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Kesirli Viskoelastik Sınır Koşullarına Sahip Euler-Bernoulli Kirişi

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Özet: *Bu çalışmada, Riemann-Liouville kesirli türevini içeren kesirli viskoelastik sınır koşullarına sahip doğrusal Euler-Bernoulli kirişinin zorlamalı titreşim analizi ele alınmıştır. Kirişin, harmonik dış kuvvetin etkisi altında olduğu kabul edilmiştir. İlk olarak, Cauchy gerilme teorisi kullanılarak kirişin matematiksel modeli elde edilmiştir. Daha sonra, geometri ve malzemeye bağımlılığı ortadan kaldırmak için hareket denklemi boyutsuzlaştırılmıştır. Analizde bir pertürbasyon yöntemi (çok zaman ölçekli metodun doğrudan uygulaması) kullanılmıştır. Yaklaşık analitik çözümler, genlik ve faz modülasyon denklemleri elde edilmiştir. k değişimlerine göre doğal frekanslar ve mod şekilleri elde edilmiştir. Kirişin belirli bir k değerinden itibaren basit-basit mesnetli bir kiriş olarak davranışı $k-\omega$ değişim grafikleriyle gösterilmiştir. Ayrıca kararlı durum çözümleri ve bunların kararlılıkları incelenmiştir.*

Anahtar Kelimeler: *Kesirli Türev, Kesirli Viskoelastik Sınır Koşulları, Çok Zaman Ölçekli Metod, Pertürbasyon Yöntemi, Euler-Bernoulli Kirişi*

Euler-Bernoulli Beam With Fractional Viscoelastic Boundary Conditions

Abstract: *In this paper, forced vibration analysis of linear Euler-Bernoulli beam with fractional viscoelastic boundary conditions, which involves Riemann-Liouville fractional derivative, is considered. The beam is assumed under the effect of external harmonic force. Firstly, the mathematical model of the beam is obtained by using Cauchy stress theory. Then, to obtain universal results by eliminating dependence on geometry and material, the equation of motion is non-dimensionalized. A perturbation method (direct application of the method of multiple scales) is employed in the analysis. The approximate analytical solutions, the amplitude and phase modulation equations are obtained. Natural frequencies and mode shapes according to k variations are obtained. The beam's behavior as a simply-simply supported beam from a certain value of k is demonstrated with $k-\omega$ variation graphics. Also steady state solutions and their stability are examined.*

Keywords: *Fractional Derivative, Fractional Viscoelastic Boundary Conditions, Method of Multiple Time Scales, Perturbation Methods, Euler–Bernoulli Beam.*

1. Introduction

Beams are widely used mechanical structural elements because they enable accurate modeling in various branches of applied science and in various engineering problems such as mechanical, aerospace, civil, biomedical engineering. Mathematical models of beams have been developed according to beam theories based on various asumptions; the most useful of these theories are Euler-Bernoulli (Reddy., 2011.; Carrera, Giunta, & Petrolo., 2011.; Bauchau & Craig, 2009), Timoshenko (Reddy., 2011.; Huang, Yang, & Luo, 2013) and higher-order shear theories (Thai & Vo, 2012; Simsek, 2010).

Boundary conditions are the constraints imposed on the ends of a beam, such as fixed, free, pinned, etc. They play an important role for determining static and dynamic behaviors of the beam, such as natural frequencies, deflection, critical loads and vibration (Reddy, Wang, & Lee, 1997; Zhao & Chang, 2021; Barari, Kaliji, Ghadimi, & Domairry, 2011). However, boundary conditions are not always constant, as they may change due to structural and environmental changes. These changes can significantly affect the performance of the entire beam structure. Therefore, it is essential to study the effects of changing boundary conditions on the beam behavior and to develop methods for detecting and compensating them (Barari, Kaliji, Ghadimi, & Domairry, 2011; Qiao & Rahmatalla, 2021).

Fractional calculus theory emerged from the origins of classical calculus, and it has been successfully used to the mathematical modeling of numerous branches of research and engineering, including mechanics, physics, biology, chemistry, economics, etc. The detailed explanations of the theory of fractional calculus and its applications are provided in (Podlubny, 1999; Miller & Ross, 1993; Debnath, 2003). One of the advantages of fractional calculus is that it has a better fit for mathematical modeling of the properties of materials, as it has the ability to describe the memory and inherited properties of some important materials and processes, such as viscoelasticity, diffusion, and fractional order control systems. The fractional derivative can be used to describe equations of motion for various mechanical problems, including oscillators, beams, plates, and shells, as well as to describe their boundary conditions, for example see (Agarwal, Ntouyas, Ahmad, & Alhothuali, 2013; Rong & Bai, 2015) .

As with most fractional differential equations, if equation and equation systems containing fractional derivative do not have analytical solutions, such differential equations are solved by various approaches and numerical techniques (Javadi, Noorian, & Iran, 2019; Lewandowski & Wielentejczyk, 2017).One of the methods used in the solution of the mentioned differential equations is the perturbation method, which is an approximate analytical method. It is based on the idea of using a simpler problem that is related to the original one and then correcting the solution by adding small terms that account for the difference between the two problems. The small terms are usually expressed as a power series in a small parameter that measures the deviation from the simpler problem. Depending on the choice of the simpler problem and the small parameter, there are different types of perturbation methods. The method of multiple scales is one of the most popular perturbation techniques used to examine approximate analytical solutions (Demir, Bildik, & Sınır, 2014; Sınır, Yıldız, & Sınır; Tang, Zhen, & Fang, 2018).

The present paper introduces the dynamic response of Euler-Bernoulli beam with fractional visco-elastic boundary conditions. Method of multiple scales is used in the analysis. The approximate analytical solutions, the amplitude and phase modulation equations are derived. Additionally, the steady state solutions and their stability are discussed. The natural frequencies and mode shapes are obtained, also are demonstrated in tables and graphs.

2. Euler-Bernoulli Beam Model and Nondimensionalization

The equation of motion of Euler-Bernoulli beam derived from Cauchy Stress theory (Carrera, Giunta, & Petrolo., 2011.) is considered as follows

$$EI\frac{\partial^4 \hat{v}}{\partial \hat{x}^4}+\rho A\frac{\partial^2 \hat{v}}{\partial \hat{t}^2}=\hat{q}. \quad (1)$$

Here E, is the Young's modulus, I is the moment of inertia of the beam cross-section, ρ is the mass density, $\hat{q}$ is the external load and A is the cross-sectional area.

While determining the viscoelastic support conditions of the model, the effects of viscoelastic support were directly applied as boundary conditions in accordance with the sign acceptance in (İnan, 1967). In this study, the visco-elastic support is used only to prevent displacement, so it does not have an anti-rotation effect. Therefore, when one assume the change of the damping term in boundary conditions with respect to time as fractional; with well-known relations for shear force and the bending moment, the boundary conditions can be written as follows:

$$\frac{\partial^2 \hat{v}}{\partial \hat{x}^2}(0,\hat{t})=0 \text{ and } EI\frac{\partial^3 \hat{v}}{\partial \hat{x}^3}(0,\hat{t})=-\hat{k}\hat{v}(0,\hat{t})-\hat{\eta}D_{\hat{t}}^{\propto}\hat{v}(0,\hat{t}) \quad (2)$$

$$\frac{\partial^2 \hat{v}}{\partial \hat{x}^2}(L,\hat{t})=0 \text{ and } EI\frac{\partial^3 \hat{v}}{\partial \hat{x}^3}(L,\hat{t})=\hat{k}\hat{v}(L,\hat{t})-\hat{\eta}D_{\hat{t}}^{\propto}\hat{v}(L,\hat{t}) \quad (3)$$

Here $\hat{k}$ is the spring coefficient, $\hat{\eta}$ is the damping coefficient and L is the length of the beam. Additionally, the symbol D_t^{α} denotes the the Riemann - Liouville fractional derivative with fractional order α is defined as (Podlubny, 1999),

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds,$$

where $n - 1 \leq \alpha < n$ $(n \in \mathbb{N}, \alpha \in \mathbb{R})$ and Γ denotes the Euler-Gamma function.

The dimensionless form of the equation of motion of beam is rewritten according to the following variables

$$v = \frac{\hat{v}}{L}, x = \frac{\hat{x}}{L}, t = \hat{t} L^2 \sqrt{\frac{pA}{\hat{E}I}} \tag{4}$$

and the dimensionless external load is obtained as $q = \frac{\hat{q}L^3}{EI}$, $k = \frac{\hat{k}L^3}{EI}$ and $\eta = \frac{\hat{\eta}L^{3+2\alpha}(\rho A)^{\alpha/2}}{(EI)^{1+\alpha/2}}$. Here, q is considered as a harmonic load. f and Ω denote amplitude and frequency of q, respectively; and viscous damping having damping coefficient η, at order ε. One finally obtains the following the dimensionless form of Eq. (1)

$$v^{IV} + \ddot{v} = \varepsilon f cos(\Omega t) \tag{5}$$

and the dimensionless boundary conditions are converted as,

$$v''(0,t) = 0 \text{ and } v'''(0,t) = -kv(0,t) - \varepsilon\eta D_t^{\alpha}\, v(0,t) \tag{6}$$

$$v''(1,t) = 0 \text{ and } v'''(1,t) = kv(1,t) + \varepsilon\eta D_t^{\alpha}\, v(1,t) \tag{7}$$

Note that, for simplicity, here and for the rest of the article, the symbols $(')$ and $(\cdot)$ are used for differentiation by spatial variable and differentiation by time, respectively.

3. Perturbation Analysis

The Method of Multiple Scales (Nayfeh, Introduction to perturbation techniques, 2011) is used to find the general solution of the Eq. (5). The solution function is assumed in the following expansion:

$$v = (x, T_0, T_1; \varepsilon) = v_0(x, T_0, T_1) + \varepsilon v_1(x, T_0, T_1) + \cdots, \tag{8}$$

where $v_0, v_1, \ldots$ are displacement functions and the T_n are defined by

$$T_0 = t, T_1 = \varepsilon t, \ldots$$

Here each variable represents a different time scale: T_0 is the fastest, T_1 is slower, and so on. By replacing the independent variable t in the Eq. (5) with $T_0, T_1, \ldots$ and applying the chain rule gives the time derivatives as follows

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \cdots, \tag{9}$$

$$\frac{d^2}{dt^2} = {D_0}^2 + 2\varepsilon D_0 D_1 + \cdots, \tag{10}$$

$$\left(\frac{d}{dt}\right)^{\alpha} = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots)^{\alpha} \tag{11}$$

$$= D_0^{\alpha} + \varepsilon(\alpha D_0^{\alpha-1} D_1) + \cdots$$

where D_n represents $\frac{\partial}{\partial T_n}$. Substituting v from (8) and time derivatives from (9)-(11) into Eqs. (5)-(7) and equating each of the coefficiets of ε^0 and ε to zero, one have following equations.

Order ε^0:

$$v_0^{IV} + D_0^2 v_0 = 0 \tag{12}$$

with the following conditions

$$v_0''(0,t) = 0 \text{ and } v_0'''(0,t) = -kv_0(0,t) \tag{13}$$

$$v_0''(1,t) = 0 \text{ and } v_0'''(1,t) = kv_0(1,t) \tag{14}$$

Order ε:

$$v_1^{IV} + D_0^2 v_1 = -2D_0 D_1 v_0 + f cos(\Omega t) \tag{15}$$

with the following conditions

$$v_1''(0,t) = 0 \text{ and } v_1'''(0,t) = -kv_1(0,t) - \eta D_t^{\alpha} v_0(0,t) \tag{16}$$

$$v_1''(1,t) = 0 \text{ and } v_1'''(1,t) = kv_1(0,t) + \eta D_t^{\alpha} v_0(1,t) \tag{17}$$

The following solution form can be proposed to the differential equation Eq. (12) in the first order:

$$v_0(x, T_0, T_1) = \left(A_n(T_0, T_1)e^{i\omega_n T_0} + \bar{A}_n(T_0, T_1)e^{-i\omega_n T_0}\right)X_n(x) \tag{18}$$

Here $A_n(T_0, T_1)$ and $\bar{A}_n(T_0, T_1)$ are complex amplitudes and their complex conjugates, respectively; $X_n(x)$ are mode shapes and ω_n are natural frequencies. Inserting Eq. (18) into the Eqs. (12)-(14) and making the necessary simplifications, $X_n(x)$ satisfies the following equations

$$X_n^{IV} + \omega_n^2 X_n = 0 \tag{19}$$

and boundary conditions

$$X''(0) = 0 \text{ and } X_n'''(0) = -kX_n(0) \tag{20}$$

$$X_n''(1) = 0 \text{ and } X_n'''(1) = kX_n(1). \tag{21}$$

Eq. (19) is a homogenous linear differential equation with constant coefficients, so the solution can be in the following form:

$$X_n(x) = C_1 \, cos\left(\sqrt{\omega_n} x\right) + C_2 \, sin\left(\sqrt{\omega_n} x\right) + C_3 \, cos\,h\left(\sqrt{\omega_n} x\right) + C_4 \, sinh\left(\sqrt{\omega_n} x\right). \tag{22}$$

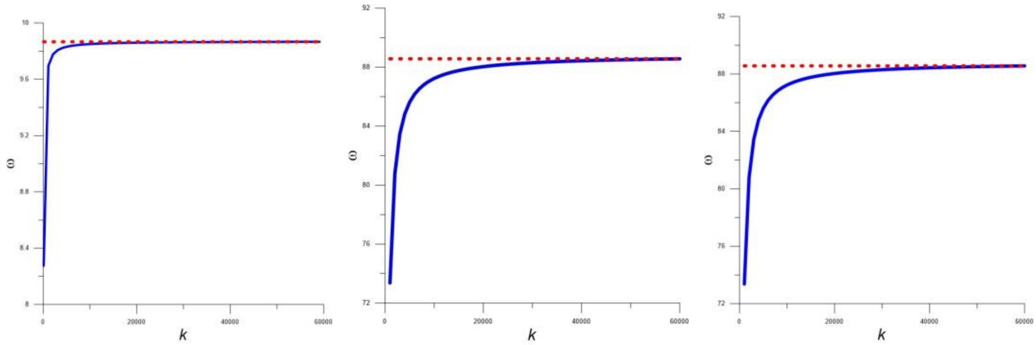
By utilizing (22) into Eqs. (20)-(21), one obtains a system of algebraic equations:

$$\begin{gathered} -C_1 + C_3 = 0 \\ -C_1 \, cos\left(\sqrt{\omega_n}\right) - C_2 \, sin\left(\sqrt{\omega_n}\right) + C_3 \, cos\,h\left(\sqrt{\omega_n}\right) + C_4 \, sinh\left(\sqrt{\omega_n}\right) = 0 \\ \omega_n\sqrt{\omega_n}(-C_1 + C_4) + k(C_1 + C_2) = 0 \\ \omega_n\sqrt{\omega_n}\left(C_1 \, sin\left(\sqrt{\omega_n}\right) - C_2 \, cos\left(\sqrt{\omega_n}\right) + C_3 \, sin\,h\left(\sqrt{\omega_n}\right) + C_4 \, cosh\left(\sqrt{\omega_n}\right)\right) \\ -k\left(cos\left(\sqrt{\omega_n}\right) + C_2 \, sin\left(\sqrt{\omega_n}\right) + C_3 \, cos\,h\left(\sqrt{\omega_n}\right)\right) = 0 \end{gathered} \tag{23}$$

The following non-trival solution conditions,

$$-2w_n^5 - 4\,cos\left(\sqrt{\omega}\right) \sinh\left(\sqrt{\omega}\right) \omega_n^{\frac{7}{2}} k + 4 \sin\left(\sqrt{\omega_n}\right) \cosh\left(\sqrt{\omega_n}\right) \omega_n^{\frac{7}{2}} k + \; 2 \cos\left(\sqrt{\omega_n}\right) cosh\sqrt{\omega_n}\,\omega_n^5 - 4k^2\omega_n^2 \, sin\left(\sqrt{\omega}\right) sinh\left(\sqrt{\omega}\right) = 0 \tag{24}$$

can be found from non-trivial solutions of the system (23). Numerical values of ω_n can be found by using (24) non-trivial solution condition. (See Fig. 1)



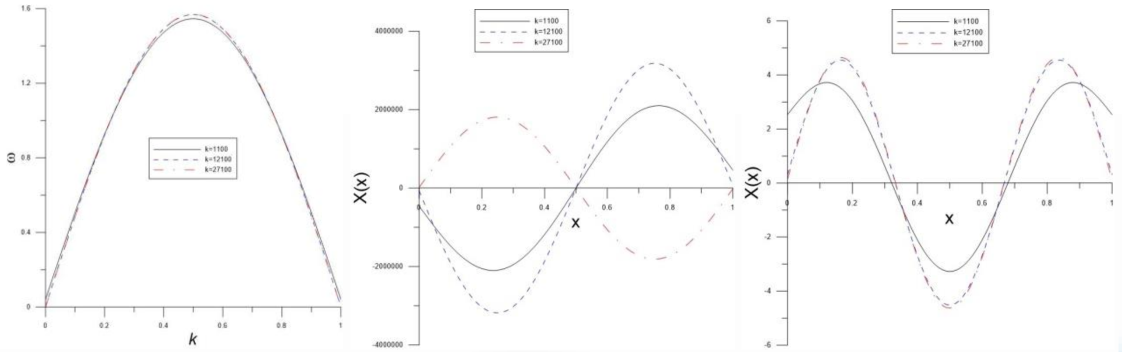
(a) (b) (c)

Figure 1. (a) 1. mode $\boldsymbol{k - \omega}$ variation, (b) 2. mode $\boldsymbol{k - \omega}$ variation, (c) 3. mode $\boldsymbol{k - \omega}$ variation.

The mode shape equation is

$$X_n = C_1 \left(\begin{array}{c} cos(\sqrt{\omega_n}x) + \frac{\sqrt{\omega_n^3}\,cos(\sqrt{\omega_n}) - \sqrt{\omega_n^3}\,cos\,h(\sqrt{\omega_n}) + 2k sinh(\sqrt{\omega_n})}{\sqrt{\omega_n^3}(sinh(\sqrt{\omega_n}) - sin(\sqrt{\omega_n}))} sin(\sqrt{\omega_n}x) \\ + cosh(\sqrt{\omega_n}x) + \frac{\sqrt{\omega_n^3}\,cos(\sqrt{\omega_n}) - \sqrt{\omega_n^3}\,cos\,h(\sqrt{\omega_n}) + 2k sin(\sqrt{\omega_n})}{\sqrt{\omega_n^3}(sinh(\sqrt{\omega_n}) - sin(\sqrt{\omega_n}))} sinh(\sqrt{\omega_n}x) \end{array} \right) \quad (25)$$

(See Fig.2)



(a) (b) (c)

Figure 2. Mode shapes (a) first-mode (b) second-mode (c) third-mode.

At the ε order of approximation, the case $\Omega \approx \omega_n$ considered. Because of the case under consideration, a detuning parameter σ_n is defined by the following primary resonance condition

$\Omega = \omega_n + \varepsilon\sigma_n$ (26)

For v_1, the solution form

$$v_1(x, T_0, T_1) = \psi_n(x, T_1)e^{i\omega_n T_0} + c.c. + \Lambda(x, T_0, T_1) \quad (27)$$

is assumed. Here Λ includes nonsecular terms (NST) and c.c. denotes complex conjugates. Substituting (18), (26) and (27) to Eqs. (15) - (17), one has

$$\psi_n^{IV} - \omega_n^2\psi_n = -2i\omega_n X_n D_1 A_n + \frac{f}{2}e^{i\sigma_n T_1} + c.c. + NST \quad (28)$$

with the following conditions

$$\psi_n(0, T_1) = 0 \text{ and } \psi_n''(0, T_1) = -k\psi_n(0) - (i\omega_n)^\alpha \eta A_n(T_1)X_n(0) \quad (29)$$

$$\psi_n''(1, T_1) = 0 \text{ and } \psi_n''(1, T_1) = k\psi_n(1) + (i\omega_n)^\alpha \eta A_n(T_1)X_n(1) \quad (30)$$

For the non-homogenous problem, to have a solution, a solvability condition should be satisfied (Nayfeh, Introduction to perturbation techniques, 2011). The solvability condition for the proposed model is

$$D_1 A_n - (K_R + iK_1)A_n + \frac{i}{2}F_n e^{i\sigma_n T_1} = 0 \quad (31)$$

Where

$$K_R = \omega\eta sin\left(\frac{\alpha\pi}{2}\right)\frac{(X_n^2(1) - X_n^2(0))}{2\int_0^1 X_n^2 dx} \quad (32)$$

$$K_1 = -\omega_n^{\alpha-1}\eta cos\left(\frac{\alpha\pi}{2}\right)\frac{(X_n^2(1) - X_n^2(0))}{2\int_0^1 X_n^2 dx} \quad (33)$$

and

$$F_n = \frac{\int_0^1 f X_n dx}{2\omega_n \int_0^1 X_n^2 dx} \tag{34}$$

All coefficients have both real and imaginary parts. Complex amplitudes can be expressed in polar form

$$A_n(T_1) = \frac{1}{2} a_n(T_1) e^{i\beta_n(T_1)}. \tag{35}$$

Inserting (35) into the solvability condition (31) and separating real and imaginary parts yields the equations for amplitude and phase modulation

$$a_n' = F_n \sin(\gamma_n) - K_R a_n \tag{36}$$

$$\gamma_n' = \sigma_n + \frac{1}{a_n} F_n \cos(\gamma_n) - K_I \tag{37}$$

Where

$$\gamma_n = \sigma_n T_1 - \beta_n. \tag{38}$$

For steady-state solutions, $a_n' = \gamma_n' = 0, a_n \neq 0$. After this condition is substituted in the amplitude and phase modulation equations,

$$\begin{aligned} a_n' &= F_n \sin(\gamma_n) - K_R a_n = 0 \\ \gamma_n' &= \sigma_n + \frac{1}{a_n} F_n \cos(\gamma_n) - K_I = 0 \end{aligned} \tag{39}$$

are obtained. Elimination of γ_n between in the system of equations (39) yields the detuning parameter

$$\sigma_n = -K_I \mp \frac{1}{a_n}\sqrt{F_n^2 - (a_n K_R)^2}. \tag{40}$$

In order to determine if the stability of fixed points, the Jacobian matrix of the system (39) is considered,

$$\begin{bmatrix} -K_R & F_n \cos(\gamma_n) \\ -\frac{1}{a_n^2} F_n \cos(\gamma_n) & -\frac{1}{a_n} F_n \sin(\gamma_n) \end{bmatrix}_{\substack{a_n = a_0 \\ \gamma_n = \gamma_0}} \tag{41}$$

For stability, the real parts of the eigenvalues of the matrix (41) must be less than zero. The eigenvalues of the matrix are calculated

$$\lambda_{1,2} = -\frac{1}{2} K_R - \frac{1}{2a_n} F_n \sin(\gamma_n) \mp \sqrt{\left(K_R - \frac{1}{a_n} F_n \sin(\gamma_n)\right)^2 - \frac{4}{{a_n}^2} F_n^2 \cos^2(\gamma_n))} \tag{42}$$

Therefore, non-trival solutions are stable.

Solution (8) was suggested for the corresponding model (5). After the necessary mathematical calculations are performed, the approximate solution of the system is obtained.

$$v(x,t;\varepsilon) = \frac{1}{2} a_n e^{i(\Omega t - \gamma_n)} X_n + O(\varepsilon) \tag{43}$$

The form of Eqs. in (39) can be used for determining stability of the non-trivial solutions. For stability of the trivial solution, these forms are not suitable. Instead of the polar representation, one chooses another transformation (Nayfeh & Mook, Nonlinear oscillations, 2008);

$$A_n = \frac{1}{2}(p_n + i q_n) e^{i\frac{\sigma_n}{2} T_1}. \tag{44}$$

Substituting (44) into (31), separating into real and imaginary parts, one finally has following equations,

$$p_n' = K_I q_n - \frac{1}{2} q_n \sigma_n - F_n \sin\left(\frac{1}{2}\sigma_n T_1\right) - K_R p_n \tag{45}$$

$$q_n' = -K_I p_n - \frac{1}{2} p_n \sigma_n - F_n \cos\left(\frac{1}{2}\sigma_n T_1\right) - K_R q_n \tag{46}$$

To determine stability, the Jacobian matrix is constructed

$$J_n = \begin{bmatrix} -K_R & K_I - \frac{1}{2}\sigma_n \\ -K_I + \frac{1}{2}\sigma_n & -K_R \end{bmatrix}_{p_n=q_n=0} . \tag{47}$$

The eigenvalues of the matrix (47) are calculated next

$$\lambda_{1,2} = -K_R \mp \left(\frac{1}{2}\sigma_n - K_I\right) \tag{48}$$

Hence, trivial solutions are stable.

4. Concluding remarks

A model of Euler-Bernoulli beam with fractional visco-elastic boundary conditions is considered. The multiple scales method is applied directly to the equation of motion. The amplitude and phase modulation equations, steady-state solutions and their stability are discussed. The natural frequencies and mode shapes of the first three natural frequencies are demonstrated in graphs. It has been observed that the obtained frequency values are compatible with the theory. The fractional derivative order α is included in the solution function and detuning parameter.

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