\mathbb{Z}} \psi\left(\frac{k}{w}\right)|\chi(wx-k)| \\ &= \sum_{k\in\mathbb{Z}} \left[1+\left(\frac{k}{w}-x\right)^2+2x\left(\frac{k}{w}-x\right)+x^2\right]|\chi(wx-k)| \\ &\leq (1+x^2)\sum_{k\in\mathbb{Z}}|\chi(wx-k)|+\frac{1}{w^2}\sum_{k\in\mathbb{Z}}|\chi(wx-k)|(k-wx)^2+\frac{2|x|}{w}\sum_{k\in\mathbb{Z}}|\chi(wx-k)||k-wx| \\ &\leq (1+x^2)\left(M_0(\chi)+\frac{1}{w^2}M_2(\chi)+\frac{2}{w}M_1(\chi)\right), \end{aligned}$$

which is desired. □

Theorem 1. *Let χ be a kernel satisfying the assumptions* $(\chi 1)$, $(\chi 2)$ *and* $(\chi 3)$ *for* $\beta = 2$. *Then, for a fixed* $w>0$, *the operator* G_w^χ *is a linear operator from* $B_{\widetilde{w}}(\mathbb{R})$ *to* $B_{\widetilde{w}}(\mathbb{R})$ *and its operator norm turns out to be*:

$$\|G_w^\chi\|_{B_{\widetilde{w}}\to B_{\widetilde{w}}} \leq \left(M_0(\chi)+\frac{1}{w^2}M_2(\chi)+\frac{2}{w}M_1(\chi)\right). \tag{3.2}$$

Proof. Let us fix $w > 0$. From (1.1), for $x \in \mathbb{R}$, we have

$$|(G_w^\chi f)(x)| \leq \sum_{k\in\mathbb{Z}} \left| \widetilde{w}\left(\frac{k}{w}\right) f\left(\frac{k}{w}\right) \right| \frac{1}{\widetilde{w}(k/w)} |\chi(wx - k)|.$$

Moreover, since $f \in B_{\widetilde{w}}(\mathbb{R})$, and recalling that $\psi := 1/\widetilde{w}$, we have

$$|(G_w^\chi f)(x)| \leq \|f\|_{\widetilde{w}} \sum_{k\in\mathbb{Z}} \psi\left(\frac{k}{w}\right) |\chi(wx - k)|$$

and using Proposition 1 we obtain

$$|(G_w^\chi f)(x)| \leq \|f\|_{\widetilde{w}} (1 + x^2) \left(M_0(\chi) + \frac{1}{w^2} M_2(\chi) + \frac{2}{w} M_1(\chi) \right),$$

which implies that

$$\frac{|(G_w^\chi f)(x)|}{(1 + x^2)} \leq \|f\|_{\widetilde{w}} \left(M_0(\chi) + \frac{1}{w^2} M_2(\chi) + \frac{2}{w} M_1(\chi) \right), \tag{3.3}$$

for every $x \in \mathbb{R}$. Since the assumption $M_2(\chi) < +\infty$ implies $M_j(\chi) < +\infty$ for $j = 0, 1$, we deduce $\|G_w^\chi f\|_{\widetilde{w}} < +\infty$, that is, $G_w^\chi f \in B_{\widetilde{w}}(\mathbb{R})$. On the other hand, taking the supremum over $x \in \mathbb{R}$ in (3.3) and the supremum with respect to $f \in B_{\widetilde{w}}(\mathbb{R})$ with $\|f\|_{\widetilde{w}} \leq 1$, we have (3.2). □

Now, the following approximation result can be established.

Theorem 2. *Let χ be a kernel satisfying the assumptions* (χ1), (χ2) *and* (χ3) *for $\beta = 2$. Moreover, let $f \in C_{\widetilde{w}}(\mathbb{R})$ be fixed. Then,*

$$\lim_{w\to\infty} (G_w^\chi f)(x) = f(x) \tag{3.4}$$

holds for every $x \in \mathbb{R}$. Moreover, if $f \in U_{\widetilde{w}}(\mathbb{R})$, then

$$\lim_{w\to\infty} \|G_w^\chi f - f\|_{\widetilde{w}} = 0. \tag{3.5}$$

Proof. First, for all $x \in \mathbb{R}$, $k \in \mathbb{Z}$ and $w > 0$, by a straightforward computation, the inequality

$$\left| f\left(\frac{k}{w}\right) - f(x) \right| \leq \widetilde{w}\left(\frac{k}{w}\right) \left| f\left(\frac{k}{w}\right) \right| \left| \frac{1}{\widetilde{w}\left(\frac{k}{w}\right)} - \frac{1}{\widetilde{w}(x)} \right| + \frac{1}{\widetilde{w}(x)} \left| \widetilde{w}\left(\frac{k}{w}\right) f\left(\frac{k}{w}\right) - \widetilde{w}(x) f(x) \right|$$

holds. Then, using (χ2) and the above inequality, we can write what follows:

$$\begin{aligned}
|(G_w^\chi f)(x) - f(x)| &\leq \sum_{k\in\mathbb{Z}} \left| f\left(\frac{k}{w}\right) - f(x) \right| |\chi(wx - k)| \\
&\leq \sum_{k\in\mathbb{Z}} |\chi(wx - k)| \left\{ \widetilde{w}\left(\frac{k}{w}\right) \left| f\left(\frac{k}{w}\right) \right| \left| \frac{1}{\widetilde{w}\left(\frac{k}{w}\right)} - \frac{1}{\widetilde{w}(x)} \right| + \frac{1}{\widetilde{w}(x)} \left| \widetilde{w}\left(\frac{k}{w}\right) f\left(\frac{k}{w}\right) - \widetilde{w}(x) f(x) \right| \right\} \\
&\leq \|f\|_{\widetilde{w}} \sum_{k\in\mathbb{Z}} |\chi(wx - k)| \left| \left(\frac{k}{w}\right)^2 - x^2 \right| + \frac{1}{\widetilde{w}(x)} \sum_{k\in\mathbb{Z}} |\chi(wx - k)| \left| \widetilde{w}\left(\frac{k}{w}\right) f\left(\frac{k}{w}\right) - \widetilde{w}(x) f(x) \right| \\
&=: I_1 + I_2.
\end{aligned}$$

Let us first consider I_1. Since $f \in C_{\widetilde{w}}(\mathbb{R})$, we obtain

$$\begin{aligned}I_1 &\le \|f\|_{\widetilde{w}} \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left(\left| \frac{k}{w} - x \right|^2 + 2|x| \left| \frac{k}{w} - x \right| \right) \\ &= \frac{\|f\|_{\widetilde{w}}}{w^2} \sum_{k\in\mathbb{Z}} |\chi(wx-k)||k-wx|^2 + \frac{2|x|\|f\|_{\widetilde{w}}}{w} \sum_{k\in\mathbb{Z}} |\chi(wx-k)||k-wx| \\ &= \frac{\|f\|_{\widetilde{w}}}{w^2} M_2(\chi) + \frac{2|x|\|f\|_{\widetilde{w}}}{w} M_1(\chi).\end{aligned}$$

Let us now consider I_2. Let $x \in \mathbb{R}$ and $\varepsilon > 0$ be fixed. Since f is continuous at x, wf is also continuous at x, hence there exists $\delta > 0$ such that $\left|\widetilde{w}\left(\frac{k}{w}\right)f\left(\frac{k}{w}\right) - \widetilde{w}(x)f(x)\right| < \varepsilon$ whenever $|k/w - x| < \delta$. Then we can write

$$\begin{aligned}I_2 &= \frac{1}{\widetilde{w}(x)} \sum_{|k-wx|\le w\delta} |\chi(wx-k)| \left| \widetilde{w}\left(\frac{k}{w}\right) f\left(\frac{k}{w}\right) - \widetilde{w}(x)f(x) \right| \\ &\quad + \frac{1}{\widetilde{w}(x)} \sum_{|k-wx|> w\delta} |\chi(wx-k)| \left| \widetilde{w}\left(\frac{k}{w}\right) f\left(\frac{k}{w}\right) - \widetilde{w}(x)f(x) \right| \\ &=: I_{2,1} + I_{2,2}.\end{aligned}$$

Hence, we can write

$$I_{2,1} < \frac{\varepsilon}{\widetilde{w}(x)} \sum_{|k-wx|\le w\delta} |\chi(wx-k)| \le \frac{\varepsilon}{\widetilde{w}(x)} M_0(\chi).$$

On the other hand, by Lemma 1 we have for sufficiently large $w > 0$ that

$$I_{2,2} \le \frac{2\,\|f\|_{\widetilde{w}}}{\widetilde{w}(x)} \sum_{|k-wx|> w\delta} |\chi(wx-k)| \le \frac{2\,\|f\|_{\widetilde{w}}}{\widetilde{w}(x)} \varepsilon.$$

Hence, we have

$$|(G_w^\chi f)(x) - f(x)| \le \frac{\|f\|_{\widetilde{w}}}{w^2} M_2(\chi) + \frac{2|x|\|f\|_{\widetilde{w}}}{w} M_1(\chi) + \frac{\varepsilon}{\widetilde{w}(x)} M_0(\chi) + \frac{2\,\|f\|_{\widetilde{w}}}{\widetilde{w}(x)} \varepsilon. \tag{3.6}$$

Taking limit of both sides as $w \to \infty$ we have (3.4).

For functions $f \in U_{\widetilde{w}}(\mathbb{R})$, let us follow the steps of above proof and replace δ with the corresponding parameter of the uniform continuity of $\widetilde{w}f$. Also considering inequality (3.6) we can write

$$\begin{aligned}\widetilde{w}(x)|(G_w^\chi f)(x) - f(x)| &\le \frac{\widetilde{w}(x)\|f\|_{\widetilde{w}}}{w^2} M_2(\chi) + \frac{2\widetilde{w}(x)|x|\|f\|_{\widetilde{w}}}{w} M_1(\chi) + \varepsilon M_0(\chi) + 2\,\|f\|_{\widetilde{w}} \varepsilon \\ &\le \frac{\|f\|_{\widetilde{w}}}{w^2} M_2(\chi) + \frac{2\,\|f\|_{\widetilde{w}}}{w} M_1(\chi) + \varepsilon(M_0(\chi) + 2\,\|f\|_{\widetilde{w}})\end{aligned} \tag{3.7}$$

and passing to the supremum in (3.7) over $x \in \mathbb{R}$ we have (3.5) for $w \to +\infty$. This completes the proof. □

The following quantitative estimate for the error of approximation can be established.

Theorem 3. *Let χ be a kernel satisfying the assumptions* (χ1), (χ2) *and* (χ3) *for $\beta = 3$. Then, for $f \in C_{\widetilde{w}}(\mathbb{R})$,*

$$\|G_w^\chi f - f\|_{\widetilde{w}} \le 16\Omega\left(f; \frac{1}{w}\right)(M_0(\chi) + M_3(\chi)), \quad w \ge 1,$$

holds.

Proof. Since the operators G_w^χ preserve constant functions, using the definition (1.1) we have for $f \in C_{\widetilde{w}}(\mathbb{R})$ that

$$|(G_w^\chi f)(x) - f(x)| = \sum_{k\in\mathbb{Z}} \left| f\left(\frac{k}{w}\right) - f(x) \right| |\chi(wx-k)|.$$

Also, from inequality (2.3), for any positive $\delta \leq 1$ we have

$$|(G_w^\chi f)(x) - f(x)| \leq 16\ (1 + x^2)\Omega(f;\ \delta)\left(M_0(\chi) + \frac{1}{(w\delta)^3}M_3(\chi)\right)$$

from which we deduce

$$\|G_w^\chi f - f\|_{\widetilde{w}} \leq 16\Omega(f;\ \delta)\left(M_0(\chi) + \frac{1}{(w\delta)^3}M_3(\chi)\right).$$

Finally, choosing $\delta = w^{-1}$, $w \geq 1$, we have

$$\|G_w^\chi f - f\|_{\widetilde{w}} \leq 16\Omega\left(f; \frac{1}{w}\right)(M_0(\chi) + M_3(\chi)).$$

which is desired. □

Corollary 1. *If we assume* $f \in C_{\widetilde{w}}^*(\mathbb{R})$ *in* Theorem 3, *by* 2 *of* Lemma 2 *we have*

$$\lim_{w\to\infty} \|G_w^\chi f - f\|_{\widetilde{w}} = 0.$$

Now, let $f \in C^r(\mathbb{R})$, $r \in \mathbb{N}$, the space of r-times continuously differentiable functions. The remainder in Taylor's formula at the point $x \in \mathbb{R}$ is given by

$$R_r(f;\ t, x) = f(t) - \sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!}(t - x)^k,$$

which can be written in the form

$$R_r(f;t, x) = \frac{(t - x)^r}{r!}(f^{(r)}(\xi) - f^{(r)}(x)), \tag{3.8}$$

where ξ is a number lying between t and x.

According to inequality (2.3), with similar method presented in [14], we can easily have the estimate

$$|R_r(f;t, x)| \leq \frac{16}{r!}(1 + x^2)\Omega(f^{(r)};\ \delta)\left(|t - x|^r + \frac{|t - x|^{r+3}}{\delta^3}\right). \tag{3.9}$$

For $j \in \mathbb{N}$, the algebraic moment of order j of a kernel χ is defined by

$$m_j(\chi, u) = \sum_{k\in\mathbb{Z}} \chi(u - k)(k - u)^j.$$

We have the following quantitative Voronovskaja-type theorem.

Theorem 4. *Let* χ *be a kernel satisfying the assumptions* (χ1), (χ2) *and* (χ3) *for* $\beta = 4$. *Furthermore, we assume in addition that the first-order algebraic moment of* χ *is constant, i.e.*:

$$m_1(\chi, x) = m_1(\chi) \in \mathbb{R}$$

for every $x \in \mathbb{R}$.

If $f' \in C_{\widetilde{w}}^*(\mathbb{R})$, *then we have for* $x \in \mathbb{R}$ *that*

$$|w[(G_w^\chi f)(x) - f(x)] - f'(x)m_1(\chi)| \leq 16(1 + x^2)\Omega(f';\ w^{-1})\{M_1(\chi) + M_4(\chi)\}. \tag{3.10}$$

If we suppose in addition $m_j(\chi, x) = 0$, *for every* $x \in \mathbb{R}$, *for* $j = 1, \ldots, r - 1$, $r \in \mathbb{N}$, *that* (χ3) *is satisfied for* $\beta = r + 3$, *and* $m_r(\chi, x) = m_r(\chi) \in \mathbb{R}$, *for every* $x \in \mathbb{R}$, *then we have for* $f^{(r)} \in C_{\widetilde{w}}^*(\mathbb{R})$ *that*

$$\left|w^r[(G_w^\chi f)(x) - f(x)] - \frac{f^{(r)}(x)}{r!}m_r(\chi)\right| \leq \frac{16}{r!}(1 + x^2)\Omega(f^{(r)};w^{-1})\{M_r(\chi) + M_{r+3}(\chi)\}. \tag{3.11}$$

Proof. Let us consider the Taylor expansion:

$$f(t) = \sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!}(t-x)^r + R_r(f;\, t, x),$$

where $R_r(f;t, x)$ is the Lagrange remainder as in (3.8). Using the above Taylor expansion in the definition of the operator $G_w^\chi f$, we can write what follows:

$$(G_w^\chi f)(x) = \sum_{k\in\mathbb{Z}} \chi(wx-k)\left\{\sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!}\left(\frac{k}{w}-x\right)^k\right\} + \sum_{k\in\mathbb{Z}} \chi(wx-k) R_r\left(f;\, \frac{k}{w}, x\right) := I_1 + I_2,$$

$x \in \mathbb{R}$, $w > 0$. Let us first consider I_1.

$$I_1 = \sum_{k=0}^{r} \frac{f^{(k)}(x)}{w^k k!} \sum_{k\in\mathbb{Z}} \chi(wx-k)(k-wx)^k = \sum_{k=0}^{r} \frac{f^{(k)}(x)}{w^k k!} m_k(\chi, wx).$$

To estimate I_2, using (3.9) we have

$$\begin{aligned}
|I_2| &\le \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left| R_r\left(f; \frac{k}{w}, x\right)\right| \\
&\le \sum_{k\in\mathbb{Z}} |\chi(wx-k)| \frac{16}{r!}(1+x^2)\Omega(f^{(r)};\, \delta)\left(\left|\frac{k}{w}-x\right|^r + \frac{\left|\frac{k}{w}-x\right|^{r+3}}{\delta^3}\right) \\
&= \frac{16}{r!}(1+x^2)\Omega(f^{(r)};\, \delta)\left\{\sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left|\frac{k}{w}-x\right|^r + \frac{1}{\delta^3}\sum_{k\in\mathbb{Z}} |\chi(wx-k)| \left|\frac{k}{w}-x\right|^{r+3}\right\} \\
&= \frac{16}{r!}(1+x^2)\Omega(f^{(r)};\, \delta)\left\{\frac{M_r(\chi)}{w^r} + \frac{1}{\delta^3}\frac{M_{r+3}(\chi)}{w^{r+3}}\right\}.
\end{aligned}$$

Hence, choosing $\delta = w^{-1}$, $w \ge 1$, we have

$$\left| w^r\left[(G_w^\chi f)(x) - \sum_{k=0}^{r} \frac{f^{(k)}(x)}{w^k k!} m_k(\chi, wx)\right]\right| \le \frac{16}{r!}(1+x^2)\Omega(f^{(r)}; w^{-1})\{M_r(\chi) + M_{r+3}(\chi)\}.$$

Now, recalling condition (χ2) and that $m_1(\chi, wx) = m_1(\chi)$, in the case $r = 1$ we immediately obtain the thesis. Furthermore, in the case $r > 1$, since $m_j(\chi, wx) = 0$ for $j = 1, \ldots, r-1$, $r \in \mathbb{N}$, $m_r(\chi, wx) = m_r(\chi)$, and $M_{r+3}(\chi) < +\infty$, we immediately have (3.11). □

Note that, for every $r \ge 1$, from Theorem 4 we immediately deduce the following Voronovskaja-type formula:

$$\lim_{w\to+\infty} w^r[(G_w^\chi f)(x) - f(x)] = \frac{f^{(r)}(x)}{r!} m_r(\chi).$$

Finally, we recall that several examples of kernel functions χ satisfying conditions (χ1), (χ2) and (χ3), with both bounded and unbounded support can be found, e.g., in [23–26]. For instance, we can mention the Jackson-type kernels, and the central B-splines of order n. We recall that the Jackson-type kernels are defined by means of (even) power of the well-known sinc-function, hence the finiteness of the discrete absolute moments (i.e., the value of the parameter β in the condition (χ3)) depends on the value of the considered power. However, in case of kernels with compact support (such as the aforementioned central B-splines) the discrete absolute moments are finite for any fixed $\beta > 0$. For more details on these respects, see, e.g., [25]. Furthermore, in [25] also a detailed description of the procedure for the computation of the discrete algebraic moments is given. Indeed, the latter is mainly based on the well-known Poisson summation formula of Fourier Analysis, based on the usual L^1-Fourier transform.

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