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On the Quaternionic B_2-Slant Helices in the Euclidean Space E^4

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Abstract. In this paper we give a new definition of harmonic curvature functions in terms of B_2 and we define a new kind of slant helix which we call quaternionic B_2-slant helix in $4-$dimensional Euclidean space E^4 by using the new harmonic curvature functions. Also we define a vector field D which we call Darboux quaternion of the real quaternionic B_2-slant helix in $4-$dimensional Euclidean space E^4 and we give a new characterization such as:

"$\alpha : I \subset \mathbb{R} \longrightarrow E^4$ is a quaternionic B_2-slant helix $\Leftrightarrow H_2^{'} - KH_1 = 0$"

where H_2, H_1 are harmonic curvature functions and K is the principal curvature function of the curve α.

Keywords. Slant helices, harmonic curvature functions, Euclidean spaces, quaternion algebra.

1. Introduction

The quaternion was introduced by Hamilton. His initial attempt to generalize the complex numbers by introducing a three-dimensional object failed in the sense that the algebra he constructed for these three-dimensional object did not have the desired properties. On the 16th October 1843 Hamilton discovered that the appropriate generalization is one in which the scalar(real) axis is left unchanged whereas the vector(imaginary) axis is supplemented by adding two further vector axis.

In [4], Özdamar and Hacısalihoğlu defined harmonic curvature functions. They generalized the inclined curves in E^3 to E^n, $n > 3$, and then gave a characterization for them: "If a curve α is an inclined curve then $\sum\limits_{i=1}^{n-2} H_i^2 =$ constant". And then many studies have been reported on generalized helices, inclined curves by using Hacısalihoğlu's harmonic curvature functions [1], [2], [3], [7].

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After them, Izumiya and Takeuchi defined a new kind of helix (slant helix) and they gave a characterization of slant helices in Euclidean 3-space E^3 [13], Kula and Yaylı investigated spherical images; the tangent indicatrix and binormal indicatrix of a slant helix [10] . In 2008, Önder *et al.* defined a new kind of slant helix in Euclidean 4-space E^4 which they called B_2-slant helix and they gave some characterizations of this slant helices in Euclidean 4-space E^4 [12] . And then in 2009, Gök *et al.* defined a new kind of slant helix in Euclidean n-space E^n, $n > 3$, which they called V_n-slant helix and they gave some characterizations of this slant helices in Euclidean n-space E^n [7] .

The theory of the Serret-Frenet formulas for quaternionic curves in $\mathbb{R}^3$ and $\mathbb{R}^4$ are given by Baharathi and Nagaraj [9] . And then, a lot of studies have been published by using this study. One of them is Karadağ and Sivridağ's study [11] where they gave many characterizations for quaternionic inclined curves in the Euclidean space $\mathbb{E}^4$, another one is Çöken and Tuna's study [1] which they gave for quaternionic inclined curves in the semi-Euclidean space $\mathbb{E}^4_2$ and the others.

In this study, we define a new kind of slant helix in Euclidean 4-space E^4, where we use the constant angle φ in between a unit and constant real quaternion X which has fixed direction and the last Frenet vector field B_2 of the curve, that is,

$$h(B_2, X) = \cos\varphi \ , \ \ \varphi \neq \frac{\pi}{2} \ , \ \ \varphi = \text{constant},$$

where h is symmetric, real valued, non-degenerate, bilinear quaternion inner product. Since Frenet vector field B_2 of a curve makes a constant angle with the real quaternion X which is fixed direction, we call this curve quaternionic B_2-slant helix in Euclidean 4-space E^4. In this paper, at first, we give, Hacısalihoğlu's harmonic curvature functions for the **real quaternionic** B_2**-slant helix** and we give some characterizations for the quaternionic B_2-slant helices in terms of the harmonic curvatures. In this case we define a new Darboux vector field D, and we give some new characterizations for B_2-slant helices. Finally, we show that if any quaternionic curve is ccr-curve (constant curvature ratio), the curve can't be quaternionic B_2-slant helix.

2. Preliminaries

Let Q_H denote a four dimensional vector space over a field H whose characteristic grater than 2. Let $e_i (1 \leq i \leq 4)$ denote a basis for the vector space. Let the rule of multiplication on Q_H be defined on $e_i (1 \leq i \leq 4)$ and extended to the whole of the vector space by distributivity as follows:

A real quaternion is defined by $q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + de_4$ where a, b, c, d are ordinary numbers. Such that

$$\begin{array}{lll} e_4 = 1, & e_1^2 = e_2^2 = e_3^2 = -1, & \\ e_1e_2 = e_3, & e_2e_3 = e_1, \ e_3e_1 = e_2, & \\ e_2e_1 = -e_3, & e_3e_2 = -e_1, \ e_1e_3 = -e_2. & \end{array} \tag{1}$$

If we denote $S_q = d$ and $\overrightarrow{V_q} = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3}$, we can rewrite a real quaternion whose basic algebraic form is $q = S_q + \overrightarrow{V_q}$ where S_q is scalar part and $\overrightarrow{V_q}$ is vectorial part of q. Using these basic products we can now expand the product of two quaternions as

$$p \times q = S_p S_q - \langle \overrightarrow{V_p},\ \overrightarrow{V_q} \rangle + S_p \overrightarrow{V_q} + S_q \overrightarrow{V_p} + \overrightarrow{V_p} \wedge \overrightarrow{V_q} \text{ for every } p, q \in Q_H, \quad (2)$$

where we have used the inner and cross products in Euclidean space E^3 [6]. There is a unique involutory antiautomorphism of the quaternion algebra, denoted by the symbol γ and defined as follows:

$$\gamma q = -a\overrightarrow{e_1} - b\overrightarrow{e_2} - c\overrightarrow{e_3} + de_4 \text{ for every } q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + de_4 \in Q_H$$

which is called the "Hamiltonian conjugation". This defines the symmetric, real valued, non-degenerate, bilinear form h as follows:

$$h(p,\ q) = \frac{1}{2}\left[\, p \times \gamma q + q \times \gamma p\right] \text{ for } p, q \in Q_H.$$

And then, the norm of any q real quaternion denoted

$$\|q\|^2 = h(q, q) = q \times \gamma q. \quad (3)$$

The concept of a spatial quaternion will be made use of throughout our work. q is called a spatial quaternion whenever $q + \gamma q = 0$ [2].

The Serret-Frenet formulae for quaternionic curves in E^3 and E^4 are as follows:

Theorem 2.1. *The three-dimensional Euclidean space E^3 is identified with the space of spatial quaternions $\{p \in Q_H \mid p + \gamma p = 0\}$ in an obvious manner. Let $I = [0, 1]$ denote the unit interval in the real line $\mathbb{R}$. Let*

$$\alpha : I \subset \mathbb{R} \longrightarrow Q_H$$

$$s \longrightarrow \alpha(s) = \sum_{i=1}^{3} \alpha_i(s)\overrightarrow{e_i},\ 1 \le i \le 3.$$

be an arc-lenghted curve with nonzero curvatures $\{k,\ r\}$ and $\{t(s),\ n(s),\ b(s)\}$ denote the Frenet frame of the curve α. Then Frenet formulas are given by

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

where k is the principal curvature, r is torsion of α [9].

Theorem 2.2. *The four-dimensional Euclidean spaces E^4 are identified with the space of unit quaternions. Let $I = [0, 1]$ denote the unit interval in the real line $\mathbb{R}$ and*

$$\beta : I \subset \mathbb{R} \longrightarrow Q_H$$

$$s \longrightarrow \beta(s) = \sum_{i=1}^{4} \alpha_i(s)\overrightarrow{e_i},\ 1 \le i \le 4,\ \overrightarrow{e_4} = 1.$$

be a smooth curve in E^4 *with nonzero curvatures* $\{K,\ k,\ r-K\}$ *and* $\{T(s),\ N(s),\ B_1(s),\ B_2(s)\}$ *denotes the Frenet frame of the curve* β. *Then Frenet formulas are given by*

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (r-K) \\ 0 & 0 & -(r-K) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

where K *is the principal curvature,* k *is torsion of* β *and* $(r-K)$ *is bitorsion of* β [9].

Definition 2.1. Let $\alpha : I \subset \mathbb{R} \longrightarrow E^4$ be a unit speed curve with nonzero curvatures $k_1(s), k_2(s)$ and $k_3(s)$ and $\{T,\ N,\ B_1,\ B_2\}$ denotes the Frenet frame of the curve α in E^4. Önder et al. defined $\alpha(s)$ as B_2-slant helix if its second binormal unit vector B_2 makes a constant angle φ with a fixed direction on a unit vector U; that is, for all $s \in I$,

$$\langle B_2, U \rangle = \cos\varphi\ ,\ \varphi \neq \frac{\pi}{2}\ ,\ \varphi = \text{constant}$$

along the curve [12].

3. Quaternionic B_2-Slant Helix and Its Harmonic Curvature Functions

In this section, we give some characterizations for quaternionic B_2-slant helix in Euclidean space E^3 and E^4 in terms of harmonic curvature function of quaternionic curve.

Definition 3.1. Let $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ be a regular real spatial quaternionic curve with arc length parameter s and $\{t(s),\ n(s),\ b(s)\}$ denotes the Frenet frame of the curve α. We call α as a real spatial quaternionic b-slant helix in E^3 if the last Frenet vector field b makes a constant angle φ with a fixed direction X, that is,

$$h(b, X) = \cos\varphi\ ,\ \varphi \neq \frac{\pi}{2}\ ,\ \varphi = \text{constant} \tag{4}$$

where X is a unit and constant real spatial quaternion for all $s \in I$.

Definition 3.2. Let $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ be a regular real spatial quaternionic curve with arc length parameter s and nonzero curvatures $\{k,\ r\}$. In that case harmonic curvature function of the curve α is

$$H : I \longrightarrow \mathbb{R}$$

$$H(s) = \frac{r(s)}{k(s)}. \tag{5}$$

Theorem 3.1. *Let* $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ *be a regular real spatial quaternionic curve in* E^3 *with arc length parameter* s *and* $\{t(s),\ n(s),\ b(s)\}$ *denotes the Frenet frame of the curve* α. *If the curve* α *is a real spatial quaternionic* b*-slant helix with* X *as its axis then we have*

$$h(t,\ X) = H.h(b,\ X) \tag{6}$$

where H *is harmonic curvature function of the curve* α.

Proof. Let $\varphi \neq \frac{\pi}{2}$ be a constant angle between the X real spatial quaternion and the binormal vectors of the curve α real spatial quaternionic b-slant helix in E^3. So, we have

$$h(b(s),\ X) = \cos\varphi \text{ , for all } s \in I, \tag{7}$$

then differentiating Eq. (7), with respect to s, we obtain

$$h(b^{'}(s),\ X) = 0$$

or from Serret-Frenet formulas of α we have

$$h(-rn,\ X) = 0$$

where $r \neq 0$, then

$$h(n,\ X) = 0. \tag{8}$$

Again, differentiating Eq. (8), with respect to s, and by using the Serret-Frenet formulas of α we have

$$h(t, X) = \frac{r}{k}.h(b, X)$$

and then Eq. (5) gives us

$$h(t,\ X) = H.h(b,\ X). \qquad \square$$

Theorem 3.2. *Let* $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ *be a regular real spatial quaternionic curve with arc length parameter* s *and* $\{t(s),\ n(s),\ b(s)\}$ *denotes the Frenet frame of the curve* α. *If the curve* α *is a real spatial quaternionic* b*-slant helix in* E^3, *the axis of* α *is*

$$X = (H(s)t(s) + b(s))h(b,\ X)$$

or

$$X = (H(s)t(s) + b(s))\cos\varphi. \tag{9}$$

where $H(s)$ *is harmonic curvature function of the curve* α.

Proof. If the axis of α real spatial quaternionic b-slant helix α is X, then we can write

$$X = \lambda_1 t + \lambda_2 n + \lambda_3 b.$$

Then by using Theorem (3.1)

$$\begin{aligned}\lambda_1 &= Hh(b,\ X),\\ \lambda_2 &= 0,\\ \lambda_3 &= h(b,\ X).\end{aligned}$$

Thus it is easy to obtain

$$X = (H(s)t(s) + b(s))\cos\varphi. \qquad \square$$

Definition 3.3. Let $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ be a regular real spatial quaternionic curve with arc length parameter s and $\{t(s),\ n(s),\ b(s)\}$ denotes the Frenet frame of the curve α and H denote the harmonic curvature function at the point $\alpha(s)$. The quaternion

$$D = H(s)t(s) + b(s) \tag{10}$$

is called a Darboux quaternion of the real spatial quaternionic b-slant helix α in E^3.

Theorem 3.3. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ be a regular real spatial quaternionic curve with arc length parameter s and $\{t(s),\ n(s),\ b(s)\}$ denotes the Frenet frame of the curve α and H denotes the harmonic curvature function at the point $\alpha(s)$. Then the curve α is a real spatial quaternionic b-slant helix in E^3 if and only if D is a constant real spatial quaternion.*

Proof. ($\Rightarrow$) Let α be a real spatial quaternionic b-slant helix in E^3 and X be the axis of α. From Theorem (3.2), we have

$$X = (H(s)t(s) + b(s))\cos\varphi = D\cos\varphi$$

where φ and X are constant and hence D is a constant real spatial quaternion.

($\Leftarrow$) Suppose that D is a constant vector field, then we have $\|D\|^2 = h(D, D)$ is constant. By using Theorem (3.2) we can write

$$\begin{aligned} \|X\|^2 &= \|D\cos\varphi\|^2 \\ &= h(D\cos\varphi, D\cos\varphi) \\ &= \cos^2\varphi . h(D, D). \end{aligned}$$

Since X is a unit real quaternion and $\|D\|$ is constant, we have $\cos\varphi = \frac{1}{\|D\|} = h(b(s), X)$ is constant. So, α is a real spatial quaternionic $b-$slant helix. This completes the proof. $\square$

Theorem 3.4. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ be a regular real spatial quaternionic curve with arc length parameter s and $\{t(s),\ n(s),\ b(s)\}$ denotes the Frenet frame of the curve α and H denotes the harmonic curvature function at the point $\alpha(s)$. If the curve α is a real spatial quaternionic b-slant helix in E^3, then $H^2(s)$ is constant.*

Proof. Let α be a real spatial quaternionic b-slant helix. Since the axis of α is $X = H(s)\cos\varphi t(s) + \cos\varphi b(s)$ unit real spatial quaternion, $\|X\|^2 = 1$. Hence, using Eq. (3) we have

$$\begin{aligned} \|X\|^2 &= h(X,\ X) = X \times \gamma X, \\ 1 &= [H(s)\cos\varphi t(s) + \cos\varphi b(s)] \times \gamma\left[H(s)\cos\varphi t(s) + \cos\varphi b(s)\right], \\ &= H^2(s)\cos^2\varphi\left\|t(s)\right\|^2 - H(s)\cos^2\varphi t(s) \times b(s) \\ &\quad - H(s)\cos^2\varphi b(s) \times t(s) + \cos^2\varphi\left\|b(s)\right\|^2, \end{aligned}$$

where since $t(s) \times b(s) = -b(s) \times t(s)$, we can write

$$\|X\|^2 = 1 = (1 + H^2(s)) \cos^2 \varphi$$

and then

$$H^2(s) = \tan^2 \varphi = \text{constant}. \qquad \square$$

Corollary 3.5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ be a regular real spatial quaternionic curve with arc length parameter s and nonzero curvatures $\{k,\ r\}$. Then the curve α is a real spatial quaternionic b-slant helix in E^3 if and only if* $\left(\frac{r(s)}{k(s)}\right)' = 0$

Proof. It is obvious from Theorem (3.3) $\square$

Definition 3.4. Let $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ be an arc-lengthen real quaternionic curve with nonzero curvatures $K, k, r - K$ and $\{T(s), N(s), B_1(s), B_2(s)\}$ denotes the Frenet frame of β. We call β as a quaternionic B_2-slant helix in Q_H if the last unit vector field B_2 makes a constant angle φ with a fixed direction X, that is

$$h(B_2,\ X) = \cos \varphi,\ \varphi \neq \frac{\pi}{2},\ \varphi = \text{constant}.$$

where X is a unit and constant real quaternion which is the axis of α for all $s \in I$.

Theorem 3.6. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^3$ be a regular real spatial quaternionic b-slant helix, such that $\alpha(s) = \alpha_1(s)\overrightarrow{e_1} + \alpha_2(s)\overrightarrow{e_2} + \alpha_3(s)\overrightarrow{e_3}, \beta(s) = \alpha_1(s)\overrightarrow{e_1} + \alpha_2(s)\overrightarrow{e_2} + \alpha_3(s)\overrightarrow{e_3} + \alpha_4(s)$ be obtained from α. Then β is a real quaternionic inclined curve in Q_H.*

Proof. Let $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ be an arc-lengthen real quaternionic curve and X be a unit and constant real spatial quaternion which is the axis of α such that $\{T(s), N(s), B_1(s), B_2(s)\}$ be Frenet apparatus at the point $\beta(s)$ of β. Then we have

$$\begin{aligned} h(T(s),\ X) &= \frac{1}{2}\left[T(s) \times \gamma X + X \times \gamma T(s)\right],\ T(s) = S_{T(s)} + \overrightarrow{V}_{T(s)} \\ &= \frac{1}{2}\left[(S_{T(s)} + \overrightarrow{V}_{T(s)}) \times \gamma X + X \times \gamma(S_{T(s)} + \overrightarrow{V}_{T(s)})\right] \end{aligned}$$

where since X is a unit and constant real spatial quaternion, $S_X = 0$ and $\gamma X = -X$. Eq. (2) gives us that

$$\begin{aligned} h(T(s),\ X) &= \frac{1}{2}\Big[-S_{T(s)}.X + \langle \overrightarrow{V}_{T(s)},\ X\rangle - \overrightarrow{V}_{T(s)} \wedge X + S_{T(s)}.X \\ &\quad + \langle \overrightarrow{V}_{T(s)},\ X\rangle - X \wedge \overrightarrow{V}_{T(s)}\Big] \\ &= \langle \overrightarrow{V}_{T(s)}, X\rangle \end{aligned}$$

Since, α is a real spatial quaternionic b-slant helix, $h(b(s), X) =$ constant and then $h(t(s), X) = \langle \overrightarrow{V}_{T(s)}, X\rangle$ =constant. So, β is a quaternionic inclined curve in Q_H. $\square$

Definition 3.5. Let $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ be an arc-lengthen real quaternionic curve with nonzero curvatures $K, k, r - K$. In that case harmonic curvature functions in terms of B_2 of β are defined by

$$H_i : I \subset \mathbb{R} \longrightarrow \mathbb{R}$$

$$\begin{aligned} H_0 &= 0, \\ H_1 &= \tfrac{r-K}{k}, \\ H_2 &= -\tfrac{1}{K} H_1' = -\tfrac{1}{K} \left(\tfrac{r-K}{k}\right)'. \end{aligned} \tag{11}$$

where K is the principal curvature, k is the torsion of β, $(r - K)$ is the bi-torsion of β.

Theorem 3.7. *Let $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ be an arc-lengthen real quaternionic curve and X be a unit and constant quaternion of Q_H.*

$$\{T(s), N(s), B_1(s), B_2(s)\}$$

denotes the Frenet frame of the curve β, $\{H_1, H_2\}$ denotes the harmonic curvature functions of the quaternionic curve β. If $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ is a quaternionic B_2-slant helix and X as its axis then we have

$$\begin{aligned} h\left(T(s),\ X\right) &= H_2 h\left(B_2(s),\ X\right), & h\left(N(s),\ X\right) &= H_1 h\left(B_2(s),\ X\right), \\ h\left(B_1(s),\ X\right) &= H_0 h\left(B_2(s),\ X\right), & h\left(B_2(s),\ X\right) &= \cos\varphi. \end{aligned} \tag{12}$$

Proof. Let $\varphi \neq \frac{\pi}{2}$ be a constant angle between the quaternion X and the last Frenet vector of the curve β quaternionic B_2-slant helix in Q_H. So, we have

$$h\left(B_2(s),\ X\right) = \cos\varphi, \text{ for all } s \in I$$

then differentiating the equation above, with respect to s, we obtain

$$h\left(B_2'(s),\ X\right) = 0$$

or from Serret-Frenet formulas of β we have

$$h\left(-(r-K)B_1(s),\ X\right) = 0$$

where $(r - K) \neq 0$, then

$$h\left(B_1(s),\ X\right) = 0 = H_0 h\left(B_2(s),\ X\right). \tag{13}$$

By taking the derivate of Eq. (13) and applying the Frenet formulas we obtain

$$\begin{aligned} h\left(B_1'(s),\ X\right) &= 0 \\ h\left(-kN(s) + (r-K)\,B_2(s),\ X\right) &= 0 \\ -kh(N(s),\ X) + (r-K)h(B_2(s),\ X) &= 0 \end{aligned}$$

and from Eq. (11), we have

$$h(N(s),\ X) = H_1 h(B_2(s),\ X). \tag{14}$$

If we take the derivative of the last equation, we get

$$\begin{aligned} h(N^{'}(s), X) &= H_1^{'} h(B_2(s),\ X) \\ h(-KT(s) + kB_1(s),\ X) &= H_1^{'} h(B_2(s),\ X) \\ -Kh(T(s),\ X) + kh(B_1(s),\ X) &= H_1^{'} h(B_2(s),\ X) \end{aligned}$$

and by using Eq. (13) and Definition (3.5) give us that

$$h(T(s),\ X) = H_2 h(B_2(s),\ X). \tag{15}$$

So, from Eq. (13), Eq. (14) , Eq. (15) the proof is completed. □

Corollary 3.8. *Let* $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ *be an arc-lengthen parameter real quaternionic curve and* X *be a unit and constant real quaternion of* Q_H*. Let* $\{T(s), N(s), B_1(s), B_2(s)\}$, $\{H_1, H_2\}$ *denote the Frenet frame and the harmonic curvature functions of the quaternionic curve, respectively. If* $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ *is a quaternionic* B_2*-slant helix, the axis of* β *is*

$$X = \{H_2 T(s) + H_1 N(s) + B_2(s)\}\, h(B_2(s),\ X)$$

or

$$X = \{H_2 T(s) + H_1 N(s) + B_2(s)\} \cos\varphi.$$

Proof. If X is the axis of a quaternionic B_2-slant helix in Q_H, then we can write

$$X = \lambda_1 T(s) + \lambda_2 N(s) + \lambda_3 B_1(s) + \lambda_4 B_2(s)$$

and then by using Theorem (3.7), we get

$$\begin{aligned} \lambda_1 &= h(T(s),\ X) = H_2 h(B_2(s),\ X)\ , \\ \lambda_2 &= h(N(s),\ X) = H_1 h(B_2(s),\ X)\ , \\ \lambda_3 &= h(B_1(s,\ X) = H_0 h(B_2(s),\ X), \\ \lambda_4 &= h(B_2(s),\ X). \end{aligned}$$

Thus we easily obtain that

$$X = \{H_2 T(s) + H_1 N(s) + B_2(s)\} \cos\varphi.$$ □

Definition 3.6. Let $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ be an arc-lengthen quaternionic curve. Let $\{T(s), N(s), B_1(s), B_2(s)\}$ denote the Frenet frame of the curve β, $\{H_1, H_2\}$ denotes the harmonic curvature functions of the quaternionic curve β. The quaternion

$$D = H_2 T(s) + H_1 N(s) + B_2(s)$$

is called the Darboux quaternion of the quaternionic B_2-slant helix β.

Theorem 3.9. *Let* $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ *be an arc-lengthen real quaternionic curve. Let* $\{T(s), N(s), B_1(s), B_2(s)\}$ *denote the Frenet frame of the curve* β*,* $\{H_1, H_2\}$ *denotes the harmonic curvature functions of the quaternionic curve* β*. Then* β *is a quaternionic* B_2*-slant helix if and only if* D *is a constant real quaternion.*

Proof. ($\Rightarrow$) Suppose that β is a quaternionic B_2-slant helix in Q_H and X is the axis of β. From Corollary (3.8), we have that

$$X = \{H_2T(s) + H_1N(s) + B_2(s)\}\cos\varphi = D\cos\varphi$$

where φ and X are constant and hence D is a constant real quaternion.
($\Leftarrow$) Let D be a constant real spatial quaternion. From Definition (3.6) and Corollary (3.8), we can write

$$X = D\cos\varphi. \tag{16}$$

Derivating of Eq. (16) with respect to s, we get

$$X' = D'\cos\varphi + D(\cos\varphi)'$$

from the hypothesis, D is a constant real spatial quaternion and

$$D(\cos\varphi)' = 0 \text{ where } D \neq 0,$$

or we get

$$\cos\varphi = \text{constant}.$$

We can define a unique axis of the quaternionic B_2-slant helix where,

$$\begin{aligned} h(B_2(s),\ X) &= h(B_2(s),\ D\cos\varphi) \\ &= \cos\varphi h(B_2(s),\ D). \end{aligned}$$

From Definition (3.6), we get

$$h(B_2(s),\ X) = \cos\varphi$$

Thus X is a constant real quaternion and β is a quaternionic B_2-slant helix in Q_H. Which completes the proof. □

Theorem 3.10. *Let $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ be an arc-lengthen real quaternionic curve. Let $\{T(s), N(s), B_1(s), B_2(s)\}$ denote the Frenet frame of the curve β, $\{H_1, H_2\}$ denotes the harmonic curvature functions of the quaternionic curve β. Then β is a quaternionic B_2-slant helix if and only if*

$$H_2' - KH_1 = 0. \tag{17}$$

Proof. ($\Rightarrow$) If we differentiate D along the curve β, we get

$$D' = H_2'T + H_2T' + H_1'N + H_1N' + B_2'.$$

The Serret-Frenet formulas and Eq. (11) give us

$$D' = \left(H_2' - KH_1\right)T. \tag{18}$$

Since β is a quaternionic B_2-slant helix, D is a constant real quaternion. Thus we can write $D' = 0$ or $\left(H_2' - KH_1\right) = 0$.

($\Leftarrow$) If Eq. $H_2' - KH_1 = 0$ we can easily see that $D' = 0$ or D is a constant real quaternion, and then from Theorem (3.9), we have that β is a quaternionic B_2-slant helix in Q_H. Which completes the proof. □

Corollary 3.11. *Let* $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ *be an arc-lengthen real quaternionic curve with nonzero curvatures* $K, k, r-K$. *Let* $\{T(s), N(s), B_1(s), B_2(s)\}$ *and* $\{H_1, H_2\}$ *be the Frenet frame and the harmonic curvature functions of the quaternionic curve, respectively. Then* β *is a quaternionic* B_2*-slant helix if and only if*

$$\left[\frac{1}{K}\left(\frac{r-K}{k}\right)'\right]' + K\left(\frac{r-K}{k}\right) = 0 \tag{19}$$

Proof. ($\Rightarrow$) Let β be a quaternionic B_2-slant helix in Q_H, then from Theorem (3.10) we have $H_2' - KH_1 = 0$. By using Definition (3.5) we have

$$\left[\frac{1}{K}\left(\frac{r-K}{k}\right)'\right]' + K\left(\frac{r-K}{k}\right) = 0.$$

($\Leftarrow$) We suppose that the equation $\left[\frac{1}{K}\left(\frac{r-K}{k}\right)'\right]' + K\left(\frac{r-K}{k}\right) = 0$ holds, then from Theorem (3.10) and Definition (3.5), it is obvious that β is a quaternionic B_2-slant helix in Q_H which completes the proof. ☐

Corollary 3.12. *Let* $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ *be an arc-lengthen real quaternionic curve with nonzero curvatures* $K, k, r-K$. *If* β *is ccr-curve (constant curvature ratio), the curve can't be quaternionic* B_2*-slant helix.*

Proof. Let β be real quaternionic ccr-curve; that is, $\frac{r}{k}$ and $\frac{r-K}{k}$ are constant. So, from Theorem (3.10) and Definition (3.5), it is obvious that the curve β can't be a quaternionic B_2-slant helix. ☐

Theorem 3.13. *Let* $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ *be an arc-lengthen real spatial quaternionic curve with nonzero curvatures* $K, k, r-K$. *If* β *is a quaternionic* B_2*-slant helix then the following condition is satisfied,*

$$H_1^2 + H_2^2 = \tan^2\varphi = constant$$

where φ *is the constant angel between the last Frenet vector* B_2 *and a constant unit real quaternion* X.

Proof. Let β be a real quaternionic B_2-slant helix. Since the axis of β is $X = H_2\cos\varphi T(s) + H_1\cos\varphi N(s) + \cos\varphi B_2(s)$ unit real quaternion, $\|X\|^2 = 1$. Hence, using Eq. (3) we have

$$\begin{aligned}
\|X\|^2 &= h(X,\ X) = X \times \gamma X,\\
1 &= [H_2\cos\varphi T(s) + H_1\cos\varphi N(s) + \cos\varphi B_2(s)]\\
&\quad \times \gamma\left[H_2\cos\varphi T(s) + H_1\cos\varphi N(s) + \cos\varphi B_2(s)\right],\\
&= H_2^2(s)\cos^2\varphi T(s) \times \gamma T(s) + H_2H_1\cos^2\varphi T(s)\\
&\quad \times \gamma N(s) + H_2\cos^2\varphi T(s) \times \gamma B_2(s) + H_1H_2\cos^2\varphi N(s)\\
&\quad \times \gamma T(s) + H_1^2(s)\cos^2\varphi N(s) \times \gamma N(s) + H_1(s)\cos^2\varphi N(s)\\
&\quad \times \gamma B_2(s) + H_2\cos^2\varphi B_2(s) \times \gamma T(s) + H_1\cos^2\varphi B_2(s)
\end{aligned}$$

$$\times \gamma N(s) + \cos^2 \varphi B_2(s) \times \gamma B_2(s)$$

where by using the properties of quaternion product we can easily write that

$$\|X\|^2 = 1 = (1 + H_1^2(s) + H_2^2(s)) \cos^2 \varphi$$

and then we get

$$H_1^2(s) + H_2^2(s) = \tan^2 \varphi = \text{constant}. \qquad \square$$

Corollary 3.14. *Let $\beta : I \subset \mathbb{R} \longrightarrow Q_H$ be an arc-lengthen quaternionic B_2-slant helix with nonzero curvatures $K, k, r - K$. Let $\{H_1, H_2\}$ be the harmonic curvature functions. By using Theorem* (3.10) *and Definition* (3.5) *the derivatives of harmonic curvatures are as follows:*

$$\begin{bmatrix} H_1' \\ H_2' \end{bmatrix} = \begin{bmatrix} 0 & K \\ -K & 0 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

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