Communications in Advanced Mathematical Sciences
ISSN (Online) 2651-4001

Vol. 8, No. 4, 247-260, 2025
https://doi.org/10.33434/cams.1758799
Research Article

Direction Curves of Vectorial Moments with Respect to an Alternative Frame in Euclidean Space

Uğur Oruç [1], Murat Turan [2]*, Sıddıka Özkaldı Karakuş[3]

Abstract
This study introduces an alternative geometric framework for analyzing curves in three-dimensional Euclidean space. By constructing a set of alternative frame vectors and examining their corresponding derivative relations, a new perspective on the differential geometry of space curves is developed. Utilizing these frame vectors and a specifically defined Darboux vector, the vectorial moment curves associated with the original curve are derived. A key contribution of this work is the identification and characterization of the dual directed curves corresponding to these vectorial moments within the alternative frame structure. This approach offers new insights into the intrinsic properties of space curves and broadens the analytical tools available for applications in theoretical and applied geometry.

Keywords: Alternative frame, Direction curves, General helices, Vectorial moment
2020 MSC: 53A04, 53Z05

[1] *Department of Mathematics, Faculty of Science, Bilecik Seyh Edebali University, Bilecik, Türkiye, uguroruc.ip@gmail.com*
[2] *Department of Mathematics, Osmaniye Korkut Ata University, Osmaniye, Türkiye, muratturan@osmaniye.edu.tr*
[3] *Department of Mathematics, Faculty of Science, Bilecik Seyh Edebali University, Bilecik, Türkiye, siddika.karakus@bilecik.edu.tr*
*Corresponding author
Received: 05 August 2025, **Accepted:** 02 December 2025, **Available online:** 05 December 2025
How to cite this article: U. Oruç, M. Turan, S. Özkaldı Karakuş, *Direction curves of vectorial moments with respect to an alternative frame in Euclidean space*, Commun. Adv. Math. Sci., 8(4) (2025), 247-260.

1. Introduction

Differential geometry, particularly the study of curves in three-dimensional Euclidean space $\mathbb{E}^3$, plays a central role in understanding the geometric and kinematic properties of space curves. One of the fundamental approaches in this area involves examining how a position vector of a curve, which varies with a curve parameter (typically arc length), generates what is known as a vectorial moment curve. This concept explores how the original curve influences the geometry of a newly defined curve, derived from the moment of the vector with respect to some point or axis. Vectorial moment curves play a crucial role not only in static and dynamic analyses in engineering but also in various fields such as physics, mathematics and computer-aided design. In physics, these curves are essential for modeling rotational motion and maintaining moment equilibrium. From a geometrical perspective, they help us to visualize and analyze the relationships between the points of force application and axes of rotation. In computer-aided engineering software, these curves enable the simulation of structural responses like torsion, bending, and rotation in advance, allowing for more precise material optimization and safety analysis. Moreover, in applications ranging from robotic arm movements to bridge design, the directional effects of moment vectors are considered to create more functional and balanced systems. In this way, vectorial moment curves serve both as theoretical constructs and interdisciplinary tools that directly contribute to design and analysis processes.

In recent years, researchers have extended this idea to the moving frames of a curve, notably the Frenet frame. In [1], the vectorial moment curves associated with the Frenet vectors tangent (T), principal normal (N), and binormal (B) of a given space curve have been investigated. By applying moment operations to these frame vectors, new curves have been defined, and their Frenet apparatus (including curvature and torsion) has been computed. This method has revealed additional geometric structures inherent in the evolution of the original curve.

Building on this foundation, Kaya and Önder [2] have proposed an alternative orthonormal frame constructed from the principal normal vector N, the unit Darboux vector W and a unit vector C, which is defined by the cross product of N and W. This frame offers a novel perspective for analyzing space curves by providing an alternative to the classical Frenet-Serret framework, especially in contexts where the original frame becomes degenerate or less informative.

The helix curve, which is one of the well-known curves in geometry, has been studied under various frameworks [3, 4]. The associated curves between different frameworks and spaces have been investigated in several studies [5–9]. For further studies on vectorial moment curves, see [10–12]. Fundamental studies on curves can be found in [13, 14].

In this study, we take these investigations a step further. First, we calculate the Frenet vectors, curvatures, and torsions of the vectorial moment curves derived not from the classical Frenet frame, but from the vectors of the alternative orthonormal frame proposed by Kaya and Önder [2]. This includes deriving new curves using the moment operation applied to the vectors N, W, C, and then fully analyzing their geometric properties through the Frenet formulas.

Next, for each of these newly generated moment curves, we construct the corresponding alternative frame vectors. This allows us to track how the alternative geometric behavior of the original frame propagates through the vectorial moment transformation. The resulting analysis reveals deeper insights into the intrinsic and extrinsic geometries of the transformed curves.

Finally, we extend the theory by constructing the dual directed curves associated with these new integral curves. Dual geometry provides a powerful algebraic tool for analyzing spatial phenomena, particularly in kinematics and screw theory, and its application here offers a more comprehensive understanding of how curve dynamics evolve under vectorial transformations.

Through this framework, we aim to bridge the concepts of vectorial moments, alternative frames, and dual geometry, enriching the geometric study of curves in $\mathbb{E}^3$ and offering a platform for further research in both theoretical and applied contexts.

2. Alternative Frame in Euclidean Space

For a unit speed curve $\alpha(s)$, its Frenet apparatus $\{T, N, B, \kappa, \tau\}$, where T, N, B, κ, τ respectively are unit tangent vector fields, principal normal vector field, binormal vector field, the curvature and the torsion, exhibit rotational motion about an axis that depends on the parameter [15],

$$T = \frac{\alpha'}{\|\alpha'\|}, \qquad N = B \wedge T, \qquad B = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \qquad \kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}.$$

The vector W is called unit Darboux vector and defined by

$$W = \frac{1}{\sqrt{\kappa^2 + \tau^2}}(\tau T + \kappa B).$$

It is obvious that the Darboux vector is orthogonal to the principal normal vector N. If we define

$$C = W \wedge N = \frac{-\kappa}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} B,$$

then $\{N, C, W\}$ are another orthonormal moving frame along the curve [16]. This frame is called an alternative frame [2].

The relationship between the alternative frame and the Frenet frame can be observed through the relations given below:

$$C = \frac{-\kappa}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} B,$$

$$W = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} T + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} B.$$

If we say

$$\beta = \sqrt{\kappa^2 + \tau^2}, \qquad \bar{\kappa} = \frac{\kappa}{\beta}, \qquad \bar{\tau} = \frac{\tau}{\beta},$$

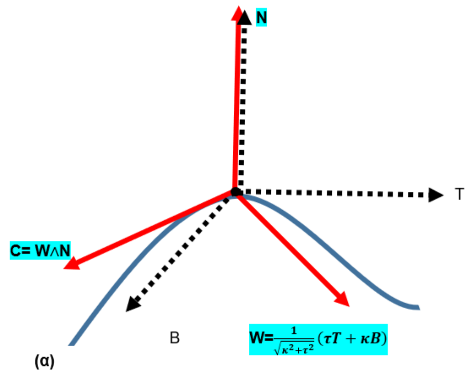

Figure 2.1. Alternative frame

then the following relations are obtained

$$\begin{cases} C = -\bar{\kappa}T + \bar{\tau}B \\ W = -\bar{\tau}T + \bar{\kappa}B \end{cases} \quad or \quad \begin{cases} T = -\bar{\kappa}C + \bar{\tau}W \\ B = -\bar{\tau}C + \bar{\kappa}W \end{cases},$$

which is also accessible in [17].

Theorem 2.1. *The relationship between the alternative frame vectors and their derivatives is given by*

$$N' = \beta C, \qquad C' = -\beta N + \gamma W, \qquad W' = -\gamma C, \qquad \gamma = \frac{\kappa^2}{\kappa^2+\tau^2}(\frac{\tau}{\kappa})',$$

[16].

Theorem 2.2. *Let the alternative frame vectors of the curve α be $\{N,C,W\}$. The Darboux vector with respect to this frame is given as [17],*

$$D = \gamma N + \beta W.$$

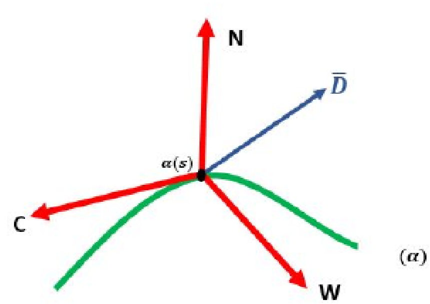

Figure 2.2. Darboux vector

Let $\{N,C,W\}$ be the alternative frame at the point $\alpha(s)$ of a unit-speed curve α. In this case, it can be expressed as

$$\alpha(s) = f(s)N(s) + g(s)C(s) + h(s)W(s).$$

If the derivative is taken,

$$\begin{aligned} f' &= g(s)\beta(s), \\ g' &= h(s)\gamma(s) - f(s)\beta(s) - \bar{\kappa}, \\ h' &= \bar{\tau}(s) - g(s)\gamma(s), \end{aligned}$$

relations between f, g, and h are observed [18].

Definition 2.3. *Let the alternative frame vectors of a unit speed regular curve α be $\{N, C, W\}$. The curve traced by the vectorial moment of the principal vector N, defined as $N^* = \alpha \wedge N$, is given by the relation, [18],*

$$\alpha_1(s) = h(s)C(s) - g(s)W(s).$$

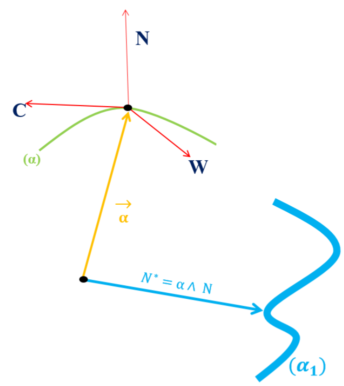

Figure 2.3. Vectorial moment curve of α_1

3. Dual Direction Curves of Vectorial Moments in Euclidean Space with Respect to Alternative Frames

In this section, some characterizations will be given by defining the $N-$dual direction curve, $C-$dual direction curve, and $W-$dual direction curve in E^3.

Definition 3.1. *Let α be an arc-length-parameterized regular curve with nonvanishing curvature and torsion, with parameter s. If the alternative frame of α is $\{N, C, W\}$, then $N-$dual direction curve of α with arc-length parameter s^* is defined as*

$$\psi_1(s^*) = \int_{s_0^*}^{s^*} (\alpha(s) \wedge N(s)) ds^*. \tag{3.1}$$

Let $\psi_1(s^*)$ be the $N-$dual direction curve of α. Let's calculate the Frenet vector fields, curvature and torsion of ψ_1.

Theorem 3.2. *Let $\alpha(s)$ be a regular curve with unit speed in $\mathbb{E}^3$, $\{N, C, W\}$ be an alternative frame and $\psi_1(s^*)$ be N-dual direction curve of α. Then the Frenet vector fields, curvature and torsion of $\psi_1(s^*)$ are given by*

$$T_{\psi_1} = \frac{hC - gW}{\sqrt{h^2 + g^2}},$$

$$N_{\psi_1} = \frac{-h\beta(g^2+h^2)N + g[g\bar{\tau} + h(f\beta + \bar{\kappa})]C + h[g\bar{\tau} + h(f\beta + \bar{\kappa})]W}{\sqrt{(g^2+h^2)\{h^2\beta^2(g^2+h^2) + [g\bar{\tau} + h(\bar{\kappa} + f\beta)]^2\}}},$$

$$B_{\psi_1} = \frac{(g\bar{\tau} + h(\bar{\kappa} + f\beta))N + gh\beta C + h^2\beta W}{\sqrt{h^2\beta^2(g^2+h^2) + (g\bar{\tau} + h(\bar{\kappa} + f\beta))^2}},$$

$$\kappa_{\psi_1} = \frac{\sigma_1\sqrt{h^2\beta^2(g^2+h^2) + (g\bar{\tau} + h(\bar{\kappa} + f\beta))^2}}{(\sqrt{h^2+g^2})^3},$$

$$\tau_{\psi_1} = \frac{\sigma_1 h\beta[h(f\beta)' - gh\beta^2 - fg\beta\gamma] - \sigma_1(fh\beta + h\bar{\kappa} + g\bar{\tau})(\tau + h\beta)'}{[(g\bar{\tau} + h(\bar{\kappa} + f\beta))^2 + (gh\beta)^2 + (h^2\beta)^2]}.$$

Proof. Let α be s arc-length parameterized curve and ψ_1 be s^* arc-length parameterized curve. Also, the alternative frame apparatus of α is $\{N, C, W, \kappa, \tau\}$ and the Frenet apparatus of ψ_1 is $\{T_{\psi_1}, N_{\psi_1}, B_{\psi_1}, \kappa_{\psi_1}, \tau_{\psi_1}\}$.
If we take the derivative of (3.1) with respect to s^* in this equation, we get

$$\begin{aligned}\psi_1'(s^*) &= \alpha(s)\wedge N(s),\\ &= h(s)C(s)-g(s)W(s).\end{aligned}$$

$$\psi_1''(s^*)=\frac{d\psi_1'(s)}{ds^*}=\frac{d\psi_1'(s)}{ds}\frac{ds}{ds^*}.$$

If we take $\dfrac{ds}{ds^*}=\sigma_1$, we obtain

$$\psi_1''(s^*)=\sigma_1(-h\beta N+\bar{\tau}C+(\bar{\kappa}+f\beta)W).$$

If the third derivative is taken

$$\begin{aligned}\psi_1'''(s^*) &= \frac{d\psi_1''(s)}{ds^*},\\ &= \frac{d\psi_1''(s)}{ds}\frac{ds}{ds^*},\\ &= P_1N+R_1C+S_1W,\end{aligned}$$

is found, where

$$\begin{aligned}P_1 &= -\sigma_1^2\tau-\sigma_1 h\beta\sigma_1'-\sigma_1^2(h\beta)',\\ R_1 &= -\sigma_1^2 h\beta^2-\sigma_1^2\gamma f\beta+\sigma_1\bar{\tau}\sigma_1',\\ S_1 &= \sigma_1^2(f\beta)'+\sigma_1\bar{\kappa}\sigma_1'+\sigma_1 f\beta\sigma_1'.\end{aligned}$$

Since it is

$$\psi_1'(s^*)=hC-gW,$$

$$\|\psi_1'(s^*)\|=\sqrt{g^2+h^2},$$

is found.

$$\begin{aligned}\psi_1'(s^*)\wedge\psi_1''(s^*) &= (hC-gW)\wedge(\sigma_1(-h\beta N+\bar{\tau}C+(\bar{\kappa}+f\beta)W)),\\ &= (\sigma_1 g\bar{\tau}+\sigma_1 h(\bar{\kappa}+f\beta))N+\sigma_1 gh\beta C+\sigma_1 h^2\beta W.\end{aligned}$$

$$\begin{aligned}\|\psi_1'(s^*)\wedge\psi_1''(s^*)\| &= \sqrt{(\sigma_1 g\bar{\tau}+\sigma_1 h(\bar{\kappa}+f\beta))^2+(\sigma_1 gh\beta)^2+(\sigma_1 h^2\beta)^2},\\ &= \sigma_1\sqrt{(g\bar{\tau}+h(\bar{\kappa}+f\beta))^2+h^2\beta^2(g^2+h^2)},\end{aligned}$$

is found.

$$\begin{aligned}det(\psi_1',\psi_1'',\psi_1''') &= \begin{vmatrix}0 & h & -g\\ -\sigma_1 h\beta & \sigma_1\bar{\tau} & \sigma_1(\bar{\kappa}+f\beta)\\ P_1 & R_1 & S_1\end{vmatrix},\\ &= \sigma_1^3 h\beta[h(f\beta)'-gh\beta^2-fg\beta\gamma]-\sigma_1^3(fh\beta+h\bar{\kappa}+g\bar{\tau})(\tau+h\beta)'.\end{aligned}$$

Using these equations, the expressions

$$T_{\psi_1}=\frac{hC-gW}{\sqrt{h^2+g^2}},$$

$$N_{\psi_1}=\frac{-h\beta(g^2+h^2)N+g[g\bar{\tau}+h(f\beta+\bar{\kappa})]C+h[g\bar{\tau}+h(f\beta+\bar{\kappa})]W}{\sqrt{(g^2+h^2)\{h^2\beta^2(g^2+h^2)+[g\bar{\tau}+h(\bar{\kappa}+f\beta)]^2\}}},$$

$$B_{\psi_1}=\frac{(g\bar{\tau}+h(\bar{\kappa}+f\beta))N+gh\beta C+h^2\beta W}{\sqrt{h^2\beta^2(g^2+h^2)+(g\bar{\tau}+h(\bar{\kappa}+f\beta))^2}},$$

$$\kappa_{\psi_1} = \frac{\sigma_1\sqrt{h^2\beta^2(g^2+h^2)+(g\bar{\tau}+h(\bar{\kappa}+f\beta))^2}}{(\sqrt{h^2+g^2})^3},$$

$$\tau_{\psi_1} = \frac{\sigma_1 h\beta[h(f\beta)'-gh\beta^2-fg\beta\gamma]-\sigma_1(fh\beta+h\bar{\kappa}+g\bar{\tau})(\tau+h\beta)'}{[(g\bar{\tau}+h(\bar{\kappa}+f\beta))^2+(gh\beta)^2+(h^2\beta)^2]},$$

can be obtained. □

Definition 3.3. *Let α be an arc-length-parameterized regular curve with nonvanishing curvature and torsion, with parameter s. If the alternative frame of α is $\{N,C,W\}$, then $C-$dual direction curve of α with arc-length parameter s^* is defined as*

$$\psi_2(s^*) = \int_{s_0^*}^{s^*}(\alpha(s)\wedge C(s))ds^*. \tag{3.2}$$

Let $\psi_2(s^*)$ be the $C-$dual direction curve of α. Let's calculate the Frenet vector fields, curvature and torsion of ψ_2.

Theorem 3.4. *Let $\alpha(s)$ be a regular curve with unit speed in $\mathbb{E}^3$, $\{N,C,W\}$ be an alternative frame and $\psi_2(s^*)$ be C-dual direction curve of α. Then the Frenet vector fields, curvature and torsion of $\psi_2(s^*)$ are given by*

$$T_{\psi_2} = \frac{-hN+fW}{\sqrt{f^2+h^2}},$$

$$N_{\psi_2} = \frac{f[gh\beta-f(\bar{\tau}-g\gamma)]N-(h\beta+f\gamma)(f^2+h^2)C+h[gh\beta-f(\bar{\tau}-g\gamma))]W}{\sqrt{(f^2+h^2)\{(h\beta+f\gamma)^2(f^2+g^2)+[gh\beta-f(\bar{\tau}-g\gamma)]^2\}}},$$

$$B_{\psi_2} = \frac{f(h\beta+f\gamma)N+(hg\beta-f(\bar{\tau}-g\gamma))C+(h(h\beta+f\gamma))W}{\sqrt{(h\beta+f\gamma)^2(f^2+g^2)+[gh\beta-f(\bar{\tau}-g\gamma)]^2}},$$

$$\kappa_{\psi_2} = \frac{\sigma_2\sqrt{(h\beta+f\gamma)^2(f^2+g^2)+[gh\beta-f(\bar{\tau}-g\gamma)]^2}}{(\sqrt{f^2+h^2})^3},$$

$$\tau_{\psi_2} = \frac{\begin{pmatrix} \sigma_2 h(h\beta+f\gamma)(g\beta)'-\sigma_2 h\gamma(h\beta+f\gamma)^2+\sigma_2 f(\bar{\tau}-g\gamma)(h\beta+f\gamma)' \\ +f(\bar{\tau}-g\gamma)(h\beta+f\gamma)\sigma_2'+\sigma_2\beta\bar{\tau}(\bar{\tau}-g\gamma)-\sigma_2 h(h\beta+f\gamma)(\bar{\tau}-g\gamma)' \\ -h(h\beta+f\gamma)(\bar{\tau}-g\gamma)\sigma_2'+\sigma_2 h\beta(h\beta+f\gamma)^2-\sigma_2 gh\beta(h\beta+f\gamma)'-\sigma_2 gh\beta^2\bar{\tau} \end{pmatrix}}{\{(h\beta+f\gamma)^2(f^2+g^2)+[gh\beta-f(\bar{\tau}-g\gamma)]^2\}}.$$

Proof. Let α be s arc-length parameterized curve and ψ_2 be s^* arc-length parameterized curve. Also, the alternative frame apparatus of α is $\{N,C,W,\kappa,\tau\}$ and the Frenet apparatus of ψ_2 is $\{T_{\psi_2},N_{\psi_2},B_{\psi_2},\kappa_{\psi_2},\tau_{\psi_2}\}$.
If we take the derivative of (3.2) with respect to s^* in this equation, we get

$$\begin{aligned}\psi_2'(s^*) &= \alpha(s)\wedge C(s), \\ &= -h(s)N(s)+f(s)W(s),\end{aligned}$$

$$\psi_2''(s^*) = \frac{d\psi_2'(s)}{ds^*} = \frac{d\psi_2'(s)}{ds}\frac{ds}{ds^*}.$$

If we take $\frac{ds}{ds^*}=\sigma_2$, we obtain

$$\psi_2''(s^*) = \sigma_2(-(\bar{\tau}-g\gamma)N-(h\beta+f\gamma)C+g\beta W).$$

If the third derivative is taken

$$\begin{aligned}\psi_2'''(s^*) &= \frac{d\psi_2''(s)}{ds^*}, \\ &= \frac{d\psi_2''(s)}{ds}\frac{ds}{ds^*}, \\ &= P_2N+R_2C+S_2W,\end{aligned}$$

is found, where

$$\begin{aligned} P_2 &= -\sigma_2\sigma_2'(\bar{\tau}-g\gamma)-\sigma_2^2(\bar{\tau}-g\gamma)'+\sigma_2^2\beta(h\beta+f\gamma), \\ R_2 &= -\sigma_2\sigma_2'(h\beta-f\gamma)-\sigma_2^2(h\beta+f\gamma)'-\sigma_2^2\beta\bar{\tau}, \\ S_2 &= \sigma_2\sigma_2' g\beta-\sigma_2^2\gamma(h\beta+f\gamma)+\sigma_2^2(g\beta)'. \end{aligned}$$

Since it is

$$\psi_2'(s^*) = -hN+fW,$$

$$\|\psi_2'(s^*)\| = \sqrt{f^2+h^2},$$

is found.

$$\begin{aligned} \psi_2'(s^*)\wedge\psi_2''(s^*) &= \sigma_2 f(h\beta+f\gamma)N+(\sigma_2 hg\beta-\sigma_2 f(\bar{\tau}-g\gamma))C+\sigma_2 h(h\beta+f\gamma)W, \\ \|\psi_2'(s^*)\wedge\psi_2''(s^*)\| &= \sigma_2\sqrt{(h\beta+f\gamma)^2(f^2+h^2)+[hg\beta+f(\bar{\tau}-g\gamma)]^2}, \end{aligned}$$

is found.

$$\begin{aligned} det(\psi_2',\psi_2'',\psi_2''') &= \begin{vmatrix} -h & 0 & f \\ -\sigma_2(\bar{\tau}-g\gamma) & -\sigma_2(h\beta+f\gamma) & \sigma_2 g\beta \\ P_2 & R_2 & S_2 \end{vmatrix}, \\ &= \sigma_2^3 h(h\beta+f\gamma)(g\beta)'-\sigma_2^3 h\gamma(h\beta+f\gamma)^2+\sigma_2^3 f(\bar{\tau}-g\gamma)(h\beta+f\gamma)' \\ &\quad +\sigma_2^2 f(\bar{\tau}-g\gamma)(h\beta+f\gamma)\sigma_2'+\sigma_2^3\beta\bar{\tau}(\bar{\tau}-g\gamma)-\sigma_2^3 h(h\beta+f\gamma)(\bar{\tau}-g\gamma)' \\ &\quad -\sigma_2^2 h(h\beta+f\gamma)(\bar{\tau}-g\gamma)\sigma_2'+\sigma_2^3 h\beta(h\beta+f\gamma)^2-\sigma_2^3 gh\beta(h\beta+f\gamma)'-\sigma_2^3 gh\beta^2\bar{\tau}. \end{aligned}$$

Using these equations, the expressions

$$T_{\psi_2} = \frac{-hN+fW}{\sqrt{f^2+h^2}},$$

$$N_{\psi_2} = \frac{f[gh\beta-f(\bar{\tau}-g\gamma)]N-(h\beta+f\gamma)(f^2+h^2)C+h[gh\beta-f(\bar{\tau}-g\gamma))]W}{\sqrt{(f^2+h^2)\{(h\beta+f\gamma)^2(f^2+g^2)+[gh\beta-f(\bar{\tau}-g\gamma)]^2\}}},$$

$$B_{\psi_2} = \frac{f(h\beta+f\gamma)N+(hg\beta-f(\bar{\tau}-g\gamma))C+(h(h\beta+f\gamma))W}{\sqrt{(h\beta+f\gamma)^2(f^2+g^2)+[gh\beta-f(\bar{\tau}-g\gamma)]^2}},$$

$$\kappa_{\psi_2} = \frac{\sigma_2\sqrt{(h\beta+f\gamma)^2(f^2+g^2)+[gh\beta-f(\bar{\tau}-g\gamma)]^2}}{(\sqrt{f^2+h^2})^3},$$

$$\tau_{\psi_2} = \frac{\begin{pmatrix} \sigma_2 h(h\beta+f\gamma)(g\beta)'-\sigma_2 h\gamma(h\beta+f\gamma)^2+\sigma_2 f(\bar{\tau}-g\gamma)(h\beta+f\gamma)' \\ +f(\bar{\tau}-g\gamma)(h\beta+f\gamma)\sigma_2'+\sigma_2\beta\bar{\tau}(\bar{\tau}-g\gamma)-\sigma_2 h(h\beta+f\gamma)(\bar{\tau}-g\gamma)' \\ -h(h\beta+f\gamma)(\bar{\tau}-g\gamma)\sigma_2'+\sigma_2 h\beta(h\beta+f\gamma)^2-\sigma_2 gh\beta(h\beta+f\gamma)'-\sigma_2 gh\beta^2\bar{\tau} \end{pmatrix}}{\{(h\beta+f\gamma)^2(f^2+g^2)+[gh\beta-f(\bar{\tau}-g\gamma)]^2\}},$$

can be obtained. □

Definition 3.5. *Let α be an arc-length-parameterized regular curve with nonvanishing curvature and torsion, with parameter s. If the alternative frame of α is $\{N,C,W\}$, then $W-$dual direction curve of α with arc-length parameter s^* is defined as*

$$\psi_3(s^*) = \int_{s_0^*}^{s^*}(\alpha(s)\wedge W(s))ds^*. \tag{3.3}$$

Let $\psi_3(s^*)$ be the $W-$dual direction curve of α. Let's calculate the Frenet vector fields, curvature and torsion of ψ_3.

Theorem 3.6. *Let $\alpha(s)$ be a regular curve with unit speed in $\mathbb{E}^3$, $\{N,C,W\}$ be an alternative frame and $\psi_3(s^*)$ be W-dual direction curve of α. Then the Frenet vector fields, curvature and torsion of $\psi_3(s^*)$ are given by*

$$T_{\psi_3} = \frac{gN - fC}{\sqrt{f^2+g^2}},$$

$$N_{\psi_3} = \frac{f(h\gamma-\bar{\kappa})N + g(h\gamma-\bar{\kappa})C - \gamma(f^2+g^2)W}{\sqrt{(f^2+g^2)((f\gamma)^2+(g\gamma)^2+(h\gamma-\bar{\kappa})^2)}},$$

$$B_{\psi_3} = \frac{f\gamma N + g\gamma C + (h\gamma-\bar{\kappa})W}{\sqrt{(f\gamma)^2+(g\gamma)^2+(h\gamma-\bar{\kappa})^2}},$$

$$\kappa_{\psi_3} = \frac{\sigma_3 f\sqrt{(f\gamma)^2+(g\gamma)^2+(h\gamma-\bar{\kappa})^2}}{\sqrt{(f^2+g^2)^3}},$$

$$\tau_{\psi_3} = \frac{\sigma_3 f^2\gamma(h\gamma-\bar{\kappa})' + \sigma_3 fg\beta\gamma(h\gamma-\bar{\kappa}) + \sigma_3 f^2 g\gamma^3 - \sigma_3 f(h\gamma-\bar{\kappa})(f\gamma)'}{f^2[(f\gamma)^2+(g\gamma)^2+(h\gamma-\bar{\kappa})^2]}.$$

Proof. Let α be s arc-length parameterized curve and ψ_3 be s^* arc-length parameterized curve. Also, the alternative frame apparatus of α is $\{N,C,W,\kappa,\tau\}$ and the Frenet apparatus of ψ_3 is $\{T_{\psi_3}, N_{\psi_3}, B_{\psi_3}, \kappa_{\psi_3}, \tau_{\psi_3}\}$.
If we take the derivative of (3.3) with respect to s^* in this equation, we get

$$\begin{aligned} \psi_3'(s^*) &= \alpha(s)\wedge W(s), \\ &= g(s)N(s) - f(s)C(s), \end{aligned}$$

is found.

$$\psi_3''(s^*) = \frac{d\psi_3'(s)}{ds^*} = \frac{d\psi_3'(s)}{ds}\frac{ds}{ds^*}.$$

If we take $\dfrac{ds}{ds^*} = \sigma_3$, we obtain

$$\psi_3''(s^*) = \sigma_3(h\gamma-\bar{\kappa})N - \sigma_3 f\gamma W.$$

If the third derivative is taken

$$\begin{aligned} \psi_3'''(s^*) &= \frac{d\psi_3''(s)}{ds^*}, \\ &= \frac{d\psi_3''(s)}{ds}\frac{ds}{ds^*}, \\ &= P_3N + R_3C + S_3W, \end{aligned}$$

is found, where

$$\begin{aligned} P_3 &= \sigma_3\sigma_3'(h\gamma-\bar{\kappa}) + \sigma_3^2(h\gamma-\bar{\kappa})', \\ R_3 &= \sigma_3^2\beta(h\gamma-\bar{\kappa}) + \sigma_3^2 f\gamma^2, \\ S_3 &= -\sigma_3\sigma_3' f\gamma - \sigma_3^2(f\gamma)'. \end{aligned}$$

Since it is,

$$\psi_3'(s^*) = gN - fC,$$

$$\|\psi_3'(s^*)\| = \sqrt{f^2+g^2},$$

is found.

$$\begin{aligned} \psi_3'(s^*)\wedge\psi_3''(s^*) &= (gN - fC)\wedge(\sigma_3(h\gamma-\bar{\kappa})N - \sigma_3 f\gamma W), \\ &= \sigma_3 f^2\gamma N + \sigma_3 fg\gamma C + \sigma_3 f(h\gamma-\bar{\kappa})W, \end{aligned}$$

$$\begin{aligned}\|\psi_3'(s^*)\wedge\psi_3''(s^*)\| &= \sqrt{(\sigma_3 f^2\gamma)^2+(\sigma_3 fg\gamma)^2+(\sigma_3 f(h\gamma-\bar\kappa))^2},\\ &= \sigma_3 f\sqrt{(f\gamma)^2+(g\gamma)^2+(h\gamma-\bar\kappa)^2},\end{aligned}$$

is found.

$$\begin{aligned}det(\psi_3',\psi_3'',\psi_3''') &= \begin{vmatrix} g & -f & 0\\ +\sigma_3(h\gamma-\bar\kappa) & 0 & -\sigma_3 f\gamma\\ P_3 & R_3 & S_3\end{vmatrix},\\ &= \sigma_3^3 f\gamma(h\gamma-\bar\kappa)'+\sigma_3^3 fg\gamma\beta(h\gamma-\bar\kappa)+\sigma_3^3 f^2g\gamma^3-\sigma_3^3 f(h\gamma-\bar\kappa)(f\gamma)'.\end{aligned}$$

Using these equations, the expressions

$$T_{\psi_3}=\frac{gN-fC}{\sqrt{f^2+g^2}},$$

$$N_{\psi_3}=\frac{f(h\gamma-\bar\kappa)N+g(h\gamma-\bar\kappa)C-\gamma(f^2+g^2)W}{\sqrt{(f^2+g^2)((f\gamma)^2+(g\gamma)^2+(h\gamma-\bar\kappa)^2)}},$$

$$B_{\psi_3}=\frac{f\gamma N+g\gamma C+(h\gamma-\bar\kappa)W}{\sqrt{(f\gamma)^2+(g\gamma)^2+(h\gamma-\bar\kappa)^2}},$$

$$\kappa_{\psi_3}=\frac{\sigma_3 f\sqrt{(f\gamma)^2+(g\gamma)^2+(h\gamma-\bar\kappa)^2}}{\sqrt{(f^2+g^2)^3}},$$

$$\tau_{\psi_3}=\frac{\sigma_3 f^2\gamma(h\gamma-\bar\kappa)'+\sigma_3 fg\beta\gamma(h\gamma-\bar\kappa)+\sigma_3 f^2g\gamma^3-\sigma_3 f(h\gamma-\bar\kappa)(f\gamma)'}{f^2[(f\gamma)^2+(g\gamma)^2+(h\gamma-\bar\kappa)^2]},$$

can be obtained. □

Definition 3.7. *Let α be an arc-length-parameterized regular curve with nonvanishing curvature and torsion, with parameter s. If the alternative frame of α is $\{N,C,W\}$, then $D-$dual direction curve of α with arc-length parameter s^* is defined as*

$$\psi_4(s^*)=\int_{s_0^*}^{s^*}(\alpha(s)\wedge D(s))ds^*. \tag{3.4}$$

Let $\psi_4(s^*)$ be the $D-$dual direction curve of α. Let's calculate the Frenet vector fields, curvature and torsion of ψ_4.

Theorem 3.8. *Let $\alpha(s)$ be a regular curve with unit speed in $\mathbb{E}^3$, $\{N,C,W\}$ be an alternative frame and $\psi_4(s^*)$ be D-dual direction curve of α. Then the Frenet vector fields, curvature and torsion of $\psi_4(s^*)$ are given by*

$$T_{\psi_4}=\frac{g\beta N+(h\gamma-f\beta)C-g\gamma W}{\sqrt{(g\beta)^2+(h\gamma-f\beta)^2+(g\gamma)^2}},$$

$$N_{\psi_4}=\frac{\left(\begin{array}{c}\left\{\begin{array}{c}-g\gamma[g\beta(g\gamma)'-g\gamma(g\beta)']-(h\gamma-f\beta)[g^2\beta^3+g\beta(h\gamma-f\beta)'\\ +g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'+\beta(h\gamma-f\beta)]\end{array}\right\}N\\ +\left\{\begin{array}{c}g\beta[g^2\beta^3+g\beta(h\gamma-f\beta)'+g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'+\beta(h\gamma-f\beta)]\\ +g\gamma[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'+g^2\gamma^3]\end{array}\right\}C\\ +\left\{\begin{array}{c}(h\gamma-f\beta)[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma\\ +g\gamma(h\gamma-f\beta)'+g^2\gamma^3]-g\beta[g\beta(g\gamma)'-g\gamma(g\beta)']\end{array}\right\}W\end{array}\right)}{\sqrt{\begin{array}{c}[(g\beta)^2+(h\gamma-f\beta)^2+(g\gamma)^2]\{[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma\\ +g\gamma(h\gamma-f\beta)'+g^2\gamma^3]^2+[g\beta(g\gamma)'-g\gamma(g\beta)']^2+[g^2\beta^3+g\beta(h\gamma-f\beta)'g^2\beta\gamma^2\\ -(h\gamma-f\beta)(g\beta)'+\beta(h\gamma-f\beta)]^2\},\end{array}}},$$

$$B_{\psi_4}=\frac{\begin{pmatrix}[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'+g^2\gamma^3]N\\+[g\beta(g\gamma)'-g\gamma(g\beta)']C\\+[g^2\beta^3+g\beta(h\gamma-f\beta)'g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'+\beta(h\gamma-f\beta)]W\end{pmatrix}}{\sqrt{\begin{matrix}[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'+g^2\gamma^3]^2\\+[g\beta(g\gamma)'-g\gamma(g\beta)']^2+[g^2\beta^3+g\beta(h\gamma-f\beta)'g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'\\+\beta(h\gamma-f\beta)]^2\end{matrix}}},$$

$$\kappa_{\psi_4}=\frac{\sigma_4\sqrt{\begin{matrix}[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'+g^2\gamma^3]^2\\+[g\beta(g\gamma)'-g\gamma(g\beta)']^2+[g^2\beta^3+g\beta(h\gamma-f\beta)'g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'\\+\beta(h\gamma-f\beta)]^2\end{matrix}}}{(\sqrt{(g\beta)^2+(h\gamma-f\beta)^2+(g\gamma)^2})^3},$$

$$\tau_{\psi_4}=\frac{\begin{pmatrix}\{g\gamma(g\beta^2+(h\gamma-f\beta)'+g\gamma^2)+(h\gamma-f\beta)(\gamma(h\gamma-f\beta)-(g\gamma)')\}P_4\\+[g\beta(g\gamma)'-g\gamma(g\beta)']R_4\\+\{g\beta(g\beta^2+(h\gamma-f\beta)'+g\gamma^2)-(h\gamma-f\beta)((g\beta)'-\beta(h\gamma-f\beta))\}S_4\end{pmatrix}}{\sigma_4^2\left\{\begin{matrix}[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'+g^2\gamma^3]^2\\+[g\beta(g\gamma)'-g\gamma(g\beta)']^2+[g^2\beta^3+g\beta(h\gamma-f\beta)'g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'\\+\beta(h\gamma-f\beta)]^2\end{matrix}\right\}}.$$

Proof. Let α be curve with arc lenght parameter s and ψ_4 be s^* arc-length parameterized curve. Also, the alternative frame apparatus of α is $\{N,C,W,\kappa,\tau\}$ and the Frenet apparatus of ψ_4 is $\{T_{\psi_4},N_{\psi_4},B_{\psi_4},\kappa_{\psi_4},\tau_{\psi_4}\}$.
If we take the derivative of (3.4) with respect to s^* in this equation, we get

$$\begin{aligned}\psi_4'(s^*) &= \alpha(s)\wedge D(s),\\ &= g\beta N+(h\gamma-f\beta)C-g\gamma W,\end{aligned}$$

is found.

$$\psi_4''(s^*)=\frac{d\psi_4'(s)}{ds^*}=\frac{d\psi_4'(s)}{ds}\frac{ds}{ds^*}.$$

If we take $\dfrac{ds}{ds^*}=\sigma_4$, we obtain

$$\psi_4''(s^*)=\sigma_4\{[(g\beta)'-\beta(h\gamma-f\beta)]N+[g\beta^2+(h\gamma-f\beta)'+g\gamma^2]C+[\gamma(h\gamma-f\beta)-(g\gamma)']W\}.$$

If the third derivative is taken

$$\begin{aligned}\psi_4'''(s^*) &= \frac{d\psi_4''(s)}{ds^*},\\ &= \frac{d\psi_4''(s)}{ds}\frac{ds}{ds^*},\\ &= P_4N+R_4C+S_4W,\end{aligned}$$

is found, where

$$\begin{aligned}P_4 &= \sigma_4[(g\beta)'-\beta(h\gamma-f\beta)]\sigma_4'+-\sigma_4^2[(g\beta)'-\beta(h\gamma-f\beta)]'-\sigma_4^2\beta[g\beta^2+(h\gamma-f\beta)'+g\gamma^2],\\ R_4 &= \sigma_4[g\beta^2+(h\gamma-f\beta)'g\gamma^2]\sigma_4'+\sigma_4^2[g\beta^2+(h\gamma-f\beta)'g\gamma^2]'+\sigma_4^2\beta[(g\beta)'-\beta(h\gamma-f\beta)]\\ &\quad -\sigma_4^2\gamma[\gamma(h\gamma-f\beta)-(g\gamma)'],\\ S_4 &= \sigma_4[\gamma(h\gamma-f\beta)-(g\gamma)']\sigma_4'+\sigma_4^2\gamma[g\beta^2+(h\gamma-f\beta)'+g\gamma^2]+\sigma_4^2[\gamma(h\gamma-f\beta)-(g\gamma)']'.\end{aligned}$$

Since it is

$$\begin{aligned}\psi_4'(s^*) &= g\beta N+(h\gamma-f\beta)C-g\gamma W,\\ \|\psi_4'(s^*)\| &= \sqrt{(g\beta)^2+(h\gamma-f\beta)^2+(g\gamma)^2},\end{aligned}$$

is found.

$$\begin{aligned}\psi_4'(s^*)\wedge\psi_4''(s^*) &= \sigma_4\{[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'+g^2\gamma^3]N\\ &+[g\beta(g\gamma)'-g\gamma(g\beta)']C+[g^2\beta^3+g\beta(h\gamma-f\beta)'+g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'\\ &+\beta(h\gamma-f\beta)]W\},\end{aligned}$$

$$\|\psi_4'(s^*)\wedge\psi_4''(s^*)\|=\sigma_4\sqrt{\begin{array}{c}[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'\\ +g^2\gamma^3]^2+[g\beta(g\gamma)'-g\gamma(g\beta)']^2+[g^2\beta^3\\ +g\beta(h\gamma-f\beta)'+g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'+\beta(h\gamma-f\beta)]^2\end{array}},$$

is found.

$$\begin{aligned}det(\psi_4',\psi_4'',\psi_4''') &= \begin{vmatrix} g\beta & h\gamma-f\beta & -g\gamma\\ (g\beta)'-\beta(h\gamma-f\beta) & g\beta^2+(h\gamma-f\beta)'+g\gamma^2 & \gamma(h\gamma-f\beta)-(g\gamma)'\\ P_4 & R_4 & S_4\end{vmatrix},\\ &=\{g\gamma(g\beta^2+(h\gamma-f\beta)'+g\gamma^2)+(h\gamma-f\beta)(\gamma(h\gamma-f\beta)-(g\gamma)')\}P_4\\ &+\{g\beta(g\gamma)'-g\gamma(g\beta)'\}R_4\\ &+\{g\beta(g\beta^2+(h\gamma-f\beta)'+g\gamma^2)-(h\gamma-f\beta)((g\beta)'-\beta(h\gamma-f\beta))\}S_4.\end{aligned}$$

Using these equations, the expressions

$$T_{\psi_4}=\frac{g\beta N+(h\gamma-f\beta)C-g\gamma W}{\sqrt{(g\beta)^2+(h\gamma-f\beta)^2+(g\gamma)^2}},$$

$$N_{\psi_4}=\frac{\left(\begin{array}{c}\left\{\begin{array}{c}-g\gamma[g\beta(g\gamma)'-g\gamma(g\beta)']-(h\gamma-f\beta)[g^2\beta^3+g\beta(h\gamma-f\beta)'\\ +g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'+\beta(h\gamma-f\beta)]\end{array}\right\}N\\ +\left\{\begin{array}{c}g\beta[g^2\beta^3+g\beta(h\gamma-f\beta)'+g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'+\beta(h\gamma-f\beta)]\\ +g\gamma[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'+g^2\gamma^3]\end{array}\right\}C\\ +\left\{\begin{array}{c}(h\gamma-f\beta)[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma\\ +g\gamma(h\gamma-f\beta)'+g^2\gamma^3]-g\beta[g\beta(g\gamma)'-g\gamma(g\beta)']\end{array}\right\}W\end{array}\right)}{\sqrt{\begin{array}{c}[(g\beta)^2+(h\gamma-f\beta)^2+(g\gamma)^2]\{[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma\\ +g\gamma(h\gamma-f\beta)'+g^2\gamma^3]^2+[g\beta(g\gamma)'-g\gamma(g\beta)']^2+[g^2\beta^3+g\beta(h\gamma-f\beta)'g^2\beta\gamma^2\\ -(h\gamma-f\beta)(g\beta)'+\beta(h\gamma-f\beta)]^2\},\end{array}}},$$

$$B_{\psi_4}=\frac{\left(\begin{array}{c}[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'+g^2\gamma^3]N\\ +[g\beta(g\gamma)'-g\gamma(g\beta)']C\\ +[g^2\beta^3+g\beta(h\gamma-f\beta)'g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'+\beta(h\gamma-f\beta)]W\end{array}\right)}{\sqrt{\begin{array}{c}[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'+g^2\gamma^3]^2\\ +[g\beta(g\gamma)'-g\gamma(g\beta)']^2+[g^2\beta^3+g\beta(h\gamma-f\beta)'g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'\\ +\beta(h\gamma-f\beta)]^2\end{array}}},$$

$$\kappa_{\psi_4}=\frac{\sigma_4\sqrt{\begin{array}{c}[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'+g^2\gamma^3]^2\\ +[g\beta(g\gamma)'-g\gamma(g\beta)']^2+[g^2\beta^3+g\beta(h\gamma-f\beta)'g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'\\ +\beta(h\gamma-f\beta)]^2\end{array}}}{(\sqrt{(g\beta)^2+(h\gamma-f\beta)^2+(g\gamma)^2})^3},$$

$$\tau_{\psi_4}=\frac{\left(\begin{array}{c}\{g\gamma(g\beta^2+(h\gamma-f\beta)'+g\gamma^2)+(h\gamma-f\beta)(\gamma(h\gamma-f\beta)-(g\gamma)')\}P_4\\ +[g\beta(g\gamma)'-g\gamma(g\beta)']R_4\\ +\{g\beta(g\beta^2+(h\gamma-f\beta)'+g\gamma^2)-(h\gamma-f\beta)((g\beta)'-\beta(h\gamma-f\beta))\}S_4\end{array}\right)}{\sigma_4^2\left\{\begin{array}{c}[\gamma(h\gamma-f\beta)^2-(h\gamma-f\beta)(g\gamma)'+g^2\beta^2\gamma+g\gamma(h\gamma-f\beta)'+g^2\gamma^3]^2\\ +[g\beta(g\gamma)'-g\gamma(g\beta)']^2+[g^2\beta^3+g\beta(h\gamma-f\beta)'g^2\beta\gamma^2-(h\gamma-f\beta)(g\beta)'\\ +\beta(h\gamma-f\beta)]^2\end{array}\right\}}.$$

□

Example 3.9. *A circular helix is considered as given by* $\alpha_1(s) = \left(\cos\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}\right)$. *One can calculate* $N(s)$, $N-$*dual direction curve* $\psi_1(s)$ *as follows:*

$$N(s) = \left(-\cos\frac{s}{\sqrt{3}}, -\sin\frac{s}{\sqrt{3}}, 0\right), \psi_1(s) = \int \alpha_1(s) \wedge N(s) ds.$$

Then the Frenet vector fields, curvature and torsion of ψ_1,

$$T_{\psi_1} = \left(\cos\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}}, 0\right), \qquad N_{\psi_1} = \left(-\sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}}, 0\right), \qquad B_{\psi_1} = (0,0,1),$$

$$\kappa_{\psi_1} = \frac{\sqrt{3}}{9}, \qquad \tau_{\psi_1} = 0,$$

can be found.

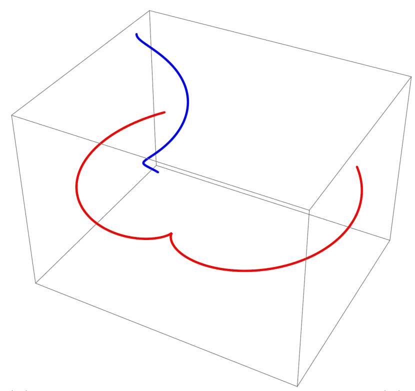

Figure 3.1. Plot of the helix curve $\alpha_1(s)$ (blue) and its moment curve $\psi_1(s)$ (red) over the interval $s \in [-6.5,\ 6.5]$.

Example 3.10. *A circular helix is considered as given by* $\alpha_2(s) = \left(\cos\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}\right)$. *One can calculate* $C(s)$, $C-$*dual direction curve* $\psi_2(s)$ *as follows:*

$$C(s) = \left(\sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}}, 0\right), \psi_2(s) = \int \alpha_2(s) \wedge C(s) ds.$$

Then the Frenet vector fields, curvature and torsion of ψ_2,

$$T_{\psi_2} = \left(\frac{\cos\frac{s}{\sqrt{3}}}{2}, \frac{\sin\frac{s}{\sqrt{3}}}{2}, -\frac{1}{2}\right), \qquad N_{\psi_2} = \left(\frac{\sqrt{3}+1}{4}\sin\frac{s}{\sqrt{3}}, -\frac{\sqrt{3}+1}{4}\cos\frac{s}{\sqrt{3}}, 0\right),$$

$$B_{\psi_2} = \frac{\sqrt{3}}{2}\left(\cos\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}}, \frac{\sqrt{3}}{3}\right), \qquad \kappa_{\psi_2} = \frac{1}{4}, \qquad \tau_{\psi_2} = -\frac{\sqrt{3}}{4},$$

can be found.

Example 3.11. *A circular helix is considered as given by* $\alpha_3(s) = \left(\cos\frac{s}{\sqrt{3}}, \sin\frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}\right)$. *One can calculate* $W(s)$, $W-$*dual direction curve* $\psi_3(s)$ *as follows:*

$$W(s) = \left(\sin\frac{s}{\sqrt{3}}, \cos\frac{s}{\sqrt{3}}, 0\right), \psi_3(s) = \int \alpha_3(s) \wedge W(s) ds.$$

Then the Frenet vector fields, curvature and torsion of ψ_3,

$$T_{\psi_3}=\left(\frac{\cos\frac{s}{\sqrt{3}}}{2},\frac{\sin\frac{s}{\sqrt{3}}}{2},-\frac{1}{2}\right),\qquad N_{\psi_3}=\left(\frac{\sqrt{3}+1}{4}\sin\frac{s}{\sqrt{3}},-\frac{\sqrt{3}+1}{4}\cos\frac{s}{\sqrt{3}},0\right),$$

$$B_{\psi_3}=\frac{\sqrt{3}}{2}\left(\cos\frac{s}{\sqrt{3}},\sin\frac{s}{\sqrt{3}},\frac{\sqrt{3}}{3}\right),\qquad \kappa_{\psi_3}=\frac{1}{4},\qquad \tau_{\psi_3}=-\frac{\sqrt{3}}{4},$$

can be found.

Example 3.12. *A circular helix is considered as given by* $\alpha_4(s)=\left(\cos\frac{s}{\sqrt{3}},\sin\frac{s}{\sqrt{3}},\frac{s}{\sqrt{3}}\right)$. *One can calculate* $D(s)$, $D-$*dual direction curve* $\psi_4(s)$ *as follows:*

$$D(s)=\left(\sin\frac{s}{\sqrt{3}},\cos\frac{s}{\sqrt{3}},0\right),\ \psi_4(s)=\int\alpha_4(s)\wedge D(s)ds.$$

Then the Frenet vector fields, curvature and torsion of ψ_4*,*

$$T_{\psi_4}=\left(\frac{\cos\frac{s}{\sqrt{3}}}{2},\frac{\sin\frac{s}{\sqrt{3}}}{2},-\frac{\sqrt{3}}{6}\right),\qquad N_{\psi_4}=(0,0,0),\qquad B_{\psi_4}=\frac{\sqrt{3}}{2}\left(\cos\frac{s}{\sqrt{3}},\sin\frac{s}{\sqrt{3}},\frac{\sqrt{3}}{3}\right),$$

$$\kappa_{\psi_4}=\frac{\sqrt{2}}{4},\qquad \tau_{\psi_4}=-\frac{\sqrt{6}}{4},$$

can be found.

4. Conclusion

In this study, we have constructed and analyzed the vectorial moment curves of space curves with respect to an alternative orthonormal frame in Euclidean 3-space. By deriving the Frenet apparatus of these moment curves, and further extending the framework to their associated dual directed curves, we have demonstrated how alternative frames enrich the classical geometric perspective. This approach not only provides new insights into the intrinsic and extrinsic properties of space curves but also establishes connections between vectorial moments, dual geometry, and alternative frame theory.

The results highlight that the alternative frame, compared to the traditional Frenet frame, can offer more robust tools in situations where the classical structure degenerates or loses descriptive power. Thus, the study underscores the necessity of adopting alternative frameworks for a deeper and more comprehensive understanding of curve dynamics in both theoretical and applied contexts. In future work, one may extend these results to higher-dimensional settings, non-Euclidean geometries, or applied areas such as kinematics, robotics, and computer-aided geometric design.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Authors' Contributions: SÖK and UO constructed the outline of the paper, SÖK completed the details of the paper. MT was a major contributor in writing and visualize the manuscript. SÖK, MT and UO made writing, review and editing. All authors have read and approved the manuscript.

Artificial Intelligence Statement: No artificial intelligence tools were used in the preparation of this manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] Y. Tuncer, *Vectorial moments of curves in Euclidean 3-space*, Int. J. Geom. Methods Mod. Phys., **14**(2) (2017), Article ID 1750020. https://doi.org/10.1142/s0219887817500207

[2] O. Kaya, M. Önder, *New partner curves in the Euclidean 3-space E^3*, Int. J. Geom., **6**(2) (2017), 41–50.

[3] M. Barros, *General helices and a theorem of Lancret*, Proc. Am. Math. Soc., **125**(5) (1997), 1503–1509. https://doi.org/10.1090/s0002-9939-97-03692-7

[4] Ü. Çiftçi, *A generalization of Lancret's theorem*, J. Geom. Phys., **59**(12) (2009), 1597–1603. https://doi.org/10.1016/j.geomphys.2009.07.016

[5] J. Choi, Y. Kim, *Associated curves of a Frenet curve and their applications*, Appl. Math. Comput., **218**(18) (2012), 9116–9124. https://doi.org/10.1016/j.amc.2012.02.064

[6] S. Deshmukh, B. Chen, A. Alghanemi, *Natural mates of Frenet curves in Euclidean 3-space*, Turk. J. Math., **42**(5) (2018), 2826–2840. https://doi.org/10.3906/mat-1712-34

[7] T. Körpınar, M. Sarıaydın, E. Turhan, *Associated curves according to Bishop frame in Euclidean 3-space*, Adv. Model. Optim., **15**(3) (2013), 713–717.

[8] S. K. Nurkan, İ. A. Güven, M. K. Karacan, *Characterizations of adjoint curves in Euclidean 3-space*, Proc. Natl. Acad. Sci. India Sect. A Phys. Sci., **89**(1) (2019), 155–161. https://doi.org/10.1007/s40010-017-0425-y

[9] S. Yılmaz, *Characterizations of some associated and special curves to type-2 Bishop frame in E^3*, Kırklareli Univ. J. Eng. Sci., **1** (2015), 66–77.

[10] S. Şenyurt, H. Şardağ, O. Çakır, *On vectorial moment of the Darboux vector*, Konuralp J. Math., **8**(1) (2020), 144–151.

[11] S. Kaya, İ. A. Güven, *Construction of vectorial moments via direction curves*, AIMS Math., **8** (2023), 12857–12871. https://doi.org/10.3934/math.2023648

[12] N. Karagöz, *Yönlü vektörel moment eğrileri*, Master's Thesis, Gaziantep University, 2024.

[13] D. Struik, *Lectures on Classical Differential Geometry*, Dover Publications, New York, 1988.

[14] R. Millman, G. Parker, *Elements of Differential Geometry*, Pearson, London, 1977.

[15] A. Sabuncuoğlu, *Diferansiyel Geometri*, Nobel Yayınları, Ankara, 2014.

[16] A. Gray, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd ed., CRC Press, Boca Raton, FL, 1997.

[17] S. Şenyurt, *D-Smarandache curves according to the Sabban frame of the spherical indicatrix curve*, Turk. J. Math. Comput. Sci., **9** (2018), 39–49.

[18] H. Şardağ, *Alternatif çatının vektörel moment eğrileri üzerine*, Master's Thesis, Ordu University, 2019.