Received: 28 December 2020 | Revised: 31 March 2021 | Accepted: 5 June 2021

DOI: 10.1002/mma.7633

SPECIAL ISSUE PAPER

WILEY

On the evolution of framed curves in Euclidean 3-space

Önder Gökmen Yıldız

Department of Mathematics, Bilecik Şeyh Edebali University, Bilecik, Turkey

Correspondence
Önder Gökmen Yıldız, Department of Mathematics, Bilecik Şeyh Edebali University, Bilecik, Turkey.
Email: ogokmen.yildiz@bilecik.edu.tr

Communicated by: F. Usta

In this paper, we study the evolution of framed curves, which may have singular points. We express flows of framed curves in Euclidean 3-space. We obtain necessary and sufficient conditions as partial differential equations involving the framed curvatures for the inextensible flow of framed curve. Also, we give examples for the evolution equation of framed curvatures.

KEYWORDS

evolution, framed curve, inextensible flow, singular points

MSC CLASSIFICATION

53A04; 58K05

1 | INTRODUCTION

Curve theory is a classical object in differential geometry. There are numerous studies in the literature on Frenet curves that are useful to investigate geometric properties of the regular curves. However, recently, curves with singular points, whose Frenet frame cannot be constructed, are frequently appeared in applications. Honda and Takahashi established the notion of framed curves to investigate the geometric properties of curves with singular points.[1] Framed curves are a natural generalization of Frenet curves. After that, Honda and Takahashi introduced framed immersion.[2] For more details on the notion of framed curves and framed surface, see several studies.[3–8]

The time evolution of geometric locus is investigated by using its flow. So, flow is used to examining the evolution of geometric locus. There have been various studies on flows of curves. If arclength of curve is preserved, its flow is said to be inextensible. Physically, inextensible flows of curves and surfaces can be described by motions in which no strain energy is induced. For example, the swinging motion of a cord of fixed length or of a piece of paper carried by a storm can be described by inextensible flows of curves and surfaces. There are many physical applications concerning flows.[9–12] The time evolution of a curve or surface has been studied by mathematicians,[13–15] but firstly, Kwon and Park introduced inextensible flows of plane curves.[16] Then, Kwon and Park introduced inextensible flows of curves and developable surfaces.[17] After that, inextensible flows were studied in many different spaces.[18–22]

In this paper, we define the flow of curve with singular points, which is a generalization of Kwon and Park.[16] We give necessary and sufficient conditions for the inextensible flow of framed curve. Also, we obtain the evolution equation of framed curvatures. Finally, we give examples of the evolution equation of framed curvatures.

2 | PRELIMINARIES

In this section, we introduce a brief summary of the notion of framed curves. Let $\mathbb{R}^3$ be the three-dimensional Euclidean space with inner product given by

$$< x, y >= \sum_{i=1}^{3} x_i y_i,$$

where $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)\in\mathbb{R}^3$, and norm of $x\in\mathbb{R}^3$ is given by $\|x\|=\sqrt{<x,x>}$.

Let $V_{3,2}$ be a three-dimensional smooth manifold as follows:

$$V_{3,2}=\left\{\eta=(\eta_1,\eta_2)\in\mathbb{R}^3\times\mathbb{R}^3 \,\middle|\, <\eta_i,\eta_j>=\delta_{ij},i,j=1,2\right\}.$$

By using $\eta=(\eta_1,\eta_2)\in V_{3,2}$, we can define a unit vector $\nu=\eta_1\times\eta_2$ such that $\det(t,\eta_1,\eta_2)=1$.

Definition 1. $(\gamma,\eta):I\to\mathbb{R}^3\times V_{3,2}$ is said to be a framed curve if $<\gamma'(s),\eta_i(s)>=0$ for all $s\in I$ and $i=1,2$. If there exists $\eta:I\to V_{3,2}$ such that (γ,η) is a framed curve, $\gamma:I\to\mathbb{R}^3$ is called as a framed base curve.[1]

Let (γ,η) be a framed curve. There exists $\nu(s)=\eta_1(s)\times\eta_2(s)$. Then we can write the Frenet–Serret-type formula as follows:

$$\begin{aligned}\eta_1'(s)&=l(s)\eta_2(s)+m(s)\nu(s),\\ \eta_2'(s)&=-l(s)\eta_1(s)+n(s)\nu(s),\\ \nu'(s)&=-m(s)\eta_1(s)-n(s)\eta_2(s),\end{aligned}$$

where $l(s)=<\eta_1'(s),\eta_2(s)>$, $m(s)=<\eta_1'(s),\nu(s)>$ and $n(s)=<\eta_2'(s),\nu(s)>$.

There exists a smooth mapping $\alpha:I\to\mathbb{R}$ such that $\gamma'(s)=\alpha(s)\nu(s)$. $(l(s),m(s),n(s),\alpha(s))$ are called curvatures of γ at $\gamma(s)$. Clearly, s_0 is a singular point of γ if and only if $\alpha(s_0)=0$. $(l(s),m(s),n(s),\alpha(s))$ are useful to investigate the framed curve and its singularities.

Theorem 1. *Let $(l(s),m(s),n(s),\alpha(s)):I\to\mathbb{R}^3$ be a smooth mapping. There exists a framed curve $(\gamma,\eta):I\to\mathbb{R}^3\times V_{3,2}$ whose associated curvature of the framed curve is $(l(s),m(s),n(s),\alpha(s))$.*[1]

Theorem 2. *Given framed curves (γ,η) and $(\widetilde{\gamma},\tilde{\eta})$, they are congruent as framed curves, if their curvatures coincide.*[1]

Let $(\gamma,\eta):I\to\mathbb{R}^3\times V_{3,2}$ be a framed curve with $(l(s),m(s),n(s),\alpha(s))$. $\widetilde{\eta}=(\tilde{\eta}_1,\tilde{\eta}_2)$ is defined by

$$\begin{bmatrix}\tilde{\eta}_1'\\ \tilde{\eta}_2'\end{bmatrix}=\begin{bmatrix}\cos\varphi(s) & -\sin\varphi(s)\\ \sin\varphi(s) & \cos\varphi(s)\end{bmatrix}\begin{bmatrix}\tilde{\eta}_1\\ \tilde{\eta}_2\end{bmatrix},$$

where $\varphi(s)$ is a smooth function.

If

$$\begin{aligned}m(s)\sin\varphi(s)&=-n(s)\cos\varphi(s),\\ m(s)&=-p(s)\cos\varphi(s)\end{aligned}$$

and

$$n(s)=p(s)\sin\varphi(s)$$

are satisfied for given smooth functions $\varphi(s)$ and $p(s)$, the Frenet–Serret formula of $(\gamma,\tilde{\eta}_1,\tilde{\eta}_2)$ is given by

$$\begin{bmatrix}\nu'(s)\\ \tilde{\eta}_1'(s)\\ \tilde{\eta}_2'(s)\end{bmatrix}=\begin{bmatrix}0 & p(s) & 0\\ -p(s) & 0 & q(s)\\ 0 & -q(s) & 0\end{bmatrix}\begin{bmatrix}\nu(s)\\ \tilde{\eta}_1(s)\\ \tilde{\eta}_2(s)\end{bmatrix},$$

where $p(s)=\|\nu'(s)\|$ and $q(s)=l(s)-\varphi'(s)$. $(p(s),q(s),\alpha(s))$ are called framed curvatures of $(\gamma,\tilde{\eta})$ and the following equations

$$\kappa(s)=\frac{p(s)}{|\alpha(s)|},\quad \tau(s)=\frac{q(s)}{\alpha(s)}$$

hold.[1]

3 | EVOLUTION OF FRAMED CURVES BY FLOW

Let $(\gamma,\tilde{\eta}):I\times J\to\mathbb{R}^3\times V_{3,2}$ be a one-parameter family of framed curve. The speed of γ and arclength variation function of γ are given, respectively

$$\sigma=\left\|\frac{\partial\gamma}{\partial u}\right\|$$

and

$$s(u,t) = \int_0^u \left\| \frac{\partial \gamma}{\partial u} \right\| du = \int_0^u \sigma(u,t) du. \tag{1}$$

Then $\frac{\partial \gamma}{\partial s} = \frac{1}{\sigma} \frac{\partial}{\partial u}$ and $ds = \sigma du$.

Definition 2. Given a framed curve γ, any flow of framed curve can be represented as

$$\frac{\partial \gamma}{\partial t} = f\nu + g\tilde{\eta}_1 + h\tilde{\eta}_2,$$

where f, g, and h are smooth and called scalar speeds of γ.

Example 3. Let the framed curve $(\gamma, \tilde{\eta}_1, \tilde{\eta}_2) : I \to \mathbb{R}^3 \times V_{3,2}$ be

$$\gamma(u) = \left(\frac{u^2}{2}, \frac{u^3}{3}, \frac{u^4}{4} \right),$$
$$\tilde{\eta}_1(u) = \frac{1}{\sqrt{1+u^2}}(-u, 1, 0)$$

and

$$\tilde{\eta}_2(u) = \frac{1}{\sqrt{(1+u^2)(1+u^2+u^4)}} \left(-u^2, -u^3, 1+u^2 \right),$$

then

$$\nu(u) = \frac{1}{\sqrt{1+u^2+u^4}} \left(1, u, u^2 \right).$$

For given $f = u^2$, $g = u^2$, and $h = u^2$, the graph of flow of initial framed curve is in Figure 1.
For given $f = \cos^2 u$, $g = \sin^2 u$, and $h = \cos u \sin u$, the graph of flow of initial framed curve is in Figure 2.
For given $f = \cos u$, $g = \sin u$, and $h = \cos u$, the graph of flow of initial framed curve is in Figure 3.

The requirement that a framed curve not to be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial \sigma(u,t)}{\partial t} du = 0, u \in I.$$

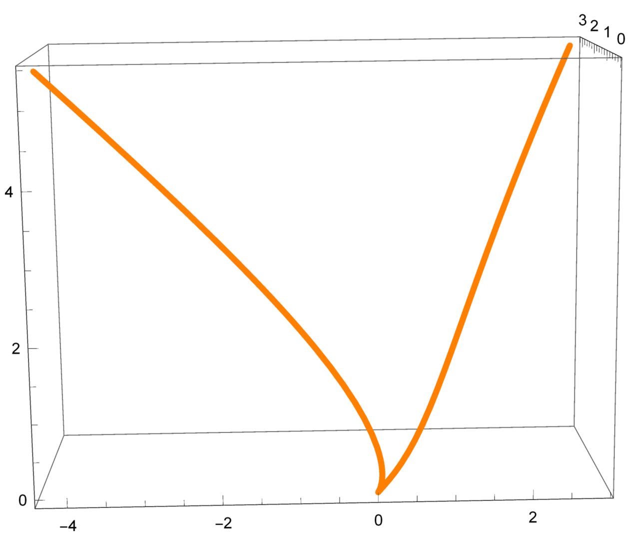


FIGURE 1 The flow of initial framed curve for $-2 \le t \le 2$ [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 2 The flow of initial framed curve for $-2 \le t \le 2$ [Colour figure can be viewed at wileyonlinelibrary.com]

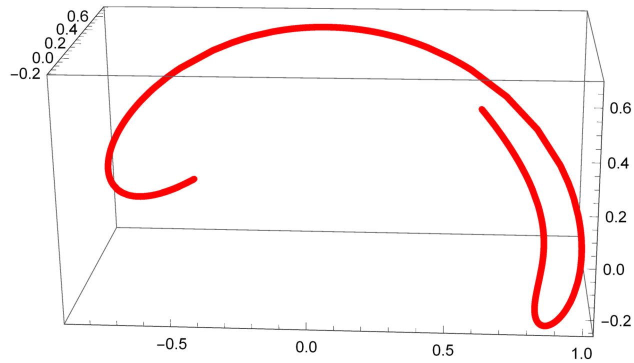
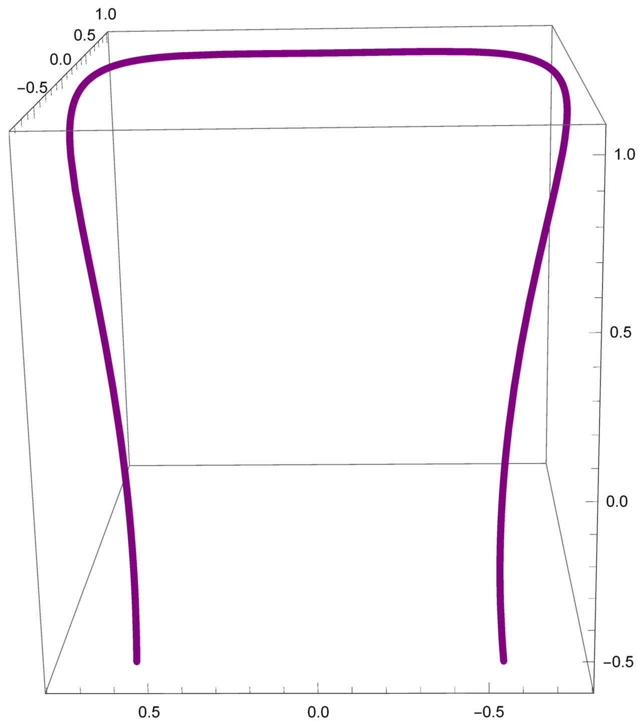


FIGURE 3 The flow of initial framed curve for $-2 \le t \le 2$ [Colour figure can be viewed at wileyonlinelibrary.com]



Definition 3. Given a framed curve γ, its flow is said to be inextensible if

$$\frac{\partial}{\partial t}\left\|\frac{\partial \gamma}{\partial u}\right\| = 0.$$

Lemma 1.

$$\frac{\partial \sigma}{\partial t} = \sigma\alpha\left(\frac{\partial f}{\partial s} - gp\right). \tag{2}$$

Proof. Since $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ are commutative and $\sigma^2 = \left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u}\right\rangle$, we get

$$\begin{aligned} 2\sigma\frac{\partial \sigma}{\partial t} &= \frac{\partial}{\partial t}\left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u}\right\rangle \\ &= 2\left\langle \frac{\partial \gamma}{\partial u}, \frac{\partial}{\partial u}(f\nu + g\tilde{\eta}_1 + h\tilde{\eta}_2)\right\rangle \\ &= 2\sigma^2\alpha\left\langle \nu, \left(\frac{\partial f}{\partial s} - gp\right)\nu + \left(fp + \frac{\partial g}{\partial s} - hq\right)\tilde{\eta}_1 + \left(gq + \frac{\partial h}{\partial s}\right)\tilde{\eta}_2\right\rangle. \end{aligned}$$

Thus, we obtain

$$\frac{\partial \sigma}{\partial t} = \sigma \alpha \left(\frac{\partial f}{\partial s} - gp \right).$$

□

Theorem 4. *The flow of framed curve γ is inextensible if and only if*

$$\frac{\partial f}{\partial s} = gp.$$

Proof. Assume that the flow of framed curve is inextensible. By using Equations (1) and (2), we obtain

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial \sigma}{\partial t} du = 0.$$

This clearly forces

$$\frac{\partial f}{\partial s} = gp.$$

Conversely, by following a similar way as above, the proof is complete. □

Lemma 2.

$$\begin{aligned}
\frac{\partial \nu}{\partial t} &= \frac{1}{\alpha}\left(fp + \frac{\partial g}{\partial s} - hq \right) \tilde{\eta}_1 + \frac{1}{\alpha}\left(gq + \frac{\partial h}{\partial s} \right) \tilde{\eta}_2, \\
\frac{\partial \tilde{\eta}_1}{\partial t} &= -\frac{1}{\alpha}\left(fp + \frac{\partial g}{\partial s} - hq \right) \nu + \Psi \tilde{\eta}_2, \\
\frac{\partial \tilde{\eta}_2}{\partial t} &= -\frac{1}{\alpha}\left(gq + \frac{\partial h}{\partial s} \right) \nu - \Psi \tilde{\eta}_1,
\end{aligned}$$

where $\Psi = \left\langle \frac{\partial \tilde{\eta}_1}{\partial t}, \tilde{\eta}_2 \right\rangle$ *dır.*

Proof. By considering $\frac{\partial}{\partial t}\left(\frac{\partial \gamma}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial \gamma}{\partial t}\right)$, we get

$$\frac{\partial \nu}{\partial t} = \frac{1}{\alpha}\left(fp + \frac{\partial g}{\partial s} - hq \right) \tilde{\eta}_1 + \frac{1}{\alpha}\left(gq + \frac{\partial h}{\partial s} \right) \tilde{\eta}_2$$

and $\frac{\partial \alpha}{\partial t} = 0$. Now, differentiating of $\langle \nu, \tilde{\eta}_1 \rangle$ and $\langle \nu, \tilde{\eta}_2 \rangle$ with respect to t, we obtain

$$\frac{\partial \tilde{\eta}_1}{\partial t} = -\frac{1}{\alpha}\left(fp + \frac{\partial g}{\partial s} - hq \right) \nu + \Psi \tilde{\eta}_2$$

and

$$\frac{\partial \tilde{\eta}_2}{\partial t} = -\frac{1}{\alpha}\left(gq + \frac{\partial h}{\partial s} \right) \nu - \Psi \tilde{\eta}_1.$$

□

From proof of Lemma 2, we have the following corollary:

Corollary 1. *Curvature α is preserved; that is, $\frac{\partial \alpha}{\partial t}$ is always zero.*

Theorem 5. *Assume that the flow of framed curve γ is inextensible. Then, the evolutions of the curvatures are*

$$\begin{aligned}\frac{\partial p}{\partial t} &= -\frac{\alpha'}{\alpha^2}\left(fp+\frac{\partial g}{\partial s}-hq\right)\\ &\quad+\frac{1}{\alpha}\left(\frac{\partial f}{\partial s}p+f\frac{\partial p}{\partial s}+\frac{\partial^2 g}{\partial s^2}-\frac{2\partial h}{\partial s}q-h\frac{\partial q}{\partial s}-gq^2\right),\\ \frac{\partial q}{\partial t} &= \frac{\partial \Psi}{\partial s}+\frac{1}{\alpha}\left(gpq+\frac{\partial h}{\partial s}p\right),\\ \Psi p &= \frac{1}{\alpha}\left(fpq+2\frac{\partial g}{\partial s}q-hq^2+g\frac{\partial q}{\partial s}+\frac{\partial^2 h}{\partial s^2}\right)-\frac{\alpha'}{\alpha^2}\left(gq+\frac{\partial h}{\partial s}\right).\end{aligned}$$

Proof. Since $\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial s}\right)=\frac{\partial}{\partial s}\left(\frac{\partial v}{\partial t}\right)$, we have

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial v}{\partial t}\right) &= \frac{\partial}{\partial t}\left(\frac{1}{\alpha}\left(fp+\frac{\partial g}{\partial s}-hq\right)\tilde{\eta}_1+\frac{1}{\alpha}\left(gq+\frac{\partial h}{\partial s}\right)\tilde{\eta}_2\right)\\ &= \frac{1}{\alpha}\left(-fp^2-\frac{\partial g}{\partial s}p+hpq\right)v\\ &\quad+\left(-\frac{\alpha'}{\alpha^2}\left(fp+\frac{\partial g}{\partial s}-hq\right)+\frac{1}{\alpha}\left(\frac{\partial f}{\partial s}p+f\frac{\partial p}{\partial s}+\frac{\partial^2 g}{\partial s^2}-2\frac{\partial h}{\partial s}q-h\frac{\partial q}{\partial s}-gq^2\right)\right)\tilde{\eta}_1\\ &\quad+\left(\frac{1}{\alpha}\left(fpq+2\frac{\partial g}{\partial s}q+g\frac{\partial q}{\partial s}-hq^2+\frac{\partial^2 h}{\partial s^2}\right)-\frac{\alpha'}{\alpha^2}\left(gq+\frac{\partial h}{\partial s}\right)\right)\tilde{\eta}_2\end{aligned}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial s}\right)=\frac{\partial}{\partial t}(p\tilde{\eta}_1)=-\frac{1}{\alpha}\left(fp^2+\frac{\partial g}{\partial s}p-hpq\right)v+\frac{\partial p}{\partial t}\tilde{\eta}_1+\Psi p\tilde{\eta}_2.$$

Then, we obtain

$$\frac{\partial p}{\partial t}=-\frac{\alpha'}{\alpha^2}\left(fp+\frac{\partial g}{\partial s}-hq\right)+\frac{1}{\alpha}\left(\frac{\partial f}{\partial s}p+f\frac{\partial p}{\partial s}+\frac{\partial^2 g}{\partial s^2}-\frac{2\partial h}{\partial s}q-h\frac{\partial q}{\partial s}-gq^2\right)$$

and

$$\Psi p=\frac{1}{\alpha}\left(fpq+2\frac{\partial g}{\partial s}q-hq^2+g\frac{\partial q}{\partial s}+\frac{\partial^2 h}{\partial s^2}\right)-\frac{\alpha'}{\alpha^2}\left(gq+\frac{\partial h}{\partial s}\right).$$

Noting that $\frac{\partial}{\partial t}\left(\frac{\partial \tilde{\eta}_2}{\partial s}\right)=\frac{\partial}{\partial s}\left(\frac{\partial \tilde{\eta}_2}{\partial t}\right)$, we have

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial \tilde{\eta}_2}{\partial t}\right) &= \frac{\partial}{\partial s}\left(-\frac{1}{\alpha}\left(gq+\frac{\partial h}{\partial s}\right)v-\Psi\widetilde{\eta}_1\right)\\ &= \left(\frac{\alpha'}{\alpha^2}\left(gq+\frac{\partial h}{\partial s}\right)-\frac{1}{\alpha}\left(\frac{\partial g}{\partial s}q+g\frac{\partial q}{\partial s}+\frac{\partial^2 h}{\partial s^2}\right)+\Psi p\right)v\\ &\quad+\left(-\frac{\partial \Psi}{\partial s}-\frac{1}{\alpha}\left(gpq+\frac{\partial h}{\partial s}p\right)\right)\tilde{\eta}_1-(\Psi p)\,\widetilde{\eta}_2\end{aligned}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial \tilde{\eta}_2}{\partial s}\right) = \frac{\partial}{\partial t}\left(-q\tilde{\eta}_2\right)$$
$$= \frac{1}{\alpha}\left(fpq + \frac{\partial g}{\partial s}q - hq^2\right)\nu - \frac{\partial q}{\partial s}\tilde{\eta}_1 - (\Psi q)\,\tilde{\eta}_2.$$

Thus, we obtain

$$\frac{\partial q}{\partial t} = \frac{\partial \Psi}{\partial s} + \frac{1}{\alpha}\left(gpq + \frac{\partial h}{\partial s}p\right).$$

□

Example 6. Consider the framed curve

$$\gamma(u) = \left(\frac{2}{3}\sec\left(-2\sqrt{2}\cos\frac{u}{2}\right)\cos u - \frac{1}{3}\sec\left(-2\sqrt{2}\cos\frac{u}{2}\right)\cos 2u,\right.$$
$$\times \frac{2}{3}\sec\left(-2\sqrt{2}\cos\frac{u}{2}\right)\sin u - \frac{1}{3}\sec\left(-2\sqrt{2}\cos\frac{u}{2}\right)\sin 2u,$$
$$\left.\times \frac{2\sqrt{2}}{3}\sec\left(-2\sqrt{2}\cos\frac{u}{2}\right)\sin\frac{u}{2}\right)$$

with curvatures

$$\alpha(u) = \sqrt{2}\sin\frac{u}{2}\sec^2\left(-2\sqrt{2}\cos\frac{u}{2}\right),$$
$$p(u) = \sqrt{2}\cos\frac{u}{2}\cos\left(-2\sqrt{2}\cos\frac{u}{2}\right),$$

and

$$q(u) = \sqrt{2}\cos\frac{u}{2}\sin\left(-2\sqrt{2}\cos\frac{u}{2}\right).$$

For given $f = \cos\frac{u}{2}$, $g = \sin\frac{u}{2}$, and $h = \cos\frac{u}{2}$, by using Theorem 5, the graphs of evolution equation of $p(u,t)$ and $q(u,t)$ are in Figures 4 and 5.

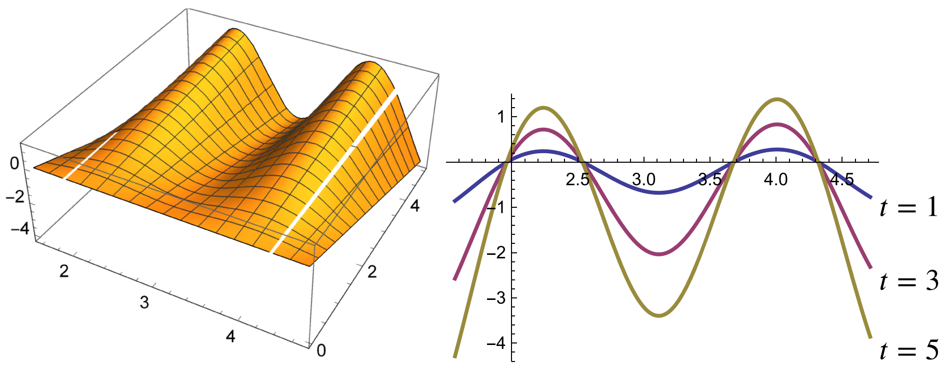


FIGURE 4 The evolution of $p(u,t)$ for $\frac{\pi}{2} \leq u \leq \frac{3\pi}{2}, 0 \leq t \leq 5$ [Colour figure can be viewed at wileyonlinelibrary.com]

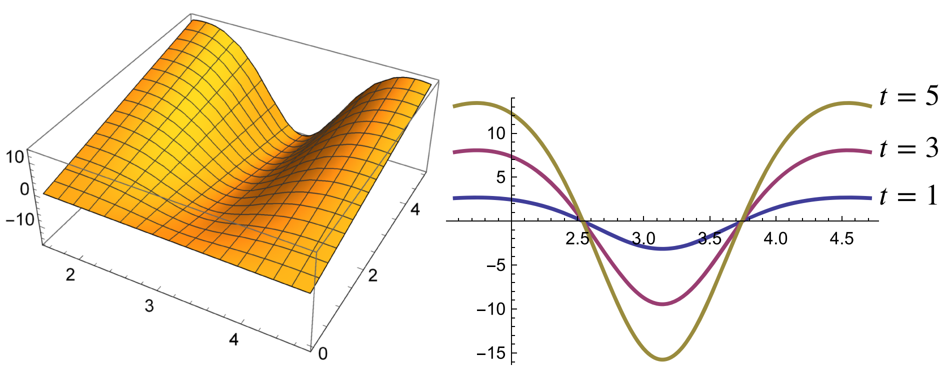


FIGURE 5 The evolution of $q(u,t)$ for $\frac{\pi}{2} \leq u \leq \frac{3\pi}{2}, 0 \leq t \leq 5$ [Colour figure can be viewed at wileyonlinelibrary.com]

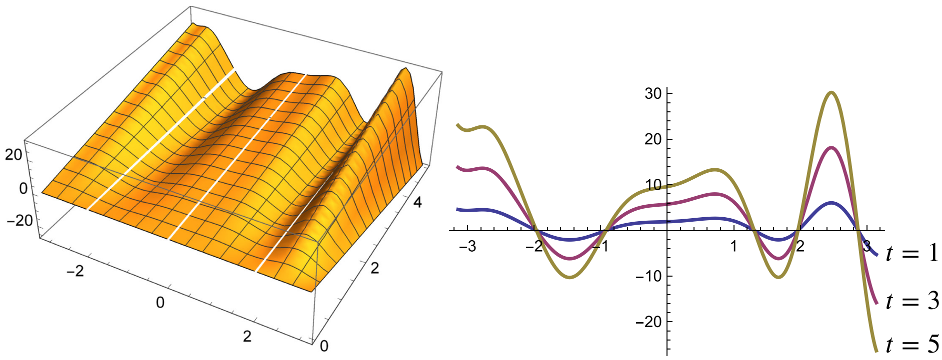


FIGURE 6 The evolution of $p(u, t)$ for $-\pi \leq u \leq -\pi$, $0 \leq t \leq 5$ [Colour figure can be viewed at wileyonlinelibrary.com]

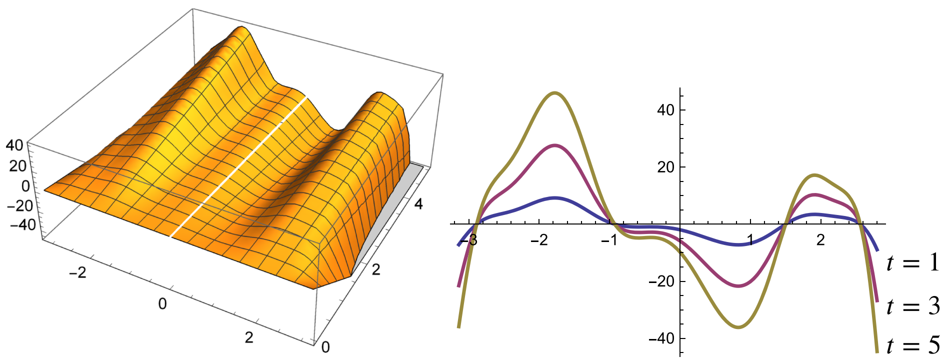


FIGURE 7 The evolution of $q(u, t)$ for $-\pi \leq u \leq -\pi$, $0 \leq t \leq 5$ [Colour figure can be viewed at wileyonlinelibrary.com]

For given $f = u^2 \cos \frac{u}{2}$, $g = u^2 \sin \frac{u}{2}$, and $h = u^2 \cos \frac{u}{2} \sin \frac{u}{2}$, by using Theorem 5, the graphs of evolution equation of $p(u, t)$ and $q(u, t)$ are in Figures 6 and 7.

ACKNOWLEDGEMENTS

The author thanks the editors and referees for their valuable comments and contributions.

CONFLICT OF INTEREST

The author declares no potential conflict of interest.

FUNDING INFORMATION

None reported.

ORCID

Önder Gökmen Yıldız https://orcid.org/0000-0002-2760-1223

REFERENCES

1. Honda S, Takahashi M. Framed curves in the Euclidean space. *Adv Geom*. 2016;16(3):265-276.
2. Honda S, Takahashi M. Evolutes and focal surfaces of framed immersions in the Euclidean space. *Proc R Soc Edinburgh Sect A: Math*. 2019;150:497-516.
3. Honda S. Rectifying developable surfaces of framed base curves and framed helices. Singularities in generic geometry. *Math Soc Japan*. 2018;78:273-292.
4. Fukunaga T, Takahashi M. Existence and uniqueness for Legendre curves. *J Geom*. 2013;104:297-307.
5. Fukunaga T, Takahashi M. Framed surfaces in the Euclidean space. *Bull Braz Math Soc*. 2019;50(1):37-65.
6. Fukunaga T, Takahashi M. Framed surfaces and one-parameter families of framed curves in Euclidean 3-space. *J Singularities*. 2020;21:30-49.
7. Wang Y, Pei D, Gao R. Generic properties of framed rectifying curves. *Mathematics*. 2019;7:37.

8. Yıldız ÖG, Akyigit M, Tosun M. On the trajectory ruled surfaces of framed base curves in the Euclidean space. *Math Meth Appl Sci*. 2020;44:7463-7470.
9. Chirikjian G, Burdick J. A modal approach to hyper-redundant manipulator kinematics. *EEE Trans Robot Autom*. 1994;10:343-354.
10. Kass M, Witkin A, Terzopoulos D. Snakes: active contour models. *Int J Comput Vision*. 2009;1(4):321-331.
11. Desbrun M, Cani-Gascuel MP. Active implicit surface for animation. In GI—Graphics Interface; June 18–20, Vancouver, British Columbia, Canada; 1998.
12. Lu HQ, Todhunter JS, Sze TW. Congruence conditions for nonplanar developable surfaces and their application to surface recognition. *CVGIP Image Underst*. 1993;58(3):265-285.
13. Gage M, Hamilton RS. The heat equation shrinking convex plane curves. *J Differential Geom*. 1986;23:69-96.
14. Gage M. On an area-preserving evolution equation for plane curves. *Contemp Math*. 1986;51:51-62.
15. Grayson M. The heat equation shrinks embedded plane curves to round points. *J Differential Geom*. 1987;26:285-314.
16. Kwon DY, Park FC. Evolution of inelastic plane curves. *Appl Math Lett*. 1999;12(6):115-119.
17. Kwon DY, Park FC, Chi DP. Inextensible flows of curves and developable surfaces. *Appl Math Lett*. 2005;18(10):1156-1162.
18. Gürbüz N. Inextensible flows of spacelike, timelike and null curves. *Int J Contemp Math Sci*. 2009;4(32):1599-1604.
19. Uçum A, Erdem HA, İlarslan K. Inextensible flows of partially null and pseudo null curves in semi-euclidean 4-space with index 2. *Novi Sad J Math*. 2016;46(1):115-129.
20. Yıldız ÖG, Ersoy S, Masal M. A note on inextensible flows of curves on oriented surface. *CUBO A Math J*. 2014;16(3):11-19.
21. Yıldız ÖG, Tosun M, Karakuş S. A note on inextensible flows of curves in E^n. *Int Electron J Geom*. 2013;6(2):118-124.
22. Kızılay A, Yıldız ÖG, Okuyucu OZ. Evolution of quaternionic curve in the semi-Euclidean space E_2^4. *Math Meth Appl Sci*. 2020:1-10.

How to cite this article: Yıldız ÖG. On the evolution of framed curves in Euclidean 3-space. *Math Meth Appl Sci*. 2022;45(18):12158-12166. https://doi.org/10.1002/mma.7633