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ORIGINAL PAPER

On electromagnetic curves and geometric phase associated with frontals in de-Sitter 2-space

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Abstract: In this work, we introduce a Berry phase model which is very important in physics relative to the frontals in de-Sitter 2-space. With this approach, we examine the relationship between a singular curve and Fermi–Walker's law and Berry's phase law in de-Sitter 2-space. We give the Rytov curves by using an orthonormal frame of spacelike frontals and timelike frontals with Fermi–Walker parallelism. We express parametric representations of Rytov curves associated with a singular optical fiber. We examine electromagnetic curves associated with orthonormal vectors of spacelike frontals and timelike frontals in de-Sitter 2-space. Hence, we express the magnetic field equations, Lorentz force equations and magnetic trajectory equations along a singular optical fiber. Finally, we give examples that support the theories both physically and geometrically.

Keywords: Berry phase; de-Sitter 2-space; Electromagnetic curves; Spacelike frontals; Timelike frontals

1. Introduction

Space curves and moving frames are very interesting in mathematics and physics. In physics, studies on optics, geometric phases, electromagnetism and Lorentz force are common. The Berry phase model is a quantum phase effect introduced by Berry [1]. Berry phase equations explain the relationships between geometry and physics quite well. The structure of polarized light waves along an optical fiber, magnetic field effect and Berry phase relationship are interesting. Ross, Kugler and Shtrikman considered an optical fiber as a space curve and they gave geometric phase models with the help of the moving frames of the space curves [2, 3]. Dandoloff and Zakrzewski further developed this approach [4]. Berry phase models and electromagnetism studies using moving frames of space curves in differential geometry are very common. In [5], Berry phase models are given with an optical fiber by using a Bishop frame in three-dimensional Riemann manifolds. In [6], Berry phase form and electromagnetic curves are given with an optical fiber according to an alternative moving frame. In [7], the geometric phase model is given for the quasi-adapted frame which more advantageous frame than the Frenet-frame. In [8], hybrid electric and magnetic phase concepts are generalized with the help of a hybrid frame. The studies [9, 10] are examples of Berry phase studies outside the Riemann manifold. In addition, studies of the Hashimoto map for pseudo-null curves concerning the classical frame or Bishop frame have interesting results between physics and geometry in non-Euclidean space [11, 12]. In addition, the study given for null curves lying on timelike ruled surfaces gives important data about physics and general relativity [13]. Moreover, interesting studies such as Heisenberg magnetic flux and Heisenberg antiferromagnetic flux in Minkowski space are expressed in [14–19].

Almost all of the Berry phase studies are based on the fact that the optical fiber is a regular space curve. However, in [20], a geometric phase model and electromagnetic curves are constructed along the optical fiber, which is a singular curve on a sphere. Also in [21], a phase model is constructed after defining a moving frame in Lie groups for framed curves, which is a special singular curve. Then, in addition to Euclidean space, electromagnetic curves in Sitter-2 space are investigated for spherical regular curves [22, 23].

Moreover, Rytov's law is explained by the rotation of the polarization plane along an optical fiber in [24–26]. Studies in the literature are based on generating Rytov curves using Fermi–Walker parallelism. On the other hand,

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Lorentz equations, Lorentz force and magnetic trajectories are dealt with closed 2-forms in Riemann or semi-Riemann manifolds. Basic studies for Lorentz equations and magnetic trajectories in three-dimensional Riemann and semi-Riemann manifolds are introduced in [27–30].

This study is based on Legendre curve theory. Legendre curves or more generally, framed curves are space curves that have singular points and can be defined by a moving frame [31]. Thanks to the definition of these curves, curve types, surface types and curve pairs have been studied that cannot be examined with the classical Frenet frame. Physically, the use of these curves gave good results in terms of singularity theory and physics. Examples of these studies can be given as [20, 21, 32, 33] studies. There are extensive studies in Euclidean space, de-Sitter 2-space and hyperbolic space for Legendre curves that can have singular points and can be defined by a moving frame [34–42].

In this study, we contribute to the studies of singular optical fiber. Inspired by the geometric phase study along an optical fiber with singular points on a sphere [20], we define Berry phase models, electromagnetic curves and new Rytov curves along an optical fiber with singular points in de-Sitter 2-space. As a corollary of Legendre curve theory, we consider the optical fiber as both spacelike frontal and timelike frontal in de-Sitter 2-space. Therefore, the study is a generalization of phase models for regular curves in de-Sitter 2-space [22].

2. Spacelike and timelike frontals in de-Sitter 2-space

In this section, we recall an orthonormal frame for spacelike and timelike frontals which may have singular points in de-Sitter 2-space. First, let us give some background on Minkowski space [43]:

The Lorentz–Minkowski space is the metric space $\mathbb{R}^3_1 = (\mathbb{R}^3, \langle , \rangle_L)$ where the metric $\langle , \rangle_L$ is

$$\begin{aligned}\langle \rho, \varrho \rangle_L =& -\rho_1\varrho_1 + \rho_2\varrho_2 + \rho_3\varrho_3,\\ \rho =&(\rho_1, \rho_2, \rho_3), \varrho = (\varrho_1, \varrho_2, \varrho_3)\end{aligned}$$

which is called the Lorentzian metric. A non-zero vector $\rho \in \mathbb{R}^3_1$ is said spacelike, timelike, lightlike if $\langle \rho, \rho \rangle_L > 0$, $\langle \rho, \rho \rangle_L < 0$ or $\langle \rho, \rho \rangle_L = 0$, respectively. The norm of the vector $\rho \in \mathbb{R}^3_1$ is denoted by $\|\rho\| = \sqrt{|\langle \rho, \rho \rangle_L|}$. The vector product $\rho \wedge_L \varrho$ of $\rho \in \mathbb{R}^3_1$ and $\varrho \in \mathbb{R}^3_1$ is given by

$$\rho \wedge_L \varrho = \begin{vmatrix} -e_1 & e_2 & e_3 \\ \rho_1 & \rho_2 & \rho_3 \\ \varrho_1 & \varrho_2 & \varrho_3 \end{vmatrix}$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3_1$. Therefore, we give the following classifications:

(i) If ρ is a timelike vector, ϱ is a spacelike vector and $\rho \wedge_L \varrho = \upsilon$, we have $\upsilon \wedge_L \rho = \varrho$ and $\varrho \wedge_L \upsilon = -\rho$.
(ii) If ρ is a spacelike vector, ϱ is a timelike vector and $\rho \wedge_L \varrho = \upsilon$, we have $\upsilon \wedge_L \rho = -\varrho$ and $\varrho \wedge_L \upsilon = \rho$.
(iii) If ρ is a spacelike vector, ϱ is a spacelike vector and $\rho \wedge_L \varrho = \upsilon$, we have $\upsilon \wedge_L \rho = -\varrho$ and $\varrho \wedge_L \upsilon = -\rho$.

On the other hand, the hyperbolic 2-space, de-Sitter 2-space and lightcone at the origin are defined by

$$\begin{aligned} H^2(-1) =& \{\rho \in \mathbb{R}^3_1 | \quad \langle \rho, \rho \rangle_L = -1\},\\ S^2_1 =& \{\rho \in \mathbb{R}^3_1 | \quad \langle \rho, \rho \rangle_L = 1\},\\ LC^* =& \{\rho \in \mathbb{R}^3_1 \setminus \{0\} | \quad \langle \rho, \rho \rangle_L = 0\}\end{aligned}$$

respectively.

Definition 1 [36] γ_1 is called spacelike frontal in S^2_1 if there exists a smooth mapping $\mu_1 : I \to H^2(-1)$ where $(\gamma_1, \mu_1) : I \to \Delta_1$ satisfies $\langle \gamma_1(s), \mu_1(s) \rangle_L = 0$ and $\langle \gamma'_1(s), \mu_1(s) \rangle_L = 0$ for all $s \in I$. Then, (γ_1, μ_1) is called spacelike Legendrian curve in Δ_1. Also, if (γ_1, μ_1) is an immersion, γ_1 is called is a spacelike front in S^2_1 where

$$\Delta_1 = \{(a, b) | \langle a, b \rangle_L = 0\} \subset S^2_1 \times H^2(-1).$$

On the other hand, $\{\gamma_1, \mu_1, \nu_1\}$ is called spacelike de-Sitter Legendrian Frenet frame where $\nu_1(s) = \gamma_1(s) \wedge_L \mu_1(s) \in S^2_1$. Therefore, the Frenet-Serret type formulas are written by

$$\begin{pmatrix} \gamma'_1(s) \\ \mu'_1(s) \\ \nu'_1(s) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_1(s) \\ 0 & 0 & n_1(s) \\ -m_1(s) & n_1(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma_1(s) \\ \mu_1(s) \\ \nu_1(s) \end{pmatrix}. \tag{1}$$

Then, (m_1, n_1) is called the spacelike de-Sitter Legendrian curvature. Also, it is clear that s_0 is a singular point of γ_1 if and only if $m_1(s_0) = 0$ and s_0 is a singular point of μ_1 if and only if $n_1(s_0) = 0$. Moreover, if $(m_1(s_0), n_1(s_0)) \neq 0$ for every $s \in I$, γ_1 is called a spacelike front in S^2_1.

Definition 2 [36] γ_2 is called timelike frontal in S^2_1 if there exists a smooth mapping $\mu_2 : I \to S^2_1$ such that $(\gamma_2, \mu_2) : I \to \Delta_2$ satisfies $\langle \gamma_2(s), \mu_2(s) \rangle_L = 0$ and $\langle \gamma'_2(s), \mu_2(s) \rangle_L = 0$ for all $s \in I$. Then, (γ_2, μ_2) is called timelike Legendrian curve in Δ_2. Also, if (γ_2, μ_2) is an immersion, γ_2 is called is a timelike front in S^2_1 where

$$\Delta_2 = \{(a, b) | \langle a, b \rangle_L = 0\} \subset S^2_1 \times S^2_1.$$

Then, $\{\gamma_2, \mu_2, \nu_2\}$ is called timelike de-Sitter Legendrian Frenet frame where $\nu_2(s) = \gamma_2(s) \wedge_L \mu_2(s) \in H^2(-1)$. Consequently, the Frenet-Serret type formulas are given by

$$\begin{pmatrix} \gamma_2'(s) \\ \mu_2'(s) \\ \nu_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_2(s) \\ 0 & 0 & n_2(s) \\ m_2(s) & n_2(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma_2(s) \\ \mu_2(s) \\ \nu_2(s) \end{pmatrix}.$$

Therefore, (m_2, n_2) is called the timelike de-Sitter Legendrian curvature. Also, it is seen that s_0 is a singular point of γ_2 if and only if $m_2(s_0) = 0$ and s_0 is a singular point of μ_2 if and only if $n_2(s_0) = 0$. Also, if $(m_2(s_0), n_2(s_0)) \neq 0$ for every $s \in I$, γ_2 is called a timelike front in S_1^2.

3. Berry phase model of polarized light wave along for spacelike frontals in de-Sitter 2-space

In this section, we consider an optical fiber as a spacelike frontal with singular points. We give a Berry phase model using the orthonormal frame for spacelike frontals in de-Sitter 2-space. The direction of Σ, which indicates the direction of a polarized light wave, with Σ being an electric field, can be represented as the linear composition of the frame elements $\{\gamma_1, \mu_1, \nu_1\}$ along the spacelike frontal optical fiber. Therefore, we get

$$\frac{d\Sigma}{ds} = \alpha_1\gamma_1(s) + \alpha_2\mu_1(s) + \alpha_3\nu_1(s) \tag{2}$$

where $\alpha_1, \alpha_2, \alpha_3$ are differentiable functions.

Since γ_1 is an optical fiber, we examine the direction of polarized light in two different classes with orthogonal frame $\{\gamma_1, \mu_1, \nu_1\}$ for spacelike frontal γ_1 in de-Sitter 2-space:

(i). $\Sigma \perp \mu_1$,
(ii). $\Sigma \perp \nu_1$.

3.1. Berry phase model of polarized light where $\Sigma \perp \mu_1$

Let us suppose that

$$\langle \Sigma, \mu_1(s) \rangle_L = 0. \tag{3}$$

By differentiating Eq. (3) and with the help of Eq. (1), we have

$$\langle \frac{d\Sigma}{ds}, \mu_1(s) \rangle_L = -n_1(s)\langle \Sigma, \nu_1(s) \rangle_L. \tag{4}$$

Therefore, we can write

$$\alpha_2 = n_1(s)\langle \Sigma, \nu_1(s) \rangle_L.$$

If we consider that there is no loss of mechanism along an optical fiber, we can write $\langle \Sigma, \Sigma \rangle_L = k$ where k is a constant. Therefore, we get

$$\langle \frac{d\Sigma}{ds}, \Sigma \rangle_L = 0$$

and

$$\alpha_1 \langle \Sigma, \gamma_1(s) \rangle_L + \alpha_3 \langle \Sigma, \nu_1(s) \rangle_L = 0.$$

Then, we can represent by

$$\alpha_1 = \lambda\langle \Sigma, \nu_1(s) \rangle_L, \alpha_3 = -\lambda\langle \Sigma, \gamma_1(s) \rangle_L$$

where $\langle \Sigma, \gamma_1(s) \rangle_L \neq 0$ and $\langle \Sigma, \nu_1(s) \rangle_L \neq 0$. By according to Eq. (2), we get

$$\begin{aligned} \frac{d\Sigma}{ds} =& n_1(s)\langle \Sigma, \nu_1(s) \rangle_L \mu_1(s) + \lambda\langle \Sigma, \nu_1(s) \rangle_L \gamma_1(s) \\ & - \lambda\langle \Sigma, \gamma_1(s) \rangle_L \nu_1(s). \end{aligned} \tag{5}$$

Since $\nu_1(s) \wedge_L \gamma_1(s) = -\mu_1(s)$ for spacelike frontals, we have

$$\frac{d\Sigma}{ds} = n_1(s)\langle \Sigma, \nu_1(s) \rangle_L \mu_1(s) - \lambda(\Sigma \wedge_L \mu_1(s)). \tag{6}$$

The statement $\Sigma \wedge_L \mu_1(s)$ refers to the rotation for the vector μ_1. If we suppose μ_1 is parallel transported, we have $\lambda = 0$. Then, we get

$$\frac{d\Sigma}{ds} = n_1(s)\langle \Sigma, \nu_1(s) \rangle_L \mu_1(s).$$

On the other hand, we can write

$$\Sigma = \langle \Sigma, \gamma_1(s) \rangle_L \gamma_1(s) + \langle \Sigma, \nu_1(s) \rangle_L \nu_1(s). \tag{7}$$

By differentiating Eq. (7), we get

$$\begin{pmatrix} \langle \Sigma, \gamma_1(s) \rangle_L' \\ \langle \Sigma, \nu_1(s) \rangle_L' \end{pmatrix} = \begin{pmatrix} 0 & m_1(s) \\ -m_1(s) & 0 \end{pmatrix} \begin{pmatrix} \langle \Sigma, \gamma_1(s) \rangle_L \\ \langle \Sigma, \nu_1(s) \rangle_L \end{pmatrix}.$$

Since μ_1 is a timelike vector, we can write it as $\langle \Sigma, \Sigma \rangle_L = 1$. Also Σ can be denoted as follows:

$$\Sigma = \cos\Psi(s)\nu_1(s) + \sin\Psi(s)\gamma_1(s). \tag{8}$$

By differentiating Eq. (8), we get

$$\begin{aligned} \frac{d\Sigma}{ds} =& n_1(s)\cos\Psi(s)\mu_1(s) \\ & - \sin\Psi(s)\Big(\Psi'(s) - m_1(s)\Big)\nu_1(s) \\ & + \cos\Psi(s)\Big(\Psi'(s) - m_1(s)\Big)\gamma_1(s). \end{aligned}$$

Then, by using Eq. (8) we get

$$\begin{aligned} \frac{d\Sigma}{ds} =& n_1(s)\langle \Sigma, \nu_1(s) \rangle_L \mu_1(s) \\ & + (\Psi'(s) - m_1(s))\Sigma \wedge_L \mu_1(s). \end{aligned}$$

Consequently, $\Psi'(s) = m_1(s)$. By according to the Eq. (8), we can write

$$\Sigma = \cos\left(\int m_1(s)ds\right)v_1(s) + \sin\left(\int m_1(s)ds\right)\gamma_1(s).$$

Then, Σ moves the parallel transport along μ_1. Now, we examine the relationship of this motion with the Fermi–Walker law. The Fermi–Walker derivative equation along the timelike unit vector μ_1 for spacelike frontals in de-Sitter 2-space is expressed by [44]

$$\frac{d\Sigma}{ds_{\mathbf{FW}}} = \frac{d\Sigma}{ds} + \langle \Sigma, \mu_1(s)\rangle_L \frac{d\mu_1}{ds} - \left\langle \Sigma, \frac{d\mu_1}{ds}\right\rangle_L \mu_1(s)$$

where $\{\gamma_1, \mu_1, v_1\}$ is orthonormal frame for spacelike frontals. Since $\langle \Sigma, \mu_1(s)\rangle_L = 0$, we get

$$\frac{d\Sigma}{ds_{\mathbf{FW}}} = \frac{d\Sigma}{ds} - n_1(s)\langle \Sigma, v_1(s)\rangle_L \mu_1(s).$$

Then, we have

$$\frac{d\Sigma}{ds} = n_1(s)\langle \Sigma, v_1(s)\rangle_L \mu_1(s).$$

So, based on what was obtained above, we can obtain the following corollary:

Corollary 1 *If Σ satisfies the Fermi–Walker parallelism, the optical fiber is Σ_{μ_1}-Rytov curve with condition*

$\langle \Sigma, \mu_1(s)\rangle_L = 0$, and the parametric representation of the Σ_{μ_1}-Rytov curve is given by

$$\begin{aligned}\Sigma_{\mu_1} =& \gamma_1(s) + \cos\left(\int m_1(s)ds\right)v_1(s) \\ &+ \sin\left(\int m_1(s)ds\right)\gamma_1(s).\end{aligned} \tag{9}$$

3.2. Berry phase model of polarized light where $\Sigma \perp v_1$

Since $\Sigma \perp v_1$, we get

$$\langle \Sigma, v_1(s)\rangle_L = 0. \tag{10}$$

By differentiating Eq. (10), we get

$$\begin{aligned}\left\langle \frac{d\Sigma}{ds}, v_1(s)\right\rangle_L =& m_1(s)\langle \Sigma, \gamma_1(s)\rangle_L \\ &- n_1(s)\langle \Sigma, \mu_1(s)\rangle_L.\end{aligned} \tag{11}$$

By using Eqs. (2), (10) and (11), we get

$$\alpha_3 = m_1(s)\langle \Sigma, \gamma_1(s)\rangle_L - n_1(s)\langle \Sigma, \mu_1(s)\rangle_L.$$

From our assumption of $\langle \Sigma, \Sigma\rangle_L$ =constant similar to the other section, we get $\langle \frac{d\Sigma}{ds}, \Sigma\rangle_L = 0$. Then, we have

$$\alpha_1\langle \Sigma, \gamma_1(s)\rangle_L + \alpha_2\langle \Sigma, \mu_1(s)\rangle_L = 0.$$

If $\langle \Sigma, \gamma_1(s)\rangle_L \neq 0$ and $\langle \Sigma, \mu_1(s)\rangle_L \neq 0$, we can write

$$\alpha_1 = \lambda\langle \Sigma, \mu_1(s)\rangle_L, \alpha_2 = -\lambda\langle \Sigma, \gamma_1(s)\rangle_L.$$

The structure $\frac{d\Sigma}{ds}$ is written by

$$\begin{aligned}\frac{d\Sigma}{ds} =& (m_1(s)\langle \Sigma, \gamma_1(s)\rangle_L - n_1(s)\langle \Sigma, \mu_1(s)\rangle_L)v_1(s) \\ &+ \lambda\langle \Sigma, \mu_1(s)\rangle_L \gamma_1(s) - \lambda\langle \Sigma, \gamma_1(s)\rangle_L \mu_1(s).\end{aligned} \tag{12}$$

On the other hand, since $\mu_1 \wedge_L \gamma_1 = -v_1$ for spacelike frontals, we get

$$\begin{aligned}\frac{d\Sigma}{ds} =& (m_1(s)\langle \Sigma, \gamma_1(s)\rangle_L - n_1(s)\langle \Sigma, \mu_1(s)\rangle_L)v_1(s) \\ &- \lambda(\Sigma \wedge_L v_1(s)).\end{aligned}$$

If v_1 is parallel transported, we have $\lambda = 0$. Therefore, we get

$$\frac{d\Sigma}{ds} = (m_1(s)\langle \Sigma, \gamma_1(s)\rangle_L - n_1(s)\langle \Sigma, \mu_1(s)\rangle_L)v_1(s).$$

Since μ_1 is a timelike vector, we can write

$$\Sigma = \langle \Sigma, \gamma_1(s)\rangle_L \gamma_1(s) - \langle \Sigma, \mu_1(s)\rangle_L \mu_1(s). \tag{13}$$

By differentiating Eq. (13), we have

$$\begin{pmatrix} \langle \Sigma, \gamma_1(s)\rangle_L^{'} \\ \langle \Sigma, \mu_1(s)\rangle_L^{'} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \langle \Sigma, \gamma_1(s)\rangle_L \\ \langle \Sigma, \mu_1(s)\rangle_L \end{pmatrix}.$$

Since v_1 is spacelike vector, $\langle \Sigma, \Sigma\rangle_L = 1$ or $\langle \Sigma, \Sigma\rangle_L = -1$. Therefore, we can write the following equation with $\langle \Sigma, \Sigma\rangle_L = 1$.

$$\Sigma = \cosh \Psi(s)\gamma_1(s) + \sinh \Psi(s)\mu_1(s). \tag{14}$$

By differentiating Eq. (14), we get

$$\begin{aligned}\frac{d\Sigma}{ds} =& (m_1(s)\langle \Sigma, \gamma_1(s)\rangle_L - n_1(s)\langle \Sigma, \mu_1(s)\rangle_L)v_1(s) \\ &+ \Psi^{'}(s)\Sigma \wedge_L v_1(s).\end{aligned}$$

Therefore, we get $\Psi^{'}(s) = 0$ and $\Psi = constant$. With the help of Eq. (14), the expression associated with polarization vector is given by

$$\Sigma = \cosh \Psi\gamma_1(s) + \sinh \Psi\mu_1(s)$$

where Ψ is a constant.

The Fermi–Walker derivative equation along the unit spacelike vector v_1 is expressed by [44]

$$\frac{d\Sigma}{ds_{\mathbf{FW}}} = \frac{d\Sigma}{ds} \pm \langle \Sigma, v_1(s)\rangle_L \frac{dv_1}{ds} + \langle \Sigma, \frac{dv_1}{ds}\rangle_L v_1(s)$$

where $\{\gamma_1, \mu_1, v_1\}$ is orthonormal frame for spacelike frontals. Since $\langle \Sigma, v_1(s)\rangle_L = 0$, we have

$$\frac{d\Sigma}{ds_{\mathbf{FW}}} = \frac{d\Sigma}{ds} - m_1(s)\langle\Sigma,\gamma_1(s)\rangle_L v_1(s) + n_1(s)\langle\Sigma,\mu_1(s)\rangle_L v_1(s). \tag{15}$$

Then, we have

$$\frac{d\Sigma}{ds} = m_1(s)\langle\Sigma,\gamma_1(s)\rangle_L v_1(s) - n_1(s)\langle\Sigma,\mu_1(s)\rangle_L v_1(s). \tag{16}$$

Consequently, we can give the following corollary:

Corollary 2 *If Σ satisfies the Fermi–Walker parallelism, the optical fiber is Σ_{v_1}-Rytov curve with condition $\langle\Sigma, v_1(s)\rangle_L = 0$, and the parametric representation of the Σ_{v_1}-Rytov curve is given by*

$$\Sigma_{v_1} = \gamma_1(s) + \cosh\psi\gamma_1(s) + \sinh\psi\mu_1(s) \tag{17}$$

where ψ is a constant.

Similarly, for $\langle\Sigma,\Sigma\rangle_L = -1$, the position of the polarization vectors is examined by swapping the places of the hyperbolic functions and the Rytov curve is given.

4. Electromagnetic curves and trajectories for spacelike frontals in de-Sitter 2-space

In physics, a polarized particle creates an electromagnetic field. In this section, we give the electromagnetic field equations with the $\langle\Sigma,\mu_1\rangle_L = 0$ and the condition $\langle\Sigma, v_1\rangle_L = 0$ along a spacelike frontal optical fiber. Moreover, we produce the Lorentz force which provides the equation $\Phi(\Sigma) = \Gamma \times \Sigma = \frac{d\Sigma}{ds}$ where W is a Killing magnetic vector field [5]. Hence, we introduce $\Sigma_{\mu_1}M$-trajectories and $\Sigma_{v_1}M$-trajectories along a spacelike frontal optical fiber.

4.1. Electromagnetic curves associated with an optical fiber for spacelike frontals with $\Sigma \perp \mu_1$

Suppose that $\langle\Sigma,\mu_1\rangle_L = 0$. Then, by using Eq. (5), we have

$$\frac{d\Sigma}{ds} = n_1(s)\langle\Sigma, v_1(s)\rangle_L\mu_1(s) + \lambda\langle\Sigma, v_1(s)\rangle_L\gamma_1(s) - \lambda\langle\Sigma,\gamma_1(s)\rangle_L v_1(s). \tag{18}$$

On the other hand, since

$$\begin{aligned}\langle\Phi(\Sigma),\gamma_1(s)\rangle_L &= -\langle\Phi(\gamma_1(s)),\Sigma\rangle_L,\\ \langle\Phi(\Sigma),\mu_1(s)\rangle_L &= -\langle\Phi(\mu_1(s)),\Sigma\rangle_L,\\ \langle\Phi(\Sigma),v_1(s)\rangle_L &= -\langle\Phi(v_1(s)),\Sigma\rangle_L.\end{aligned} \tag{19}$$

By combining Eqs. (18) and (19), we get

$$\begin{pmatrix}\Phi(\gamma_1(s))\\ \Phi(\mu_1(s))\\ \Phi(v_1(s))\end{pmatrix} = \begin{pmatrix}0 & 0 & -\lambda\\ 0 & 0 & n_1(s)\\ \lambda & n_1(s) & 0\end{pmatrix}\begin{pmatrix}\gamma_1(s)\\ \mu_1(s)\\ v_1(s)\end{pmatrix}.$$

Consequently, the magnetic vector field Γ is given by

$$\Gamma = b_1\gamma_1(s) + b_2\mu_1(s) + b_3 v_1(s) \tag{20}$$

where

$$\begin{aligned}\Phi(\gamma_1(s)) &= \Gamma\wedge_L\gamma_1(s),\\ \Phi(\mu_1(s)) &= \Gamma\wedge_L\mu_1(s),\\ \Phi(v_1(s)) &= \Gamma\wedge_L v_1(s).\end{aligned} \tag{21}$$

Then, it is clear that from Eqs. (20) and (21), the magnetic vector field of the $\Sigma_{\mu_1}-$trajectories are denoted by

$$\Gamma = n_1(s)\gamma_1(s) + \lambda\mu_1(s).$$

If $\Sigma \perp \mu_1$, the matrix representation of Lorentz force equations are given by

$$\begin{pmatrix}\Phi(\gamma_1(s))\\ \Phi(\mu_1(s))\\ \Phi(v_1(s))\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & n_1(s)\\ 0 & n_1(s) & 0\end{pmatrix}\begin{pmatrix}\gamma_1(s)\\ \mu_1(s)\\ v_1(s)\end{pmatrix}$$

and $\Sigma_{\mu_1}-$trajectories are magnetic trajectories of the Γ magnetic field if and only if

$$\Gamma = n_1(s)\gamma_1(s).$$

Therefore, Γ is the Killing magnetic vector field of the $\Sigma_{\mu_1}-$trajectories of Φ.

4.2. Electromagnetic curves associated with an optical fiber for spacelike frontals with $\Sigma \perp v_1$

Since $\langle\Sigma, v_1\rangle_L = 0$, we have

$$\frac{d\Sigma}{ds} = (m_1(s)\langle\Sigma,\gamma_1(s)\rangle_L - n_1(s)\langle\Sigma,\mu_1(s)\rangle_L)v_1(s) + \lambda\langle\Sigma,\mu_1(s)\rangle_L\gamma_1(s) - \lambda\langle\Sigma,\gamma_1(s)\rangle_L\mu_1(s). \tag{22}$$

Then, we get

$$\begin{pmatrix}\Phi(\gamma_1(s))\\ \Phi(\mu_1(s))\\ \Phi(v_1(s))\end{pmatrix} = \begin{pmatrix}0 & -\lambda & m_1(s)\\ -\lambda & 0 & n_1(s)\\ -m_1(s) & n_1(s) & 0\end{pmatrix}\begin{pmatrix}\gamma_1(s)\\ \mu_1(s)\\ v_1(s)\end{pmatrix}.$$

By according to Eqs. (20), (21) and (22), the magnetic vector field Γ is given by

$$\Gamma = n_1(s)\gamma_1(s) - m_1(s)\mu_1(s) + \lambda v_1(s).$$

If $\Sigma \perp v_1$, the matrix representation of Lorentz force equations are given by

$$\begin{pmatrix} \Phi(\gamma_1(s)) \\ \Phi(\mu_1(s)) \\ \Phi(\nu_1(s)) \end{pmatrix} = \begin{pmatrix} 0 & 0 & m_1(s) \\ 0 & 0 & n_1(s) \\ -m_1(s) & n_1(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma_1(s) \\ \mu_1(s) \\ \nu_1(s) \end{pmatrix}.$$

It is said that, $\Sigma_{\nu_1}-$trajectories are magnetic trajectories of the Γ magnetic field if and only if

$$\Gamma = n_1(s)\gamma(s) - m_1(s)\mu_1(s).$$

5. Berry phase model of polarized light wave along for timelike frontals in de-Sitter 2-space

In this part, we consider an optical fiber as a timelike frontal with singular points. We give a Berry phase model using the orthonormal frame for timelike frontals in de-Sitter 2-space. Similar to the previous section, the direction of Σ, which indicates the direction of a polarized light wave, with Σ being an electric field, can be represented as the linear composition of the frame elements $\{\gamma_2, \mu_2, \nu_2\}$ along the timelike frontal optical fiber. Then, we have

$$\frac{d\Sigma}{ds} = \alpha_1\gamma_2(s) + \alpha_2\mu_2(s) + \alpha_3\nu_2(s) \tag{23}$$

where $\alpha_1, \alpha_2, \alpha_3$ are differentiable functions.

Since γ_2 is an optical fiber, we examine the direction of polarized light in two different classes with orthogonal frame$\{\gamma_2, \mu_2, \nu_2\}$ for timelike frontal γ_2 in de-Sitter 2-space:

(i). $\Sigma \perp \mu_2$,
(ii). $\Sigma \perp \nu_2$.

5.1. Berry phase model of polarized light where $\Sigma \perp \mu_2$

By using the condition, $\langle \Sigma, \mu_2(s)\rangle_L = 0$, we have

$$\langle \frac{d\Sigma}{ds}, \mu_2(s)\rangle_L = -n_2(s)\langle \Sigma, \mu_2(s)\rangle_L. \tag{24}$$

From Eqs. (23), (24), we get

$$\alpha_2 = -n_2(s)\langle \Sigma, \mu_2(s)\rangle_L.$$

Since $\langle \Sigma, \Sigma\rangle_L$ =constant, similar to the previous sections, we get

$$\begin{aligned} \frac{d\Sigma}{ds} =& \lambda\langle \Sigma, \nu_2(s)\rangle_L\gamma_2(s) - \lambda\langle \Sigma, \gamma_2(s)\rangle_L\nu_2(s) \\ &- n_2(s)\langle \Sigma, \nu_2(s)\rangle_L\mu_2(s) \end{aligned} \tag{25}$$

where $\langle \Sigma, \gamma_2(s)\rangle_L \neq 0$ and $\langle \Sigma, \nu_2(s)\rangle_L \neq 0$. Since $\nu_2 \wedge_L \gamma_2 = -\mu_2$ for timelike frontals, we get

$$\frac{d\Sigma}{ds} = -n_2(s)\langle \Sigma, \nu_2(s)\rangle_L\mu_2(s) - \lambda(\Sigma \wedge_L \mu_2(s)).$$

Suppose that μ_2 is parallel transported (i.e. $\lambda = 0$). Then, we have

$$\frac{d\Sigma}{ds} = -n_2(s)\langle \Sigma, \nu_2(s)\rangle_L\mu_2(s).$$

The polarization vector Σ is written by

$$\Sigma = \langle \Sigma, \gamma_2(s)\rangle_L\gamma_2(s) - \langle \Sigma, \nu_2(s)\rangle_L\nu_2(s) \tag{26}$$

for a timelike vector ν_2. By differentiating Eq. (26), we get

$$\begin{pmatrix} \langle \Sigma, \gamma_2(s)\rangle_L' \\ \langle \Sigma, \nu_2(s)\rangle_L' \end{pmatrix} = \begin{pmatrix} 0 & m_2(s) \\ m_2(s) & 0 \end{pmatrix} \begin{pmatrix} \langle \Sigma, \gamma_2(s)\rangle_L \\ \langle \Sigma, \nu_2(s)\rangle_L \end{pmatrix}.$$

Since μ_2 is a spacelike vector, $\langle \Sigma, \Sigma\rangle_L = 1$ or $\langle \Sigma, \Sigma\rangle_L = -1$. Then, we can denote by

$$\Sigma = \sinh \Psi(s)\nu_2(s) + \cosh \Psi(s)\gamma_2(s) \tag{27}$$

where $\langle \Sigma, \Sigma\rangle_L = 1$. By differentiating Eq. (27), we find

$$\begin{aligned} \frac{d\Sigma}{ds} =& -n_2(s)\langle \Sigma, \nu_2(s)\rangle\mu_2(s) \\ &- \left(\Psi'(s) + m_2(s)\right)\Sigma \wedge_L \mu_2(s). \end{aligned}$$

Then, we get $\Psi'(s) + m_2(s) = 0$. By using Eq. (27), the polarization vector is written by

$$\begin{aligned} \Sigma =& \sinh\left(\int -m_2(s)ds\right)\nu_2(s) \\ &+ \cosh\left(\int -m_2(s)ds\right)\gamma_2(s). \end{aligned}$$

The Fermi–Walker derivative equation along the unit spacelike vector μ_2 is obtained by [44]

$$\frac{d\Sigma}{ds}_{\mathbf{FW}} = \frac{d\Sigma}{ds} - \langle \Sigma, \mu_2(s)\rangle_L\frac{d\mu_2}{ds} + \langle \Sigma, \frac{d\mu_2}{ds}\rangle_L\mu_2(s)$$

where $\{\gamma_2, \mu_2, \nu_2\}$ is orthonormal frame for timelike frontals. Since $\langle \Sigma, \mu_2(s)\rangle_L = 0$, we get

$$\frac{d\Sigma}{ds}_{\mathbf{FW}} = \frac{d\Sigma}{ds} + n_2(s)\langle \Sigma, \nu_2(s)\rangle_L\mu_2(s). \tag{28}$$

Then, we have

$$\frac{d\Sigma}{ds} = -n_2(s)\langle \Sigma, \nu_2(s)\rangle_L\mu_2(s). \tag{29}$$

Corollary 3 *If Σ satisfies the Fermi–Walker parallelism, the optical fiber is Σ_{μ_2}-Rytov curve with condition $\langle \Sigma, \mu_2(s)\rangle_L = 0$, and the parametric representation of the Σ_{μ_2}-Rytov curve is given by*

$$\begin{aligned}\Sigma_{\mu_2} =&\gamma_2(s) + \sinh\left(\int -m_2(s)ds\right)v_2(s)\\ &+ \cosh\left(\int -m_2(s)ds\right)\gamma_2(s).\end{aligned} \tag{30}$$

Similarly, for $\langle\Sigma,\Sigma\rangle_L = -1$, the position of the polarization vectors is examined by swapping the places of the hyperbolic functions sinh and cosh and the Rytov curve is written.

5.2. Berry phase model of polarized light where $\Sigma \perp v_2$

Assume that, $\langle\Sigma, v_2(s)\rangle_L = 0$, we have

$$\begin{aligned}\langle\frac{d\Sigma}{ds}, v_2(s)\rangle_L =& - m_2(s)\langle\Sigma,\gamma_2(s)\rangle_L\\ &- n_2(s)\langle\Sigma,\mu_2(s)\rangle_L.\end{aligned} \tag{31}$$

By using Eqs. (23), (31), we get

$$\alpha_3 = m_2(s)\langle\Sigma,\gamma_2(s)\rangle_L + n_2(s)\langle\Sigma,\mu_2(s)\rangle_L.$$

Since $\langle\Sigma,\Sigma\rangle_L$ =constant, we can write

$$\begin{aligned}&\frac{d\Sigma}{ds} = \lambda\langle\Sigma,\mu_2(s)\rangle_L\gamma_2(s) - \lambda\langle\Sigma,\gamma_2(s)\rangle_L\mu_2(s)\\ &+(m_2(s)\langle\Sigma,\gamma_2(s)\rangle_L + n_2(s)\langle\Sigma,\mu_2(s)\rangle_L)v_2(s)\end{aligned} \tag{32}$$

where $\langle\Sigma,\gamma_2(s)\rangle_L \neq 0$ and $\langle\Sigma,\mu_2(s)\rangle_L \neq 0$. Since $\mu_2 \wedge_L \gamma_2 = -v_2$ for timelike frontals, we have

$$\begin{aligned}\frac{d\Sigma}{ds} =&(m_2(s)\langle\Sigma,\gamma_2(s)\rangle_L + n_2(s)\langle\Sigma,\mu_2(s)\rangle_L)v_2(s)\\ &- \lambda(\Sigma \wedge_L v_2(s)).\end{aligned}$$

Let us assume that v_2 is parallel transported (i.e. $\lambda = 0$). Then, we get

$$\frac{d\Sigma}{ds} = (m_2(s)\langle\Sigma,\gamma_2(s)\rangle_L + n_2(s)\langle\Sigma,\mu_2(s)\rangle_L)v_2(s).$$

We can write the polarization vector Σ by

$$\Sigma = \langle\Sigma,\gamma_2(s)\rangle_L\gamma_2(s) + \langle\Sigma,\mu_2(s)\rangle_L\mu_2(s) \tag{33}$$

for spacelike vectors γ_2 and μ_2. By differentiating Eq. (33), we have

$$\begin{pmatrix}\langle\Sigma,\gamma_2(s)\rangle_L'\\ \langle\Sigma,\mu_2(s)\rangle_L'\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\langle\Sigma,\gamma_2(s)\rangle_L\\ \langle\Sigma,\mu_2(s)\rangle_L\end{pmatrix}.$$

Since v_2 is a timelike vector, we get $\langle\Sigma,\Sigma\rangle_L = 1$. Then, we can represent by

$$\Sigma = \cos\Psi(s)\gamma_2(s) + \sin\Psi(s)\mu_2(s). \tag{34}$$

By differentiating Eq. (34), we get

$$\begin{aligned}\frac{d\Sigma}{ds} =&(m_2(s)\langle\Sigma,\gamma_2(s)\rangle_L + n_2(s)\langle\Sigma,\mu_2(s)\rangle_L)v_2(s)\\ &+ \Psi'(s)\Sigma \wedge_L v_2(s).\end{aligned}$$

Therefore, we get $\Psi'(s) = 0$. Then, we can write the polarization vector by

$$\Sigma = \cos\Psi\gamma_2(s) + \sin\Psi\mu_2(s)$$

where Ψ =constant. The Fermi–Walker derivative equation along the unit timelike vector v_2 is given by [44]

$$\frac{d\Sigma}{ds}_{\mathbf{FW}} = \frac{d\Sigma}{ds} + \langle\Sigma, v_2(s)\rangle_L\frac{dv_2}{ds} - \left\langle\Sigma,\frac{dv_2}{ds}\right\rangle_L v_2(s)$$

where $\{\gamma_2,\mu_2,v_2\}$ is orthonormal frame for timelike frontals. Since $\langle\Sigma, v_2(s)\rangle_L = 0$, we get

$$\begin{aligned}\frac{d\Sigma}{ds}_{\mathbf{FW}} =&\frac{d\Sigma}{ds} - (m_2(s)\langle\Sigma,\gamma_2(s)\rangle_L\\ &+ n_2(s)\langle\Sigma,\mu_2(s)\rangle_L)v_2(s).\end{aligned} \tag{35}$$

Consequently, we have

$$\frac{d\Sigma}{ds} = (m_2(s)\langle\Sigma,\gamma_2(s)\rangle_L + n_2(s)\langle\Sigma,\mu_2(s)\rangle_L)v_2(s).$$

Corollary 4 *If Σ satisfies the Fermi–Walker parallelism, the optical fiber is Σ_{v_2}-Rytov curve with condition $\langle\Sigma, v_2(s)\rangle_L = 0$, and the parametric representation of the Σ_{v_2}-Rytov curve is given by*

$$\Sigma_{v_2} = \gamma_2(s) + \cos\Psi\gamma_2(s) + \sin\Psi\mu_2(s) \tag{36}$$

where Ψ =constant.

6. Electromagnetic curves and trajectories for timelike frontals in de-Sitter 2-space

In this section, we introduce electromagnetic field equations and electromagnetic trajectories along the optical fiber, which is a timelike frontal, similar to the fourth section.

6.1. Electromagnetic curves of a light wave in an optical fiber for timelike frontals with $\Sigma \perp \mu_2$

By using equation $\langle\Sigma,\mu_2\rangle_L = 0$, we have

$$\begin{aligned}\frac{d\Sigma}{ds} =&\lambda\langle\Sigma, v_2(s)\rangle_L\gamma_2(s) - \lambda\langle\Sigma,\gamma_2(s)\rangle_L v_2(s)\\ &- n_2(s)\langle\Sigma, v_2(s)\rangle_L\mu_2(s).\end{aligned} \tag{37}$$

Similar to previous section, we get

$$\begin{aligned}\langle \Phi(\Sigma), \gamma_2(s)\rangle_L &= -\langle \Phi(\gamma_2(s)), \Sigma\rangle_L,\\ \langle \Phi(\Sigma), \mu_2(s)\rangle_L &= -\langle \Phi(\mu_2(s)), \Sigma\rangle_L,\\ \langle \Phi(\Sigma), \nu_2(s)\rangle_L &= -\langle \Phi(\nu_2(s)), \Sigma\rangle_L.\end{aligned} \tag{38}$$

By combining Eqs. (37) and (38), we get

$$\begin{pmatrix}\Phi(\gamma_2(s))\\ \Phi(\mu_2(s))\\ \Phi(\nu_2(s))\end{pmatrix} = \begin{pmatrix}0 & 0 & -\lambda\\ 0 & 0 & n_2(s)\\ -\lambda & n_2(s) & 0\end{pmatrix}\begin{pmatrix}\gamma_2(s)\\ \mu_2(s)\\ \nu_2(s)\end{pmatrix}.$$

Also, we can write the magnetic vector field Γ by

$$\Gamma = b_1\gamma_2(s) + b_2\mu_2(s) + b_3\nu_2(s) \tag{39}$$

where

$$\begin{aligned}\Phi(\gamma_2(s)) &= \Gamma \wedge_L \gamma_2(s),\\ \Phi(\mu_2(s)) &= \Gamma \wedge_L \mu_2(s),\\ \Phi(\nu_2(s)) &= \Gamma \wedge_L \nu_2(s).\end{aligned} \tag{40}$$

Then, it is seen that from Eqs. (39) and (40), the magnetic vector field of the Σ_{μ_2}−trajectories are given by

$$\Gamma = n_2(s)\gamma_2(s) + \lambda\mu_2(s).$$

Since $\Sigma \perp \mu_2$, the matrix representation of Lorentz force equations are given by

$$\begin{pmatrix}\Phi(\gamma_2(s))\\ \Phi(\mu_2(s))\\ \Phi(\nu_2(s))\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & n_2(s)\\ 0 & n_2(s) & 0\end{pmatrix}\begin{pmatrix}\gamma_2(s)\\ \mu_2(s)\\ \nu_2(s)\end{pmatrix}$$

and Σ_{μ_2}−trajectories are magnetic trajectories of the Γ magnetic field if and only if

$$\Gamma = n_2(s)\gamma_2(s).$$

Therefore, Γ is the Killing magnetic vector field of the Σ_{μ_2}−trajectories of Φ.

6.2. Electromagnetic curves of a light wave in an optical fiber for timelike frontals with $\Sigma \perp \nu_2$

Since $\langle \Sigma, \nu_2\rangle_L = 0$, we have

$$\begin{aligned}\frac{d\Sigma}{ds} =& \lambda\langle \Sigma, \mu_2(s)\rangle_L\gamma_2(s) - \lambda\langle \Sigma, \gamma_2(s)\rangle_L\mu_2(s)\\ &+ (m_2(s)\langle \Sigma, \gamma_2(s)\rangle_L + n_2(s)\langle \Sigma, \mu_2(s)\rangle_L)\nu_2(s).\end{aligned} \tag{41}$$

Then, we have

$$\begin{pmatrix}\Phi(\gamma_2(s))\\ \Phi(\mu_2(s))\\ \Phi(\nu_2(s))\end{pmatrix} = \begin{pmatrix}0 & -\lambda & m_2(s)\\ \lambda & 0 & n_2(s)\\ m_2(s) & n_2(s) & 0\end{pmatrix}\begin{pmatrix}\gamma_2(s)\\ \mu_2(s)\\ \nu_2(s)\end{pmatrix}.$$

Similar to previous section, the magnetic vector field Γ is written by

$$\Gamma = n_2(s)\gamma_2(s) - m_2(s)\mu_2(s) + \lambda\nu_2(s).$$

If $\Sigma \perp \nu_2$, the matrix representation of Lorentz force equations are given by

$$\begin{pmatrix}\Phi(\gamma_2(s))\\ \Phi(\mu_2(s))\\ \Phi(\nu_2(s))\end{pmatrix} = \begin{pmatrix}0 & 0 & m_2(s)\\ 0 & 0 & n_2(s)\\ m_2(s) & n_2(s) & 0\end{pmatrix}\begin{pmatrix}\gamma_2(s)\\ \mu_2(s)\\ \nu_2(s)\end{pmatrix}.$$

It is clear that, Σ_{ν_2}−trajectories are magnetic trajectories of the Γ magnetic field if and only if

$$\Gamma = n_2(s)\gamma_2(s) - m_2(s)\mu_2(s).$$

Example 1 Let $\gamma_1 : [0, 2\pi) \rightarrow S_1^2$ be a spacelike front as an optical fiber in de-Sitter 2-space defined by

$$\begin{aligned}\gamma_1(s) =& \Big(\frac{3}{4}\sinh s - \frac{1}{4}\sinh 3s, \frac{3}{4}\cosh s - \frac{1}{4}\cosh 3s,\\ &\frac{\sqrt{3}}{2}\cosh s\Big).\end{aligned} \tag{42}$$

Also, since $\gamma_1'(0) = (0, 0, 0)$, $\gamma_1(s)$ is a non-regular curve in de-Sitter 2-space. Therefore, we can write

$$\begin{aligned}\mu_1(s) =& \Big(-\frac{3}{4}\cosh s - \frac{1}{4}\cosh 3s, -\frac{3}{4}\sinh s - \frac{1}{4}\sinh 3s,\\ &\frac{\sqrt{3}}{2}\sinh s\Big),\\ \nu_1(s) =& \Big(-\frac{\sqrt{3}}{2}\sinh 2s, -\frac{\sqrt{3}}{2}\cosh 2s, \frac{1}{2}\Big)\end{aligned}$$

for $\mu_1 : [0, 2\pi) \rightarrow H^2(-1)$ and $\nu_1 : [0, 2\pi) \rightarrow S_1^2$. Then, $(\gamma_1, \mu_1) : [0, 2\pi) \rightarrow \Delta_1$ is a spacelike Legendrian curve. Moreover, the spacelike Legendrian curvatures are given by

$$m_1(s) = \sqrt{3}\sinh s, \quad n_1(s) = \sqrt{3}\cosh s.$$

It is clear that (γ_1, μ_1) is a Legendre immersion. Therefore, the front γ_1 is considered an optical fiber. From Eqs. (9) and (17), Σ_{μ_1}-Rytov curve and Σ_{ν_1}-Rytov curve are given. The two cases of the Berry phase model with Rytov curves are given in Figs. 1 and 2. In Figs. 1 and 2, the blue curves on the Lorentzian sphere represent optical fibers with spacelike fronts in de-Sitter 2-space. It is clear from the figure that optical fibers have singular points. The Σ_{μ_1} and Σ_{ν_1} Rytov- curves formed along singular optical fibers are black curves and can be singular or regular. The red vectors represent the polarization vectors. Also, optical fiber curve and Rytov curves are plotted in the interval (-2.2).

Example 2 Let $\gamma_2 : [0, 2\pi) \rightarrow S_1^2$ be a timelike front as an optical fiber in de-Sitter 2-space defined by

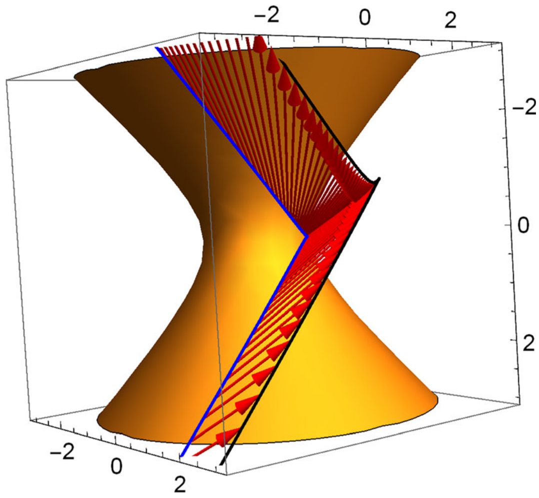

Fig. 1 Polarization vectors and Rytov curves along an optical fiber with a spacelike front when $\Sigma \perp \nu_1$ in the Lorentzian sphere in de-Sitter 2-space

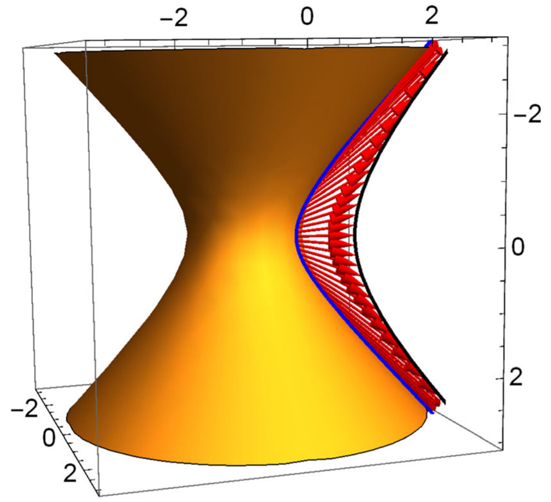

Fig. 3 Polarization vectors and Rytov curves along an optical fiber with a timelike front when $\Sigma \perp \nu_2$ in the Lorentzian sphere in de-Sitter 2-space

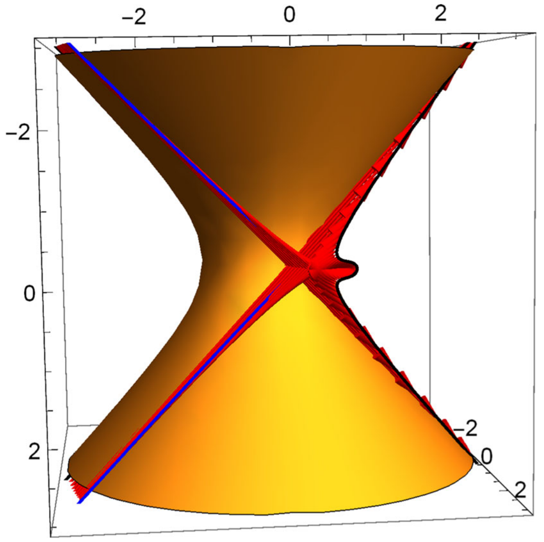

Fig. 2 Polarization vectors and Rytov curves along an optical fiber with a spacelike front when $\Sigma \perp \mu_1$ in the Lorentzian sphere in de-Sitter 2-space

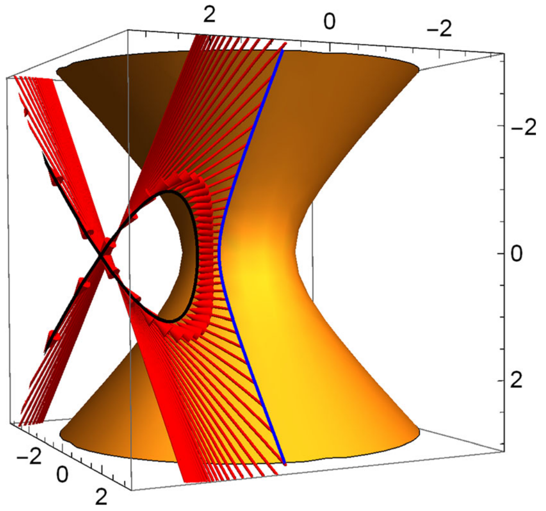

Fig. 4 Polarization vectors and Rytov curves along an optical fiber with a timelike front when $\Sigma \perp \mu_2$ in the Lorentzian sphere in de-Sitter 2-space

$$\gamma_2(s) = \left(\frac{1}{\sqrt{2}}\sinh 2s, \frac{1}{\sqrt{2}}\cosh 2s, \frac{1}{\sqrt{2}}\right) \tag{43}$$

which is a timelike front, as an optical fiber in de-Sitter 2-space. Since $\gamma_2'(s) \neq (0,0,0)$ for every s, $\gamma_2(s)$ is a regular curve in de-Sitter 2-space. Then, we get

$$\begin{aligned}\mu_2(s) =& \left(\frac{1}{\sqrt{2}}\sinh 2s, \frac{1}{\sqrt{2}}\cosh 2s, -\frac{1}{\sqrt{2}}\right),\\ \nu_2(s) =& (\cosh 2s, \sinh 2s, 0)\end{aligned}$$

for $\mu_2 : [0, 2\pi) \to S_1^2$ and $\nu_2 : [0, 2\pi) \to H^2(-1)$. Therefore, $(\gamma_2, \mu_2) : [0, 2\pi) \to \Delta_2$ is a timelike Legendrian curve. On the other hand, the timelike Legendrian curvatures are given by

$$m_2(s) = \sqrt{2}, \quad n_2(s) = \sqrt{2}.$$

It is seen that (γ_2, μ_2) is a Legendre immersion. Therefore, the front γ_2 is considered an optical fiber. From Eqs. (30) and (36), Σ_{μ_2}-Rytov curve and Σ_{ν_2}-Rytov curve are obtained. In Figs. 3 and 4, the blue curves in the Lorentzian sphere also indicate the timelike front in Sitter 2-space. Specifically, this curve is a regular curve. The black curves represent the Σ_{μ_2} and Σ_{ν_2} Rytov curves. As can be seen from the figure, these two curves are regular. Red vectors describe polarization vectors. The two cases of the Berry phase model with Rytov curves are given in Figs. 3 and 4. Also, optical fiber curve and Rytov curves are plotted in the interval (-2.2).

7. Conclusions

In this article, we construct Berry phase models along optical fibers that are spacelike and timelike frontal in de-Sitter 2-space, and we introduce their electromagnetic trajectories and electromagnetic curves. We think that this article will contribute to both singularity theory and physics, due to the scarcity of singular point curve studies and the interdisciplinary approach in this field. This article can also be considered for framed curves, which is a more general structure in Minkowski space. In addition, in line with the results obtained in this study and based on the Lorentz force equations, concepts such as Maxwell's equations, the concept of angular momentum, and electromagnetic energy can be examined both in Euclidean space, Minkowski space or in Sitter 2-space, inspired by the study [22]. In addition, flow equations of spherical Legendrian frames can be generated with the help of the moving frame of singular curves in the respective spaces. Like the study [45], magnetic flux surfaces can be examined in detail thanks to the structures obtained. On the other hand, the physical structure of ferromagnetic equations for regular curves in Sitter 2-space is quite interesting [47]. In this study, since the basic concepts for singular curves in de-Sitter 2-spaces are given, spherical magnetic ferromagnetic systems and ferromagnetic equations can be studied in these curves in the future.

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