Fuzzy Soft Element and Its Application to Decision-making

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Abstract. This paper proposes the novel concept of fuzzy soft elements, thereby extending the traditional notion of fuzzy soft points through the more flexible assignment of alternatives to attributes. Consequently, this concept offers a new vantage point for understanding fuzzy soft set operations. Through this, after some of the properties of fuzzy soft set operations are given, a decision-making example is presented that demonstrates the practical application.

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1. Introduction

Both fuzzy sets [19] and soft sets [11], developed to handle uncertainty, possess their own unique advantages. Soft sets provide a flexible model for uncertainty using parameters, without the need for a membership function. Fuzzy soft sets [8], on the other hand, combine these two structures; they bring together the concept of graded membership from fuzzy sets with the flexible parameters of soft sets to create a richer framework for modelling complex and uncertain systems.

There are various approaches to performing set operations on fuzzy soft sets, inspired by methods from fuzzy set and soft set theories. Additionally, different interpretations exist regarding the members of a fuzzy soft set [1, 8, 10, 18]. These interpretations are typically expressed through the membership degrees of an element (or alternative) in universe of discourse with respect to a specific parameter (or attribute), and both theoretical and practical applications are developed accordingly. This study proposes a novel approach to the definition of fuzzy soft element, based on the notion of soft element introduced by Das and Samanta [4]. For further studies related to the soft element, see [5–7, 15–17]. Essentially, the proposed model provides a way to determine the sensitivity of alternatives for each descriptive attribute. Therefore, it facilitates achieving more accurate results in decision-making problems within this framework by enabling different evaluation strategies. Topology, metric structures and various applications can be developed using the basic method discussed in this study. Using the basic method discussed in this paper, studies on topology, metric structures and some applications can be developed in a manner similar to those in [2, 3, 12].

In this paper, the necessary basic definitions and notations are first provided, after which the notion of fuzzy soft element is introduced and illustrated with an example. Then, the set operations in fuzzy soft sets are redefined based on this notion, and some of their properties are shown by revealing its relations with previous definitions. Finally, to demonstrate its practical applicability, a decision-making method based on the fuzzy soft element is proposed. Here, a decision is modelled not as a single element of the universe of discourse but as a fuzzy soft element, which provides an alternative for each descriptive attribute.

1.1. Preliminaries.

In this section, the following definitions and notations are based on [13, 20] for fuzzy sets, [4, 11, 15] for soft sets and [1, 8, 14] for fuzzy soft sets.

Definition 1.1. *A fuzzy set $A = \{u^{\mu_A(u)} : u \in U\}$ on a universe of discourse U is defined by a membership function $\mu_A : U \to [0, 1]$, with the membership value $\mu_A(u)$ indicating the degree to which $u \in U$ belongs to the fuzzy set A. The set of all fuzzy sets on U is denoted by $F(U)$.*

A fuzzy point x^t in U is a fuzzy set with membership function

$$\mu_{x^t}(u) = \begin{cases} t \in (0,1] & \text{when } u = x, \\ 0 & \text{otherwise.} \end{cases}$$

A fuzzy point x^t is said to be member of a fuzzy set A, denoted by $x^t \in A$, iff $\mu_{x^t}(u) \leq \mu_A(u)$ for each $u \in U$. Also, the set of all fuzzy points of a fuzzy set A is represeted by $FP(A)$.

Definition 1.2. *Let $A, B \in F(U)$ be two fuzzy sets. Then,*

- *$A = \emptyset$ and $A = U$ iff $\mu_A = 0$ and $\mu_A = 1$ for each $u \in U$, respectively.*
- *$A \subseteq B$ iff $\mu_A(u) \leq \mu_B(u)$ for each $u \in U$.*
- *$A = B$ iff $\mu_A(u) = \mu_B(u)$ for each $u \in U$.*
- *$C = A \vee B$ iff $\mu_C(u) = \mu_A(u) \vee \mu_B(u)$, i.e. $\mu_C(u) = \max\{\mu_A(u), \mu_B(u)\}$ for each $u \in U$.*
- *$D = A \wedge B$ iff $\mu_D(u) = \mu_A(u) \wedge \mu_B(u)$, i.e. $\mu_D(u) = \min\{\mu_A(u), \mu_B(u)\}$ for each $u \in U$.*
- *$E = A^C$ iff $\mu_E(u) = 1 - \mu_A(u)$ for each $u \in U$.*

Definition 1.3. *A pair (G, P) is called a soft set on U with a parameters set P, where $G : P \to P(U)$ is a mapping with $P(U)$ being the power set of U. The set of all soft set on U with a parameters set P is denoted by $S_P(U)$ or simply $S(U)$.*

A function $\varepsilon : P \to U$ is called a soft element of U and ε is said to be member of (G, P) if $\varepsilon(\alpha) \in G(\alpha)$ for each $\alpha \in P$. The class of soft elements of (G, P) is denoted by $SE(G, P)$. Also, the soft elements are denoted by $\tilde{x}$ and the soft elements such that $\tilde{x} = c$ for all $\alpha \in P$ and for a constant $c \in U$ are denoted by $\bar{c}$.

Definition 1.4. *A pair (g, P) is called a fuzzy soft set on U with a parameters set P, where $g : P \to F(U)$. The set of all fuzzy soft sets on U with a parameters set P is denoted by $FS_P(U)$ or simply $FS(U)$.*

A special type of fuzzy soft set $(g, P) \in FS(U)$ is called fuzzy soft point, denoted by P_x^λ, such that for a fixed $x \in U$ and $\lambda \in (0, 1]$, $\mu_{g(\alpha)}(u) = \lambda$ if $u = x$, and $\mu_{g(\alpha)}(u) = 0$ if $u \neq x$ for $\alpha \in P$. Also, for a fuzzy soft set $(g, P) \in FS(U)$, a fuzzy soft point P_x^λ is a member of (g, P) if $\lambda \leq \mu_{g(\alpha)}(u)$ for each $\alpha \in P$.

Definition 1.5. *The fuzzy soft set (g, P) is said to be a null fuzzy soft set if $g(\alpha) = \emptyset$ and an absolute fuzzy soft set if $g(\alpha) = U$ for each $\alpha \in P$, denoted by $\tilde{\mathbf{\Phi}}$ and $\tilde{\mathbf{U}}$, respectively.*

Definition 1.6. *The union and intersection of $(g, P), (h, P) \in FS(U)$ are fuzzy soft sets defined as*

$$(g, P)\tilde{\vee}(h, P) = \{(\alpha, g(\alpha) \vee h(\alpha)) : \forall \alpha \in P,\ g(\alpha), h(\alpha) \in F(U)\} \tag{1}$$

and

$$(g, P)\tilde{\wedge}(h, P) = \{(\alpha, g(\alpha) \wedge h(\alpha)) : \forall \alpha \in P,\ g(\alpha), h(\alpha) \in F(U)\}, \tag{2}$$

respectively. The complement of (g, P) is denoted by $(g, P)^C$, where $(g, P)^C(\alpha) = g(\alpha)^C$ for each $\alpha \in P$.

Example 1.7. *Let $P = \{\alpha, \beta, \gamma\}$ and $U = \{u, v, w\}$. The fuzzy soft sets*

$$(g_1, P) = \{(\alpha, \{u^{0.1}, v^{0.7}, w^{0.5}\}), (\beta, \{u^{0.8}, v^{0.2}, w^{0.6}\}), (\gamma, \{u^{0.9}, v^{0.8}, w^{0.1}\})\},$$

$$(g_2, P) = \{(\alpha, \{u^{0.5}, w^{0.5}\}), (\beta, \{u^{0.1}, v^{0.2}, w^{0.3}\}), (\gamma, \{v^{0.5}\})\},$$

$$(g_1, P)\tilde{\vee}(g_2, P) = \{(\alpha, \{u^{0.5}, v^{0.7}, w^{0.5}\}), (\beta, \{u^{0.8}, v^{0.2}, w^{0.6}\}), (\gamma, \{u^{0.9}, v^{0.8}, w^{0.1}\})\},$$

$$(g_1, P)\tilde{\wedge}(g_2, P) = \{(\alpha, \{u^{0.1}, w^{0.5}\}), (\beta, \{u^{0.1}, v^{0.2}, w^{0.3}\}), (\gamma, \{v^{0.5}\})\},$$

$$(g_1, P)^C = \{(\alpha, \{u^{0.9}, v^{0.3}, w^{0.5}\}), (\beta, \{u^{0.2}, v^{0.8}, w^{0.4}\}), (\gamma, \{u^{0.1} v^{0.2}, w^{0.9}\})\}$$

are represented in tabular form as in Table 1 and 2.

(g_1, P)	α	β	γ
u	0.1	0.8	0.9
v	0.7	0.2	0.8
w	0.5	0.6	0.1

(g_2, P)	α	β	γ
u	0.5	0.1	0
v	0	0.2	0.5
w	0.5	0.3	0

Table 1. The tabular representation of the fuzzy soft sets (g_1, P) and (g_2, P)

$(g_1,P)\tilde{\vee}(g_2,P)$	α	β	γ	$(g_1,P)\tilde{\wedge}(g_2,P)$	α	β	γ	$(g_1,P)^C$	α	β	γ
u	0.5	0.8	0.9	u	0.1	0.1	0	u	0.9	0.2	0.1
v	0.7	0.2	0.8	v	0	0.2	0.5	v	0.3	0.8	0.2
w	0.5	0.6	0.1	w	0.5	0.3	0	w	0.5	0.4	0.9

Table 2. The tabular representation of the union, intersection and complement of fuzzy soft sets (g_1, P) and (g_2, P)

It will be used solely g for a fuzzy soft set hereafter at its place (g, P) for simplicity.

2. Main results

Definition 2.1. *Let U be a universe and P be a parameters set. A fuzzy soft element $\tilde{\mathbf{x}}$ is a fuzzy soft set for which $\tilde{\mathbf{x}} : P \to FP(U)$ and a fuzzy soft element $\tilde{\mathbf{x}}$ is said to be a member of a fuzzy soft set $g \in FS(U)$, denoted by $\tilde{\mathbf{x}}\tilde{\in}g$, iff $\tilde{\mathbf{x}}(\alpha) \in g(\alpha)$ i.e. $\mu_{\tilde{\mathbf{x}}(\alpha)}(u) \leq \mu_{g(\alpha)}(u)$ for each $\alpha \in P$ and $u \in U$. The class of fuzzy soft elements is denoted by $FSE(g)$. Also, a fuzzy soft element $\tilde{\mathbf{x}}\tilde{\in}g$ is called a constant fuzzy soft element of g if there exists a $u \in U$ such that for each $\alpha \in P$, $\mu_{\tilde{\mathbf{x}}(\alpha)}(u) = 1$ and $\mu_{\tilde{\mathbf{x}}(\alpha)}(u') = 0$ for each $u' \in U - \{u\}$. The class of all constant fuzzy soft element of g is denoted by $\mathfrak{c}_g$. A constant fuzzy soft element can be considered the most desirable, competent or unique object that fulfils all attributes.*

Example 2.2. *Suppose that a technology company wants to distribute IT support requests to technicians. Let*

$$U = \{u = \textit{technician1}, v = \textit{technician2}, w = \textit{technician3}, x = \textit{technician4}\}$$

be a set of technicians and

$$P = \{\alpha = \textit{security breach}, \beta = \textit{network issues}, \gamma = \textit{software bugs}\}$$

be a set of support requests.

Each technician is specialised to a certain extent in different requests, which can be represented by a fuzzy soft set such as

$$g = \{(\alpha, \{u^{0.7}, v^1, w^{0.2}, x^{0.4}\}), (\beta, \{u^1, v^1, w^{0.3}, x^{0.5}\}), (\gamma, \{v^1, w^1\})\},$$

where $\mu_{g(\alpha)}(u)$ indicates the degree of service capacity in each support request for each technician.

g	α	β	γ
u	0.7	1	0
v	1	1	1
w	0.2	0.3	1
x	0.4	0.5	0

Table 3. The tabular representation of g

Each fuzzy soft element of g represents a scenario in which IT support requests are distributed to technicians according to prioritisation, workload balance, etc. Then,

$$\mathfrak{c}_{\tilde{\mathbf{U}}} = \begin{Bmatrix} \tilde{\mathbf{x}}_1 = \{(\alpha, u^1), (\beta, u^1), (\gamma, u^1)\}, & \tilde{\mathbf{x}}_3 = \{(\alpha, w^1), (\beta, w^1), (\gamma, w^1)\}, \\ \tilde{\mathbf{x}}_2 = \{(\alpha, v^1), (\beta, v^1), (\gamma, v^1)\}, & \tilde{\mathbf{x}}_4 = \{(\alpha, x^1), (\beta, x^1), (\gamma, x^1)\} \end{Bmatrix},$$
$$\mathfrak{c}_g = \{\tilde{\mathbf{x}}_2 = \{(\alpha, v^1), (\beta, v^1), (\gamma, v^1)\}\}$$

and some of the fuzzy soft elements of g and their tabular forms can be given as follows.

$$\tilde{\mathbf{x}}_5 = \{(\alpha, v^{0.6}), (\beta, u^{0.3}), (\gamma, w^{0.1})\},$$
$$\tilde{\mathbf{x}}_6 = \{(\alpha, u^{0.3}), (\beta, x^{0.5}), (\gamma, v^{0.2})\},$$
$$\tilde{\mathbf{x}}_7 = \{(\alpha, x^{0.2}), (\beta, w^{0.2}), (\gamma, w^{0.6})\}.$$

$\tilde{\mathbf{x}}_5$	α	β	γ	$\tilde{\mathbf{x}}_6$	α	β	γ	$\tilde{\mathbf{x}}_7$	α	β	γ
u	0	0.3	0	u	0.3	0	0	u	0	0	0
v	0.6	0	0	v	0	0	0.2	v	0	0	0
w	0	0	0.1	w	0	0	0	w	0	0.2	0.6
x	0	0	0	x	0	0.5	0	x	0.2	0	0

Table 4. The tabular representation of the fuzzy soft elements $\tilde{\mathbf{x}}_5$, $\tilde{\mathbf{x}}_6$ and $\tilde{\mathbf{x}}_7$

Definition 2.3. *Let $\mathfrak{b}$ be a class of fuzzy soft elements of $\tilde{\mathbf{U}}$. The fuzzy soft set $FSS(\mathfrak{b})$ produced by the class of fuzzy soft elements $\mathfrak{b}$ is defined by*

$$g = FSS(\mathfrak{b}) = \{(\alpha, g(\alpha)) : \forall \alpha \in P,\ g(\alpha) = \bigvee_{\tilde{\mathbf{x}} \in \mathfrak{b}} \tilde{\mathbf{x}}(\alpha)\}.$$

Example 2.4. *From Example 2.2, suppose that $\mathfrak{b} = \{\tilde{\mathbf{x}}_5, \tilde{\mathbf{x}}_6, \tilde{\mathbf{x}}_7\}$ is a class of fuzzy soft elements of $\tilde{\mathbf{U}}$. Then, the fuzzy soft set produced by $\mathfrak{b}$ is obtained as*

$$h = FSS(\mathfrak{b}) = \{(\alpha, \{u^{0.3}, v^{0.6}, x^{0.2}\}), (\beta, \{u^{0.3}, w^{0.2}, x^{0.5}\}), (\gamma, \{v^{0.2}, w^{0.6}\})\}.$$

h	α	β	γ
u	0.3	0.3	0
v	0.6	0	0.2
w	0	0.2	0.6
x	0.2	0.5	0

Table 5. The tabular representation of the fuzzy soft set produced by $\mathfrak{b}$

It is clear that $h \tilde{\subset} g$. Also, notice that $\mathfrak{b} \subset FSE(FSS(\mathfrak{b}))$, i.e. $\mathfrak{b}$ and $FSE(FSS(\mathfrak{b}))$ are not the same.

Definition 2.5. *Let $g, h \in FS(U)$ be two fuzzy soft sets. The fuzzy soft sets*

$$g \sqcup h = FSS\left(FSE(g) \cup FSE(h)\right)$$

and

$$g \sqcap h = FSS\left(FSE(g) \cap FSE(h)\right)$$

are called ξ-union and ξ-intersection of g and h, respectively.

The ξ-complement of g is denoted by $g^{\complement} = FSS\left(FSE(g^C)\right)$.

Example 2.6. *From Example 1.7, suppose that*

$$g_3 = \{(\alpha, \{v^{0.4}\}), (\beta, \{u^{0.4}, v^{0.1}, w^{0.5}\}), (\gamma, U)\}$$

is a fuzzy soft set in $FS(U)$. Then,

$$\begin{aligned} g_1 \sqcup g_2 &= g_1 \tilde{\vee} g_2 \\ &= \{(\alpha, \{u^{0.5}, v^{0.7}, w^{0.5}\}), (\beta, \{u^{0.8}, v^{0.2}, w^{0.6}\}), (\gamma, \{u^{0.9}, v^{0.8}, w^{0.1}\})\}, \end{aligned}$$

$$\begin{aligned} g_1 \sqcap g_2 &= g_1 \tilde{\wedge} g_2 \\ &= \{(\alpha, \{u^{0.1}, w^{0.5}\}), (\beta, \{u^{0.1}, v^{0.2}, w^{0.3}\}), (\gamma, \{v^{0.5}\})\} \end{aligned}$$

and

$$\begin{aligned} g_1^{\complement} &= g_1^C \\ &= \{(\alpha, \{u^{0.9}, v^{0.3}, w^{0.5}\}), (\beta, \{u^{0.2}, v^{0.8}, w^{0.4}\}), (\gamma, \{u^{0.1} v^{0.2}, w^{0.9}\})\}. \end{aligned}$$

But,

$$\begin{aligned} g_2 \sqcap g_3 &= \tilde{\mathbf{\Phi}}, \\ g_2 \tilde{\wedge} g_3 &= \{(\alpha, \emptyset), (\beta, \{u^{0.1}, v^{0.1}, w^{0.3}\}), (\gamma, \{v^{0.5}\})\} \end{aligned}$$

and

$$\begin{aligned} g_3^{\complement} &= \tilde{\mathbf{\Phi}}, \\ g_3^C &= \{(\alpha, \{u^{1}, v^{0.6}, w^{1}\}), (\beta, \{u^{0.6}, v^{0.9}, w^{0.5}\}), (\gamma, \emptyset)\}. \end{aligned}$$

So, $g_2 \sqcap g_3 \neq g_2 \tilde{\wedge} g_3$ and $g_3^{\complement} \neq g_3^C$.

Remark 1.
- *The ξ-union of g and h is the same as in Definition 1.6* (1).
- *The ξ-intersection of g and h are the same as as in Definition 1.6* (2) *if for each $\alpha \in P$ and for each same $u \in U$, $\mu_{g(\alpha)}(u) \neq 0$ and $\mu_{h(\alpha)}(u) \neq 0$ or $\mu_{g(\alpha)}(u) = \mu_{h(\alpha)}(u) = 0$ simultaneously. However, if for at least one parameter $\alpha \in P$ and for each same $u \in U$, $\mu_{g(\alpha)}(u) \neq 0$ and $\mu_{h(\alpha)}(u) = 0$ or vice versa, then they are different. In other words; if for each $\alpha \in P$, $g(\alpha) \wedge h(\alpha) \neq \emptyset$, then they mean the same. But, if for at least one parameter $\alpha \in P$, $g(\alpha) \wedge h(\alpha) = \emptyset$, then they correspond to different.*
- *The ξ-complement of g is the same as in Definition 1.6 if for each $\alpha \in P$ and for each $u \in U$, $\mu_{g(\alpha)}(u) \neq 1$. However, if for at least one parameter $\alpha \in U$ and for each $u \in U$, $\mu_{g(\alpha)}(u) = 1$, then they are different. In other words; if for each $\alpha \in P$, $g(\alpha) \neq U$, then they mean the same. But, if for at least one parameter $\alpha \in P$, $g(\alpha) = U$, then they correspond to different.*

Considering Remark 1, the class of fuzzy soft sets, denoted $FS(\tilde{\mathbf{U}})$, is identified for which either $g(\alpha) \neq \emptyset$ for each $\alpha \in P$ or $g(\alpha) = \emptyset$ for each $\alpha \in P$. Then, the following proposition is given to show the relations of Definition 1.6 and 2.5.

Proposition 2.7. *Let $g, h \in FS(\tilde{\mathbf{U}})$.*

(1) $g \vee h = g \sqcup h$.
(2) $g \sqcap h \tilde{\subset} g \tilde{\wedge} h$ *and if* $g \tilde{\wedge} h \in FS(\tilde{\mathbf{U}})$, *then* $g \sqcap h = g \tilde{\wedge} h$.
(3) $g^{\complement} \tilde{\subset} g^{C}$ *and if* $g^{C} \in FS(\tilde{\mathbf{U}})$, *then* $g^{\complement} = g^{C}$.
(4) $g \sqcup g^{\complement} \tilde{\subset} \tilde{\mathbf{U}}$ *and if* $g^{C} \in FS(\tilde{\mathbf{U}})$, *then* $g \sqcup g^{\complement} = \tilde{\mathbf{U}}$.
(5) $g \sqcap g^{\complement} = \tilde{\mathbf{\Phi}}$.
(6) $(g \sqcup h)^{C} \tilde{\supset} g^{C} \sqcap h^{C}$ *and* $(g \sqcap h)^{C} \tilde{\supset} g^{C} \sqcup h^{C}$.
(7) If $g \vee h$, $g^{C} \wedge h^{C}$, g^{C}, $h^{C} \in FS(\tilde{\mathbf{U}})$, *then* $(g \sqcup h)^{\complement} = g^{\complement} \sqcap h^{\complement}$ *and* $(g \sqcap h)^{\complement} = g^{\complement} \sqcup h^{\complement}$.
(8) If $g_i = FSS(\mathfrak{b}_i)$, $i \in I$, *then* $\bigsqcup_{i\in I} g_i = FSS\left(\bigcup_{i\in I} \mathfrak{b}_i\right)$ *and* $\bigsqcap_{i\in I} g_i \tilde{\supset} FSS\left(\bigcap_{i\in I} \mathfrak{b}_i\right)$.

Proof. The proof of 4, 5, 6, 8 follows from 1, 2, 3.

1. Since
$$(g \sqcup h)(\alpha) = \bigvee_{\tilde{\mathbf{x}} \in FSE(g) \cup FSE(h)} \tilde{\mathbf{x}}(\alpha) = \bigvee_{\tilde{\mathbf{x}} \in FSE(g)} \tilde{\mathbf{x}}(\alpha) \vee \bigvee_{\tilde{\mathbf{x}} \in FSE(h)} \tilde{\mathbf{x}}(\alpha) = g(\alpha) \vee h(\alpha)$$
for each $\alpha \in P$, then $g \sqcup h = g \tilde{\vee} h$.
2. If $g(\alpha) \wedge h(\alpha) = \emptyset$ or $g(\alpha) \wedge h(\alpha) \neq \emptyset$ for each $\alpha \in P$, then $g \tilde{\wedge} h \in FS(\tilde{\mathbf{U}})$ and so $g \tilde{\wedge} h = g \sqcap h$. If $g(\alpha) \wedge h(\alpha) \neq \emptyset$ for some $\alpha \in P$ and $g(\alpha) \wedge h(\alpha) = \emptyset$ for others, then $g \tilde{\wedge} h \neq \mathbf{\Phi}$ and $g \tilde{\wedge} h \notin FS(\tilde{\mathbf{U}})$ but $g \sqcap h = \mathbf{\Phi}$. Hence, $g \sqcap h \tilde{\subset} g \tilde{\wedge} h$.
3. If $g^{C}(\alpha) = \emptyset$ or $g^{C}(\alpha) \neq \emptyset$ for each $\alpha \in P$, then $g^{C} \in FS(\tilde{\mathbf{U}})$ and so $g^{\complement} = g^{C}$. If $g^{C} \neq \emptyset$ for some $\alpha \in P$ and $g^{C} = \emptyset$ for others, then $g^{C} \neq \mathbf{\Phi}$ and $g^{C} \notin FS(\tilde{\mathbf{U}})$ but $g^{\complement} = \mathbf{\Phi}$. Hence, $g^{\complement} \tilde{\subset} g^{C}$.
7.
$$\begin{aligned}(g \sqcup h)^{\complement} &= FSS\{\tilde{\mathbf{x}} \tilde{\in} \tilde{\mathbf{U}} : \tilde{\mathbf{x}} \tilde{\in} (g \sqcup h)^{C}\} \\ &= FSS\{\tilde{\mathbf{x}} \tilde{\in} \tilde{\mathbf{U}} : \tilde{\mathbf{x}} \tilde{\in} (g \tilde{\vee} h)^{C}\} \\ &= FSS\{\tilde{\mathbf{x}} \tilde{\in} \tilde{\mathbf{U}} : \tilde{\mathbf{x}} \tilde{\in} g^{C} \tilde{\wedge} h^{C}\} \\ &= FSS\{\tilde{\mathbf{x}} \tilde{\in} \tilde{\mathbf{U}} : \tilde{\mathbf{x}} \tilde{\in} g^{C} \sqcap h^{C}\} \\ &= FSS\{\tilde{\mathbf{x}} \tilde{\in} \tilde{\mathbf{U}} : \tilde{\mathbf{x}} \tilde{\in} g^{C} \text{ and } \tilde{\mathbf{x}} \tilde{\in} h^{C}\} \\ &= g^{\complement} \sqcap h^{\complement}\end{aligned}$$

□

2.1. **Decision-making application: Capability-workload equilibrium.** On the basis of fuzzy soft elements and within the context of Example 2.2, a decision-making method is proposed regarding technicians and support requests in order to balance capability and workload. Therefore, the following algorithm is given:

Algorithm Determining the technician assignments

Step 1. Input the set of technicians $U = \{u_1, u_2, \dots, u_n\}$, the set of support requests as $P = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ and the weights $\omega_i \in [0, 1]$, $i = 1, \dots, m$, of parameters to represent the importance of each request.

Step 2. Input the fuzzy soft set $g \in FS_P(U)$ to represent the capabilities of technicians in response to the support requests.

Step 3. Input specialization threshold T for prioritize technicians with higher expertise, specialization bonus B for specialization incentive and workload balance factor ξ to control workload and capability balancing.

Step 4. Input the set of fuzzy soft elements $\{\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \dots \tilde{\mathbf{x}}_k\} \subset FSE(g)$ to represent the intensity of assignment of request α to technician u.

Step 5. Calculate weighted capability score (WCS) for the assignment $\tilde{\mathbf{x}}_k$ with

$$WCS_k = \sum_{j=1}^{n} \sum_{i=1}^{m} \mu_{\tilde{\mathbf{x}}_k(\alpha_i)}(u_j) \cdot \mu_{g(\alpha_i)}(u_j) \cdot \omega_i \cdot (1 + \delta_{ij}),$$

where

$$\delta_{ij} = \begin{cases} B, & \text{if } \mu_{g(\alpha_i)}(u_j) \geq T, \\ 0, & \text{if } \mu_{g(\alpha_i)}(u_j) < T. \end{cases}$$

Step 6. Calculate weighted workload imbalance (WWI) for the assignment $\tilde{\mathbf{x}}_k$ with

$$WWI_k = \max_{j \in J} L_{k_j} - \min_{j \in J} L_{k_j},$$

where

$$L_{k_j} = \sum_{i=1}^{m} \mu_{\tilde{\mathbf{x}}_k(\alpha_i)}(u_j) \cdot \omega_i$$

is the workload of technician u_j in the assignment $\tilde{\mathbf{x}}_k$.

Step 7. Calculate final score for the assignment $\tilde{\mathbf{x}}_k$ with

$$S_k = WCS_k - \xi \cdot WWI_k.$$

Step 8. Rank the assignments according to their scores S_k and choose the highest one.

This algorithm evaluates the capability and availability of each technician while considering the urgency and complexity of support requests, ultimately facilitating a more efficient allocation of resources and enhancing overall service quality. By implementing this method, technician performance can be optimized and response times to support inquiries can be improved.

From Example 2.2, let $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$ and $u_1 = u$, $u_2 = v$, $u_3 = w$, $u_4 = x$. Assume that the specialization threshold $T = 0.65$, the specialization bonus $B = 0.3$ and the workload balance factor $\xi = 0.4$. In addition, let the company give the weights for each request as $\omega_1 = 0.7$, $\omega_2 = 0.5$ and $\omega_3 = 0.3$. Then, the sample calculations for $\tilde{\mathbf{x}}_6$ are given as follows:

$$L_{6_1} = \mu_{\tilde{\mathbf{x}}_6(\alpha_1)}(u_1) \cdot \omega_1 = 0.3 \cdot 0.7 = 0.21, \qquad L_{6_3} = 0,$$
$$L_{6_2} = \mu_{\tilde{\mathbf{x}}_6(\alpha_3)}(u_2) \cdot \omega_3 = 0.2 \cdot 0.3 = 0.06, \qquad L_{6_4} = \mu_{\tilde{\mathbf{x}}_6(\alpha_2)}(u_4) \cdot \omega_2 = 0.5 \cdot 0.5 = 0.25.$$

So,

$$WWI_6 = 0.25 - 0 = 0.25.$$

Then,

$$\left.\begin{aligned} \mu_{g(\alpha_1)}(u_1) &= 0.7 \geq 0.65, \\ \mu_{g(\alpha_2)}(u_4) &= 0.5 < 0.65, \\ \mu_{g(\alpha_3)}(u_2) &= 1 \geq 0.65. \end{aligned}\right\} \Rightarrow \begin{aligned} \delta_{11} &= 0.3, \\ \delta_{24} &= 0, \\ \delta_{32} &= 0.3. \end{aligned}$$

So,

$$\left.\begin{aligned} \mu_{\tilde{\mathbf{x}}_6(\alpha_1)}(u_1) \cdot \mu_{g(\alpha_1)}(u_1) \cdot \omega_1 \cdot (1 + \delta_{11}) &= 0.3 \cdot 0.7 \cdot 0.7 \cdot 1.3 = 0.1911, \\ \mu_{\tilde{\mathbf{x}}_6(\alpha_2)}(u_4) \cdot \mu_{g(\alpha_2)}(u_4) \cdot \omega_2 \cdot (1 + \delta_{24}) &= 0.5 \cdot 0.5 \cdot 0.5 \cdot 1 = 0.125, \\ \mu_{\tilde{\mathbf{x}}_6(\alpha_3)}(u_2) \cdot \mu_{g(\alpha_3)}(u_2) \cdot \omega_3 \cdot (1 + \delta_{32}) &= 0.2 \cdot 1 \cdot 0.3 \cdot 1.3 = 0.078. \end{aligned}\right\} \Rightarrow WCS_6 = 0.3941.$$

Hence, the score of $\tilde{\mathbf{x}}_6$ is

$$S_6 = WCS_6 - \xi \cdot WWI_6 = 0.3941 - 0.4 \cdot 0.25 = 0.2941.$$

Thus, it is similarly obtained that $S_5 = 0.6120$ and $S_7 = 0,2080$. As a result, $\tilde{\mathbf{x}}_5$ is chosen for the assignments.

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