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Almost paracontact structures obtained from $G_{2(2)}^*$ structures

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Abstract: In this paper, we construct almost paracontact metric structures by using the fundamental 3-forms of manifolds with $G_{2(2)}^*$ structures. The existence of certain almost paracontact metric structures is investigated due to the properties of the 2-fold vector cross-product. Furthermore, we give some relations between the classes of $G_{2(2)}^*$ structures and almost paracontact metric structures.

Key words: $G_{2(2)}^*$ structure, almost paracontact metric structure

1. Introduction

Almost paracontact structures on manifolds of odd dimension, analogues of the almost contact structures on manifolds, were first introduced by Kaneyuki and Williams in [5]. After the work of Zamkovoy in [10], almost paracontact metric structures have been a widely studied research area. In [11], almost paracontact metric structures were classified into 2^{12} classes taking into consideration the Levi-Civita covariant derivative of the fundamental 2-form of the structure.

Almost contact metric structures induced by G_2 structures were constructed by Matzeu and Munteanu in [7]; see also [1]; and the possible classes that these structures may belong to were considered in [8].

The objective of this manuscript is the investigation of almost paracontact metric structures on manifolds with structure group $G_{2(2)}^*$. First, we construct almost paracontact metric structures induced by $G_{2(2)}^*$ structures. Then we investigate the relation between the classes of almost paracontact metric structures and $G_{2(2)}^*$ structures. In addition, we give an elementary example to support the arguments of the manuscript.

2. Preliminaries

Consider \mathbb{R}^7 with the standard basis $\{e_1, \dots, e_7\}$. The fundamental 3-form on \mathbb{R}^7 is defined as

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where $\{e^1, \dots, e^7\}$ denotes the basis dual to $\{e_1, \dots, e_7\}$ and $e^{ijk} = e^i \wedge e^j \wedge e^k$. The Lie group G_2 is defined by

$$G_2 := \{f \in GL(7, \mathbb{R}) \mid f^* \varphi_0 = \varphi_0\};$$

see [3].

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A 7-dimensional oriented manifold M has a G_2 structure if and only if its structure group reduces to G_2 . Then there is a 3-form φ on M with the property that $(T_pM, \varphi_p) \cong (\mathbb{R}^7, \varphi_0)$, for all $p \in M$, said to be the fundamental 3-form or the G_2 structure on M . Manifolds (M, g) with G_2 structure were classified into 16 classes in [4].

The noncompact dual of G_2 is the group

$$G_{2(2)}^* = \{g \in GL(7, \mathbb{R}) \mid g^*\tilde{\varphi} = \tilde{\varphi}\},$$

where

$$\tilde{\varphi} = -e^{127} - e^{135} + e^{146} + e^{236} + e^{245} - e^{347} + e^{567}$$

and $\{e^1, \dots, e^7\}$ denotes the dual to the standard basis of $\mathbb{R}^{4,3} = (\mathbb{R}^7, g_{4,3})$ with the metric $g_{4,3} = (-1, -1, -1, -1, 1, 1, 1)$. A semi-Riemannian manifold M with the metric of signature $(-, -, -, -, +, +, +)$ whose structure group reduces to $G_{2(2)}^*$ is called a manifold with $G_{2(2)}^*$ structure. Similar to the G_2 case, there is the fundamental 3-form (or the $G_{2(2)}^*$ structure) $\tilde{\varphi}$ on M inducing a metric $g_{4,3}$, a volume form, and a 2-fold vector cross-product \tilde{P} on M , which can be calculated via

$$\tilde{\varphi}(X, Y, Z) = g_{4,3}(\tilde{P}(X, Y), Z); \tag{2.1}$$

see [3]. Similar to the G_2 case, a $G_{2(2)}^*$ structure $\tilde{\varphi}$ satisfying $\nabla^{g_{4,3}}\tilde{\varphi} = 0$ is called a parallel $G_{2(2)}^*$ structure and a $G_{2(2)}^*$ structure with $\nabla_X^{g_{4,3}}\tilde{\varphi}(X, Y, Z) = 0$ is called nearly parallel [6].

For convenience, throughout the paper, a $G_{2(2)}^*$ structure and the induced vector cross-product will be denoted by φ and P , respectively.

A triple (ϕ, ξ, η) on a $2n + 1$ -dimensional differentiable manifold M^{2n+1} satisfying

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.2}$$

where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, and η is a 1-form η on M , is called an almost paracontact structure on M and M is called an almost paracontact manifold. As a consequence of (2.2), one can see that $\phi(\xi) = 0$ and $\eta \circ \phi = 0$ on the almost paracontact structure (ϕ, ξ, η) .

If an almost paracontact manifold M has a semi-Riemannian metric g of signature $(n, n + 1)$ satisfying

$$g(\phi(X), \phi(Y)) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

then M is an almost paracontact metric manifold having the almost paracontact metric structure (ϕ, ξ, η, g) and g is said to be a compatible metric.

The 2-form

$$\Phi(X, Y) := g(\phi(X), Y)$$

is said to be the fundamental 2-form of the almost paracontact metric structure. It is known that on an almost paracontact metric manifold there is an orthonormal basis (called a ϕ -basis) $\{e_1, \phi e_1, \dots, e_n, \phi e_n, \xi\}$ with

$$g(e_i, e_j) = -g(\phi e_i, \phi e_j) = \delta_{ij}, \quad g(e_i, \phi e_j) = 0, \quad i, j = 1, \dots, n;$$

see [10]. For the almost contact case, see [2].

Let F be the $(0, 3)$ tensor field defined by

$$F(X, Y, Z) = (\nabla_X \Phi)(Y, Z) = g((\nabla_X \phi)Y, Z), \tag{2.4}$$

for $X, Y, Z \in TM$. It can be seen that F has the following properties:

$$F(X, Y, Z) = -F(X, Z, Y), \tag{2.5}$$

$$F(X, \phi(Y), \phi(Z)) = F(X, Y, Z) + \eta(Y)F(X, Z, \xi) - \eta(Z)F(X, Y, \xi).$$

In [11], a classification of almost paracontact metric manifolds was obtained by considering the space \mathcal{F} of tensors F that satisfy (2.5). Initially, this space was decomposed into four subspaces

$$W_1 = \left\{ F \in \mathcal{F} \left| \begin{array}{l} F(X, Y, Z) = g(\mathcal{A}_Y^F X, Z), \\ F(\xi, Y, Z) = g(\mathcal{A}_Y^F \xi, Z) = 0, \\ F(X, \xi, Z) = g(\mathcal{A}_\xi^F X, \phi(Z)) = 0 \end{array} \right. \right\}, \tag{2.6}$$

$$W_2 = \left\{ F \in \mathcal{F} \left| \begin{array}{l} F(X, Y, Z) = \eta(Y)g(\phi(\mathcal{A}_\xi^F X), Z) \\ \quad + \eta(Z)g(\phi(\mathcal{A}_\xi^F X), Y), \\ \mathcal{A}_\xi^F \xi = 0 \end{array} \right. \right\}, \tag{2.7}$$

$$W_3 = \mathcal{G}_{11} = \{F \in \mathcal{F} | F(X, Y, Z) = \eta(X)F(\xi, \phi(Y), \phi(Z))\}, \tag{2.8}$$

$$W_4 = \mathcal{G}_{12} = \{F \in \mathcal{F} | F(X, Y, Z) = \eta(X)(\eta(Y)\omega_F(Z) - \eta(Z)\omega_F(Y))\}, \tag{2.9}$$

where $\mathcal{A}_X^F Y = (\nabla_Y \phi)(X)$, $\mathcal{A}_\xi^F X = \nabla_X \xi$ and $\omega_F(X) = F(\xi, \xi, X)$. Then W_1 and W_2 were written as sums of $U(n) \times 1$ irreducible components $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$, and $\mathcal{G}_5, \dots, \mathcal{G}_{10}$ respectively, where $U(n)$ is the paraunitary group, with the following defining relations [11]:

\mathcal{G}_1	$F(X, Y, Z) = \frac{1}{2(n-1)} \{g(X, \phi Y)\theta_F(\phi Z) - g(X, \phi Z)\theta_F(\phi Y) + g(\phi X, \phi Z)\theta_F(hY) - g(\phi X, \phi Y)\theta(hZ)\}$
\mathcal{G}_2	$F(\phi X, \phi Y, Z) = -F(X, Y, Z), \quad \theta_F = 0$
\mathcal{G}_3	$F(\xi, Y, Z) = F(X, \xi, Z) = 0, \quad F(X, Y, Z) = -F(Y, X, Z)$
\mathcal{G}_4	$F(\xi, Y, Z) = F(X, \xi, Z) = 0, \quad \mathfrak{S}_{(X,Y,Z)}F(X, Y, Z) = 0$
\mathcal{G}_5	$F(X, Y, Z) = \frac{\theta_F(\xi)}{2n} [\eta(Y)g(\phi X, \phi Z) - \eta(Z)g(\phi X, \phi Y)]$
\mathcal{G}_6	$F(X, Y, Z) = -\frac{\theta_F^*(\xi)}{2n} [\eta(Y)g(X, \phi Z) - \eta(Z)g(X, \phi Y)]$
\mathcal{G}_7	$F(X, Y, Z) = -F(Y, Z, X) + F(Z, X, Y) - 2F(\phi X, \phi Y, Z),$ $= -F(\phi X, \phi Y, Z) - F(\phi X, Y, \phi Z)$ $\theta_F^*(\xi) = 0$
\mathcal{G}_8	$F(X, Y, Z) = -F(Y, Z, X) - F(Z, X, Y),$ $= -F(\phi X, \phi Y, Z) - F(\phi X, Y, \phi Z)$ $\theta_F(\xi) = 0$
\mathcal{G}_9	$F(X, Y, Z) = -F(Y, Z, X) + F(Z, X, Y) + 2F(\phi X, \phi Y, Z),$ $= F(\phi X, \phi Y, Z) + F(\phi X, Y, \phi Z)$
\mathcal{G}_{10}	$F(X, Y, Z) = -F(Y, Z, X) - F(Z, X, Y)$ $= F(\phi X, \phi Y, Z) + F(\phi X, Y, \phi Z)$
\mathcal{G}_{11}	$F(X, Y, Z) = \eta(X)F(\xi, \phi Y, \phi Z)$
\mathcal{G}_{12}	$F(X, Y, Z) = \eta(X) [\eta(Y)\omega_F(Z) - \eta(Z)\omega_F(Y)]$

where $\theta_F(X) = g^{ij}F(e_i, e_j, X)$, $\theta_F^*(X) = g^{ij}F(e_i, \phi(e_j), X)$, and $h(X) = \phi^2(X)$.

The trivial class denoted by \mathcal{G}_0 , for which the defining relation is $\nabla\Phi = 0$, is the class of paracosymplectic structures. The classes \mathcal{G}_5 and \mathcal{G}_6 correspond to α -para-Sasakian and β -para-Kenmotsu structures, respectively. Also, the defining relations of paracontact and almost K-paracontact classes are $d\eta = \Phi$ and $\nabla_\xi\Phi = 0$, respectively.

Let (M, ϕ, ξ, η, g) be an almost paracontact metric manifold. M is called normal if

$$\phi((\nabla_X\phi)(Y)) - (\nabla_{\phi X}\phi)(Y) + (\nabla_X\eta)(Y)\xi = 0; \tag{2.10}$$

see [9].

3. Almost paracontact metric structures and $G_{2(2)}^*$ structures

Consider a 7-dimensional smooth manifold M with a $G_{2(2)}^*$ -structure φ inducing the pseudo-Riemannian metric $g_{4,3}$ and the vector cross-product P . Let ξ be a nonzero vector field on M such that $g_{4,3}(\xi, \xi) = -1$. Then the quadruple (ϕ, ξ, η, g) , where the endomorphism is

$$\phi(X) = P(\xi, X) \tag{3.1}$$

and $g = -g_{4,3}$, $\eta(X) = g(\xi, X)$, is an almost paracontact metric structure on M . Indeed, we have

$$\begin{aligned} \phi^2 X &= \phi(\phi X) = \phi(P(\xi, X)) = P(\xi, P(\xi, X)) \\ &= -g_{4,3}(\xi, \xi)X + g_{4,3}(\xi, X)\xi = g(\xi, \xi)X - g(\xi, X)\xi \\ &= X - \eta(X)\xi \end{aligned}$$

and

$$\begin{aligned} g(\phi X, \phi Y) &= -g_{4,3}(P(\xi, X), P(\xi, Y)) \\ &= -g_{4,3}(\xi, \xi)g_{4,3}(X, Y) + g_{4,3}(\xi, X)g_{4,3}(\xi, Y) \\ &= -g(X, Y) + \eta(X)\eta(Y). \end{aligned}$$

Throughout the paper, unless otherwise stated, (ϕ, ξ, η, g) corresponds to the almost paracontact metric structure (a.p.m.s.) obtained by a $G_{2(2)}^*$ structure φ on M . Note that $\nabla^g = \nabla^{g_{4,3}}$ and we use the notation ∇ for the Levi-Civita covariant derivative ∇^g .

The following proposition gives a relation between the covariant derivatives of the fundamental 2-form of the almost paracontact structure and of the $G_{2(2)}^*$ structure φ .

Proposition 3.1 *For an a.p.m.s. (ϕ, ξ, η, g) on M , the equation*

$$(\nabla_X \Phi)(Y, Z) = -(\nabla_X \varphi)(\xi, Y, Z) - \varphi(\nabla_X \xi, Y, Z) \tag{3.2}$$

holds.

Proof

$$\begin{aligned} (\nabla_X \varphi)(\xi, Y, Z) &= g_{4,3}(\nabla_X P(\xi, Y), Z) - g_{4,3}(P(\nabla_X \xi, Y), Z) - g_{4,3}(P(\xi, \nabla_X Y), Z) \\ &= -g(\nabla_X(\phi Y), Z) - \varphi(\nabla_X \xi, Y, Z) + g(\phi(\nabla_X Y), Z) \\ &= -g((\nabla_X \phi)(Y), Z) - \varphi(\nabla_X \xi, Y, Z) \\ &= -(\nabla_X \Phi)(Y, Z) - \varphi(\nabla_X \xi, Y, Z). \end{aligned}$$

□

The following proposition gives a condition for almost paracontact metric structures induced by $G_{2(2)}^*$ structures to be paracontact.

Proposition 3.2 *An a.p.m.s. (ϕ, ξ, η, g) induced by a $G_{2(2)}^*$ structure is paracontact (i.e. $d\eta = \Phi$) if and only if ξ satisfies*

$$g_{4,3}(P(\xi, X), Y) = \frac{1}{2}(g_{4,3}(\nabla_X \xi, Y) - g_{4,3}(\nabla_Y \xi, X)). \tag{3.3}$$

Proof The exterior derivative of η is:

$$2d\eta(X, Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X. \tag{3.4}$$

After some calculations, the following is obtained:

$$d\eta(X, Y) = \frac{1}{2}(-g_{4,3}(\nabla_X \xi, Y) + g_{4,3}(\nabla_Y \xi, X)).$$

Besides, for the corresponding almost paracontact metric structure, we have $\Phi(X, Y) = g(\phi(X), Y) = -g_{4,3}(P(\xi, X), Y)$. Thus, $d\eta = \Phi$ if the relation (3.3) holds. \square

Theorem 1 *An a.p.m.s. (ϕ, ξ, η, g) induced by a parallel $G_{2(2)}^*$ structure φ on $(M, g_{4,3})$ (i.e. $\nabla^{g_{4,3}}\varphi = 0$) is in the class $\mathcal{G}_0(\nabla\Phi = 0)$ (paracosymplectic) if and only if the vector field ξ is parallel.*

Proof Let φ be a parallel structure; that is, $\nabla\varphi = 0$. Then from the equation (3.2), we have

$$(\nabla_X \Phi)(Y, Z) = -\varphi(Y, Z, \nabla_X \xi) = -g_{4,3}(P(Y, Z), \nabla_X \xi),$$

which implies

$$\nabla\Phi = 0 \iff \nabla\xi = 0.$$

\square

Theorem 2 *For an a.p.m.s. (ϕ, ξ, η, g) , if ξ is not parallel, then the structure is not in W_1 .*

Proof Consider the equation

$$g(\mathcal{A}'_X \xi, \phi Z) = g(\nabla_X \xi, \phi Z).$$

Letting the vector field ξ not be parallel, then there exists X_0 such that $\nabla_{X_0} \xi \neq 0$ and obviously the third condition of the defining relation (2.6) of W_1 fails. \square

Note that, under the assumption of Theorem 2, the structure is not an element of any subclass of $W_1 = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3 \oplus \mathcal{G}_4$.

Theorem 3 *If the $G_{2(2)}^*$ structure φ is nearly parallel and ξ is parallel, then (ϕ, ξ, η, g) is in W_1 .*

Proof Let φ be nearly parallel; that is,

$$(\nabla_X \varphi)(X, Y, Z) = 0,$$

and let ξ be parallel, i.e. $\nabla\xi = 0$. Then, from equation (3.2),

$$F(\xi, Y, Z) = -(\nabla_\xi \varphi)(\xi, Y, Z) - \varphi(\nabla_\xi \xi, Y, Z) = 0$$

and

$$F(X, \xi, Z) = -(\nabla_X \varphi)(\xi, \xi, Z) - \varphi(\nabla_X \xi, \xi, Z) = 0.$$

Thus, the definition of W_1 is satisfied. \square

Theorem 4 *An a.p.m.s. (ϕ, ξ, η, g) from a nearly parallel structure φ satisfies $\nabla_\xi \Phi = 0$ (almost K-parcontact) if and only if $\nabla_\xi \xi = 0$.*

Proof Let φ be nearly parallel. Then this is an immediate consequence of formula (3.2) and of the definition of the nearly parallel $G_{2(2)}^*$ structure. Indeed,

$$(\nabla_\xi\Phi)(X, Y) = -(\nabla_\xi\varphi)(\xi, X, Y) - \varphi(\nabla_\xi\xi, X, Y) = -\varphi(\nabla_\xi\xi, X, Y).$$

Then

$$\nabla_\xi\Phi = 0 \iff \nabla_\xi\xi = 0.$$

□

Note that an a.p.m.s. (ϕ, ξ, η, g) such that $\nabla_\xi\xi \neq 0$ cannot be in the class W_2 by the definition of W_2 . In addition, if ξ is not Killing, the structure is not in the class $\mathcal{G}_5 \oplus \mathcal{G}_8$.

Theorem 5 *If ξ is not parallel, then the structure (ϕ, ξ, η, g) is not an element of $W_3 (= \mathcal{G}_{11})$.*

Proof Take $Y = \xi$ in the defining relation (2.8) of the class W_3 . Then, as a consequence of the formula (3.2), the left-hand side of (2.8) is

$$\begin{aligned} (\nabla_X\Phi)(\xi, Z) &= X[\Phi(\xi, Z)] - \Phi(\nabla_X\xi, Z) - \Phi(\xi, \nabla_XZ) \\ &= g(\phi Z, \nabla_X\xi), \end{aligned}$$

while the right-hand side vanishes since $\phi(\xi) = 0$. Thus, if $\nabla_X\xi \neq 0$ (i.e. ξ is not parallel), the structure can not be in the class W_3 . □

Theorem 6 *If there exists a vector field $X \in \{\xi\}^\perp$ with the property $\nabla_X\xi \neq 0$, then the structure (ϕ, ξ, η, g) is not in $W_4 (= \mathcal{G}_{12})$.*

Proof Let $X \in \{\xi\}^\perp$ with $\nabla_X\xi \neq 0$. Take $Y = \xi$ in the defining relation (2.9) of the class W_4 . Then $\eta(X) = 0$ since $X \in \{\xi\}^\perp$, so the right-hand side of the relation (2.9) is zero. On the other hand, from formula (3.2),

$$\begin{aligned} (\nabla_X\Phi)(Y, Z) &= (\nabla_X\Phi)(\xi, Z) \\ &= X[\Phi(\xi, Z)] - \Phi(\nabla_X\xi, Z) - \Phi(\xi, \nabla_XZ) \\ &= g(\phi Z, \nabla_X\xi). \end{aligned}$$

Therefore, $(\nabla_X\Phi)(Y, Z)$ does not have to be zero since $\nabla_X\xi \neq 0$. Hence, the defining relation is not satisfied under the given conditions. □

Example 7 *Consider the seven-dimensional Lie algebra \mathfrak{L} with nonzero brackets*

$$[e_1, e_2] = e_5, \quad [e_1, e_3] = e_6.$$

Then \mathfrak{L} admits the $G_{2(2)}^$ structure*

$$\varphi = e^{567} - e^{512} - e^{534} - e^{613} + e^{624} + e^{714} + e^{723}. \tag{3.5}$$

The metric $g_{4,3}$ induced by φ is

$$g_{4,3}(x, y) = x_5y_5 + x_6y_6 + x_7y_7 - x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4$$

for any vector fields $x = \sum x_i e_i$, $y = \sum y_i e_i$; see [3]. Note that $g_{4,3}(e_i, e_i) = -1$ for $i = 1, 2, 3, 4$ and $g_{4,3}(e_i, e_i) = 1$ otherwise. The cross-product of frame elements are obtained via (2.1):

$$\begin{aligned} P(e_1, e_2) &= -e_5, & P(e_1, e_3) &= -e_6, & P(e_1, e_4) &= e_7, & P(e_1, e_5) &= -e_2, \\ P(e_1, e_6) &= -e_3, & P(e_1, e_7) &= e_4, & P(e_2, e_3) &= e_7, & P(e_2, e_4) &= e_6, \\ P(e_2, e_5) &= e_1, & P(e_2, e_6) &= e_4, & P(e_2, e_7) &= e_3, & P(e_3, e_4) &= -e_5, \\ P(e_3, e_5) &= -e_4, & P(e_3, e_6) &= e_1, & P(e_3, e_7) &= -e_2, & P(e_4, e_5) &= e_3, \\ P(e_4, e_6) &= -e_2, & P(e_4, e_7) &= -e_1, & P(e_5, e_6) &= e_7, & P(e_5, e_7) &= -e_6, P(e_6, e_7) = e_5. \end{aligned}$$

The nonzero Levi-Civita covariant derivatives evaluated by Kozsul's formula are

$$\begin{aligned} \nabla_{e_1} e_2 &= \frac{1}{2} e_5, \nabla_{e_1} e_3 = \frac{1}{2} e_6, \nabla_{e_1} e_5 = \frac{1}{2} e_2, \nabla_{e_1} e_6 = \frac{1}{2} e_3, \nabla_{e_2} e_1 = \frac{-1}{2} e_5, \nabla_{e_2} e_5 = \frac{-1}{2} e_1 \\ \nabla_{e_3} e_1 &= \frac{-1}{2} e_6, \nabla_{e_3} e_6 = \frac{-1}{2} e_1, \nabla_{e_5} e_1 = \frac{1}{2} e_2, \nabla_{e_5} e_2 = \frac{-1}{2} e_1, \nabla_{e_6} e_1 = \frac{1}{2} e_3, \nabla_{e_6} e_3 = \frac{-1}{2} e_1. \end{aligned}$$

Now we investigate the existence of certain classes on \mathfrak{L} .

Assume that a nonzero vector field $X = a_1 e_1 + \dots + a_7 e_7$ is parallel. Then,

$$\begin{aligned} \nabla_{e_1} X &= a_1 \nabla_{e_1} e_1 + a_2 \nabla_{e_1} e_2 + a_3 \nabla_{e_1} e_3 + a_4 \nabla_{e_1} e_4 + a_5 \nabla_{e_1} e_5 + a_6 \nabla_{e_1} e_6 + a_7 \nabla_{e_1} e_7 \\ &= \frac{a_2}{2} e_5 + \frac{a_3}{2} e_6 + \frac{a_5}{2} e_2 + \frac{a_6}{2} e_3 \\ &= 0 \iff a_2 = a_3 = a_5 = a_6 = 0. \end{aligned}$$

On the other hand,

$$\nabla_{e_2} X = -\frac{a_1}{2} e_5 = 0 \iff a_1 = 0 \tag{3.6}$$

and there is no other restriction on the coefficients a_i . Thus, $X = \sum a_i e_i$ is parallel iff $X = a_4 e_4 + a_7 e_7$, that is, iff $X \in \text{span}\{e_4, e_7\}$.

Note that the $G_{2(2)}^*$ structure (3.5) is neither parallel (since $(\nabla_{e_1} \varphi)(e_2, e_3, e_4) = 1 \neq 0$) nor nearly parallel (since $(\nabla_{e_1} \varphi)(e_2, e_3, e_4) + (\nabla_{e_2} \varphi)(e_1, e_3, e_4) = \frac{1}{2} \neq 0$).

Now we give an example of an a.p.m.s. such that the characteristic vector field is parallel. Let (ϕ, ξ, η, g) be the a.p.m.s. induced by the $G_{2(2)}^*$ structure (3.5), where $\xi = e_4$ and $g = -g_{4,3}$. Then from the equation (3.1), we get $\phi(e_1) = P(e_4, e_1) = -e_7$, $\phi(e_2) = -e_6$, $\phi(e_3) = e_5$, $\phi(e_4) = 0$, $\phi(e_5) = e_3$, $\phi(e_6) = -e_2$, $\phi(e_7) = -e_1$. Since $(\nabla_{e_1} \phi)(e_2) = -e_3 \neq 0$, this structure is not paracosymplectic. Theorem 1 states that an a.p.m.s. induced by a parallel $G_{2(2)}^*$ structure is paracosymplectic if and only if the characteristic vector field is parallel. This example shows that if the $G_{2(2)}^*$ structure is not parallel, we can obtain a.p.m. structures that are not paracosymplectic but have parallel characteristic vector fields.

It is easy to check that this structure is in W_1 , although the $G_{2(2)}^*$ structure is not nearly parallel, comparing with Theorem 3.

Now let (ϕ, ξ, η, g) be the a.p.m.s. induced by the $G_{2(2)}^*$ structure (3.5), where $\xi = e_2$ (ξ is not parallel in this case) and $g = -g_{4,3}$. By Theorem 2, this structure is not in W_1 . In addition, it is not in W_3 by Theorem 5. Also, since $\nabla_{e_1} e_2 = \frac{1}{2} e_5 \neq 0$, this structure is not in W_4 by Theorem 6. From the equation (3.1), we have $\phi(e_1) = e_5$,

$\phi(e_2) = 0, \phi(e_3) = e_7, \phi(e_4) = e_6, \phi(e_5) = e_1, \phi(e_6) = e_4, \phi(e_7) = e_3$. Since $(\nabla_{e_1}\phi)(e_1) = \frac{1}{2}e_2 \neq 0$, this structure is not paracosymplectic. One can check that $\nabla_{\xi}\phi = \nabla_{e_2}\phi = 0$; that is, this structure is almost-K-paracontact.

Now we investigate the existence of paracontact structures on \mathfrak{L} induced by the $G_{2(2)}^*$ structure (3.5). Let (ϕ, ξ, η, g) be such a structure with fundamental 2-form Φ ; that is, $d\eta = \Phi$. Since $de^5 = e^{12}, de^6 = e^{13}$, for $\eta = \sum b_i e^i, i = 1, \dots, 7$, we have $d\eta = b_5e^{12} + b_6e^{13} = \Phi$. This implies $\phi(e_5) = 0$. From the equation

$$g(\phi(e_5), \phi(e_5)) = -g(e_5, e_5) + \eta^2(e_5),$$

we obtain $\eta^2(e_5) = -1$, which is a contradiction. Therefore, there is no paracontact structure on \mathfrak{L} induced by the given $G_{2(2)}^*$ structure.

Finally, we study the existence of α -para-Sasakian structures on \mathfrak{L} induced by the $G_{2(2)}^*$ structure (3.5). Let (ϕ, ξ, η, g) be an α -para-Sasakian structure induced by (3.5). Note that $g = -g_{4,3}$. The characteristic vector field ξ is Killing. From the equation

$$g(\nabla_{e_i}\xi, e_j) + g(\nabla_{e_j}\xi, e_i) = 0, \tag{3.7}$$

we obtain that ξ is Killing if and only if $a_1 = a_2 = a_3 = 0$. Thus, $\xi = a_4e_4 + \dots + a_7e_7$. From the definition of an α -para-Sasakian structure, we have $\phi(X) = \frac{1}{\alpha}\nabla_x\xi$ for all vector fields X . Then $\phi(e_2) = -\frac{\alpha_5}{2\alpha}e_1$ and $\phi(e_3) = -\frac{\alpha_6}{2\alpha}e_1$. The equation

$$g(\phi(e_2), \phi(e_3)) = -g(e_2, e_3) + \eta(e_2)\eta(e_3)$$

implies $a_5a_6 = 0$. Thus, $\phi(e_1) = 0$ or $\phi(e_2) = 0$. Assume without loss of generality that $\phi(e_1) = 0$. Since

$$0 = g(\phi(e_1), \phi(e_1)) \neq -g(e_1, e_1) + \eta^2(e_1) = -1,$$

there is no α -para-Sasakian structure induced by (3.5).

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