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Framed curves in three-dimensional Lie groups and a Berry phase model

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ABSTRACT

In this study, we introduce framed curves which can have singular points in three-dimensional Lie groups. We give Frenet-Serret type formulas of framed curves with the help of a Lie curvature in three-dimensional Lie groups. Then, we define adapted frames of framed curves in three-dimensional Lie groups such that Frenet-Serret type frame and Bishop frame. Finally, as a physical application, we give Berry phase model of polarized light wave along an optical fiber in Lie groups.

1. Introduction

There have been many studies on non-regular curves in recent years and they contribute to singularity theory. Framed curves are one of them. Framed curves are smooth curves with singular points in Euclidean space. In differential geometry, a moving frame cannot be defined for a non-regular curve. However, Honda and Takahashi introduced framed curves that have both singular points and are defined by a moving frame [1]. Also, framed curves are generalizations of regular curves under linear independent conditions and Legendre curves in unit tangent bundles. Therefore, framed curves have come to an important place in the literature [2–9].

In [10], the degenerate semi-Riemannian geometry of Lie group is obtained. Çiftçi defined general helices and the generalization of Lancret's theorem in three dimensional Lie groups in [11]. In [12], the authors gave slant helices in three dimensional Lie group with a bi-invariant metric. Also, Bertrand and Mannheim curves are investigated in Lie groups [13,14]. Lie groups are an important mathematical form because they have three different structures in mathematics such that S^3, $SO(3)$ and abelian Lie groups. In addition, Lie groups have a wide range of theory and application in physics and mechanics, as well as their importance in mathematics. For example, incompressible inviscid fluid motion and rigid body motion correspond to geodesic flow of left (or right) invariant metric defined on a Lie group [15–17]. Also, in [18], the authors gave spinor equations in three-dimensional Lie groups. These studies on curves in Lie groups usually include a regular curve and moving frames constructed from these curves.

On the other hand, Berry phase or geometric phase is a quantum phase effect defined by Berry [19]. Important results have been obtained with the movements of quantum systems in space with the Berry phase. Classical examples are the

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motion of the charged particle due to the magnetic vector field and the propagation of a polarized light wave along an optical fiber. Ross, Kugler and Shtrikman consider the optical fiber as a space curve and they study the geometry of the polarization vector along an optical fiber [20], [21]. Geometric phase models along an optical fiber according to Bishop frame in 3D Riemann manifolds in [22], geometric phase model and electromagnetic curves along an optical fiber for an alternative moving frame $\{n, c, w\}$ in [23], Berry phase model according to quasi-adapted frame in [24], the electromagnetic curves of a polarized light wave in a 3D semi-Riemann manifold in [25], are given. On the other hand, basic definitions and explanations for Rytov's law, which is defined by the rotation of the polarization plane along an optical fiber, are given in [26–28].

In this study, we give the basic properties of framed curves in Lie groups, which have singular points but can have a special moving frame. Therefore, we introduce a new approach to both physical and geometrical studies of Lie groups. Also, we give Frenet-Serret type formulas of framed curves with the help of a Lie curvature in three-dimensional Lie groups. Then, we define adapted frames of framed curves in three-dimensional Lie groups such that Frenet-Serret type frame and Bishop frame. Finally, as a physical application, we give Berry phase model of polarized light wave along an optical fiber in Lie groups.

2. Preliminaries

Let G be a Lie group with a bi-invariant metric $\langle , \rangle$ and D be the Levi-Civita connection of Lie group G. If $\mathfrak{g}$ denotes the Lie algebra of G then we know that $\mathfrak{g}$ is isomorphic to T_eG where e is neutral element of G. If $\langle , \rangle$ is a bi-invariant metric on G then we have

$$\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle, \tag{1}$$

and

$$D_X Y = \frac{1}{2}[X, Y], \tag{2}$$

for all $X, Y, Z \in \mathfrak{g}$. Also the Lie bracket of two vector fields W and Z is given

$$[W, Z] = \sum_{i=1}^{n} w_i z_i [X_i, X_j], \tag{3}$$

where $W = \sum_{i=1}^{n} w_i X_i$ and $Z = \sum_{i=1}^{n} z_i X_i$ with orthonormal basis $\{X_1, X_2, ..., X_n\}$ of $\mathfrak{g}$. If $\beta : I \to G$ be an arc-lenghted regular curve, the covariant derivative of W along the curve β with the notation $D_{\beta'} W$ is given as follows

$$D_{\beta'} W = D_T W = \dot{W} + \frac{1}{2}[T, W], \tag{4}$$

where T is unit-tangent vector of β and $\dot{W} = \sum_{i=1}^{n} \frac{dw}{dt} X_i$. In addition, if W is the left-invariant vector field to the curve, then $\dot{W} = 0$ (see for details [29]).

Let G be a three-dimensional Lie group and (T, N, B, κ, τ) denote the Frenet apparatus of the regular a curve. Then the Frenet-Serret formulas of the regular a curve satisfy:

$$\begin{aligned} D_T T &= \kappa N, \\ D_T N &= -\kappa T + \tau B, \\ D_T B &= -\tau N, \end{aligned} \tag{5}$$

where D is Levi-Civita connection of Lie group G and $\kappa = \|\dot{T}\|$.

3. Framed curves in three-dimensional Lie groups

In this section, we define framed curves in a three dimensional Lie group G with a bi-invariant metric $\langle , \rangle$. We can give definition of framed curve in three-dimensional Lie groups similar to Honda and Takahashi's definition of framed curve in three-dimensional Euclidean space as follows [1]:

Definition 1. A curve $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ in three-dimensional Lie group G is a framed curve if $\langle \gamma'(s), \eta_i(s)\rangle = 0$ for all $s \in I$ and $i = 1, 2$ where

$$\Delta_G = \{\eta = (\eta_1, \eta_2) \in G \times G | \langle \eta_1, \eta_1\rangle = \langle \eta_2, \eta_2\rangle = 1, \langle \eta_1, \eta_2\rangle = 0\}.$$

We define a unit vector ν by $\nu = \eta_1 \times \eta_2$. Also, if $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ a framed curve, we accept the covariant derivative of W along the framed curve (γ, η_1, η_2) with the help of unit vector $\nu = \eta_1 \times \eta_2$ as follows

$$D_\nu W = \dot{W} + \frac{1}{2}[\nu, W]. \tag{6}$$

where $\dot{W} = \sum_{i=1}^{n} \dfrac{dw}{ds} X_i$ for orthonormal basis $\{X_1, X_2, ..., X_n\}$ of $\mathfrak{g}$. On the other hand, a smooth function $\alpha(s)$ on I is denoted by $\gamma'(s) = \alpha(s)\nu(s)$. Obviously, s_0 is a singular point of γ if and only if $\alpha(s_0) = 0$. Then the Frenet-Serret type formulas of the curve (γ, η_1, η_2) satisfy [1]:

$$\begin{aligned} D_\nu \nu &= -m\eta_1 - n\eta_2, \\ D_\nu \eta_1 &= l\eta_2 + m\nu, \\ D_\nu \eta_2 &= -l\eta_1 + n\nu, \end{aligned} \tag{7}$$

where D is Levi-Civita connection of Lie group G and $\sqrt{m^2 + n^2} = \|\dot{\nu}\|$.

Corollary 2. *If ν is the left-invariant vector field to the framed curve, then $m(s) = n(s) = 0$ for all $s \in I$.*

Proposition 3. *Let $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ be a framed curve in three-dimensional Lie groups. Then, we have the following equalities*

$$[\nu, \eta_1] = \langle [\nu, \eta_1], \eta_2 \rangle \eta_2 = 2\delta_G \eta_2,$$
$$[\nu, \eta_2] = \langle [\nu, \eta_2], \eta_1 \rangle \eta_1 = -2\delta_G \eta_1.$$

Proof. Since $[\nu, \eta_1] \in Sp\{\nu, \eta_1, \eta_2\}$, we have

$$[\nu, \eta_1] = a\nu + b\eta_1 + c\eta_2. \tag{8}$$

If we multiply the two sides of the Eq. (8) with ν, η_1 and η_2 respectively, we get

$$\begin{aligned} \langle [\nu, \eta_1], \nu \rangle &= a = 0, \\ \langle [\nu, \eta_1], \eta_1 \rangle &= b = 0, \\ \langle [\nu, \eta_1], \eta_2 \rangle &= c. \end{aligned} \tag{9}$$

By using Eqs. (8) and (9), we have

$$[\nu, \eta_1] = \langle [\nu, \eta_1], \eta_2 \rangle \eta_2.$$

We assume that $\dfrac{\langle [\nu, \eta_1], \eta_2 \rangle}{2} = \delta_G$, we get $[\nu, \eta_1] = 2\delta_G \eta_2$. On the other hand, since $[\nu, \eta_2] \in Sp\{\nu, \eta_1, \eta_2\}$, we have

$$[\nu, \eta_2] = d\nu + e\eta_1 + f\eta_2. \tag{10}$$

By using a similar method we can

$$\begin{aligned} \langle [\nu, \eta_2], \nu \rangle &= d = 0, \\ \langle [\nu, \eta_2], \eta_1 \rangle &= e, \\ \langle [\nu, \eta_2], \eta_2 \rangle &= c = 0. \end{aligned} \tag{11}$$

Therefore, we get

$$\langle [\nu, \eta_2], \eta_1 \rangle = \langle \nu, [\eta_2, \eta_1] \rangle = -\langle \nu, [\eta_1, \eta_2] \rangle = -\langle [\nu, \eta_1], \eta_2 \rangle = -2\delta_G. \tag{12}$$

From Eqs. (10) and (11), we have $[\nu, \eta_2] = -2\delta_G \eta_1$. □

Theorem 4. *Let $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ be a framed curve. The Frenet-Serret type formulas of framed curves in Lie groups are given by*

$$\begin{pmatrix} \dot{\nu} \\ \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = \begin{pmatrix} 0 & -m & -n \\ m & 0 & (l - \delta_G) \\ n & -(l - \delta_G) & 0 \end{pmatrix} \begin{pmatrix} \nu \\ \eta_1 \\ \eta_2 \end{pmatrix}, \tag{13}$$

where $m = \langle \dot{\eta}_1, \nu \rangle$, $n = \langle \dot{\eta}_2, \nu \rangle$, $l = \langle \dot{\eta}_1, \eta_2 \rangle + \delta_G$ and $\delta_G = \dfrac{1}{2} \langle [\nu, \eta_1], \eta_2 \rangle$.

Proof. By using Eq. (6), we get

$$\begin{aligned} D_\nu \nu &= \dot{\nu} + \frac{1}{2}[\nu,\nu], \\ -m\eta_1 - n\eta_2 &= \dot{\nu}, \\ D_\nu \eta_1 &= \dot{\eta}_1 + \frac{1}{2}[\nu,\eta_1], \\ l\eta_2 + m\nu &= \dot{\eta}_1 + \frac{1}{2}2\delta_G \eta_2, \\ D_\nu \eta_2 &= \dot{\eta}_2 + \frac{1}{2}[\nu,\eta_2], \\ -l\eta_1 + n\nu &= \dot{\eta}_2 - \frac{1}{2}2\delta_G \eta_1. \end{aligned} \tag{14}$$

From Eq. (14), it is seen that

$$\begin{aligned} \dot{\nu} &= -m\eta_1 - n\eta_2, \\ \dot{\eta}_1 &= m\nu + (l-\delta_G)\eta_2, \\ \dot{\eta}_2 &= n\nu - (l-\delta_G)\eta_1. \end{aligned} \tag{15}$$

Therefore, from Eq. (15), we have Eq. (13). □

Corollary 5. *Let* $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ *be a framed curve. Then, Lie curvature* $\delta_G = \frac{1}{2}\langle [\nu, \eta_1], \eta_2 \rangle$ *of framed curve is given by*

$$\delta_G = \frac{1}{2(mn' - m'n + l(m^2+n^2))}\langle \ddot{\nu}, [\nu, \dot{\nu}]\rangle + \frac{1}{4(mn' - m'n + l(m^2+n^2))}\|[\nu, \dot{\nu}]\|^2.$$

Proof. Since, according to Eq. (6), we have

$$D_\nu \nu = \dot{\nu} + \frac{1}{2}[\nu,\nu],$$

and

$$D_\nu \dot{\nu} = \ddot{\nu} + \frac{1}{2}[\nu,\dot{\nu}].$$

Then, we get

$$\begin{aligned} \dot{\nu} &= -m\eta_1 - n\eta_2, \\ \ddot{\nu} &= (-m^2 - n^2)\nu + (-m' + nl)\eta_1 + (-n' - lm)\eta_2 - \frac{1}{2}[\nu, \dot{\nu}]. \end{aligned} \tag{16}$$

By using Eq. (16), it is seen that

$$\begin{aligned} \delta_G &= \frac{1}{2(mn' - m'n + l(m^2+n^2))}\langle \ddot{\nu}, [\nu, \dot{\nu}]\rangle + \frac{1}{4(mn' - m'n + l(m^2+n^2))}\|[\nu, \dot{\nu}]\|^2, \\ &= \frac{1}{2}\langle [\nu, \eta_1], \eta_2\rangle. \end{aligned}$$ □

4. Adapted frames of framed curves in three-dimensional Lie groups

In this section, we define adapted frames of framed curves in three-dimensional Lie groups. Let $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ be a framed curve. We define $(\overline{\eta}_1, \overline{\eta}_2) \in \Delta_G$ by

$$\begin{aligned} \overline{\eta}_1 &= \cos\phi\,\eta_1 - \sin\phi\,\eta_2, \\ \overline{\eta}_2 &= \sin\phi\,\eta_1 + \cos\phi\,\eta_2, \end{aligned} \tag{17}$$

where ϕ smooth function on I. Then,

$$(\gamma, \overline{\eta}_1, \overline{\eta}_2) : I \to G \times \Delta_G$$

is a framed curve. By using Eq. (17), we get

$$\begin{aligned}\dot{\overline{\eta}}_1 &= (m\cos\phi - n\sin\phi)\nu + (l-\delta_G-\phi')\sin\phi\eta_1 + (l-\delta_G-\phi')\cos\phi\eta_2,\\ \dot{\overline{\eta}}_2 &= (m\sin\phi + n\cos\phi)\nu + (-l+\delta_G+\phi')\cos\phi\eta_1 + (l-\delta_G-\phi')\sin\phi\eta_2.\end{aligned} \tag{18}$$

i. We assume that $m\sin\phi + n\cos\phi = 0$. Let $m = -p\cos\phi$ and $n = p\sin\phi$. Then, we have

$$\begin{aligned}&\dot{\nu} = -m\eta_1 - n\eta_2 = p\cos\phi\eta_1 - p\sin\phi\eta_2 = p\overline{\eta}_1,\\ &\dot{\overline{\eta}}_1 = -p\nu + (l-\delta_G-\phi')\overline{\eta}_2,\\ &\dot{\overline{\eta}}_2 = -(l-\delta_G-\phi')\overline{\eta}_1.\end{aligned} \tag{19}$$

Then, from Eq. (19) we get

$$\begin{pmatrix}\dot{\nu}\\ \dot{\overline{\eta}}_1\\ \dot{\overline{\eta}}_2\end{pmatrix} = \begin{pmatrix}0 & p & 0\\ -p & 0 & (q-\delta_G)\\ 0 & -(q-\delta_G) & 0\end{pmatrix}\begin{pmatrix}\nu\\ \overline{\eta}_1\\ \overline{\eta}_2\end{pmatrix}. \tag{20}$$

where $q = l - \phi'$.

Corollary 6. *If $\delta_G = 0$, the frame (20) corresponds to the Frenet-type frame of framed curves in three-dimensional Euclidean space [6].*

ii. Suppose that $l - \delta_G - \phi' = 0$. From, Eq. (18),

$$\begin{aligned}&\dot{\nu} = -\widetilde{m}\overline{\eta}_1 - \widetilde{n}\overline{\eta}_2,\\ &\dot{\overline{\eta}}_1 = \widetilde{m}\nu,\\ &\dot{\overline{\eta}}_2 = \widetilde{n}\nu.\end{aligned}$$

It can be seen from here

$$\begin{pmatrix}\dot{\nu}\\ \dot{\overline{\eta}}_1\\ \dot{\overline{\eta}}_2\end{pmatrix} = \begin{pmatrix}0 & -\widetilde{m} & -\widetilde{n}\\ \widetilde{m} & 0 & 0\\ \widetilde{n} & 0 & 0\end{pmatrix}\begin{pmatrix}\nu\\ \overline{\eta}_1\\ \overline{\eta}_2\end{pmatrix}, \tag{21}$$

where

$$\begin{pmatrix}\widetilde{m}\\ \widetilde{n}\end{pmatrix} = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}m\\ n\end{pmatrix}.$$

Corollary 7. *If $\delta_G = 0$, the frame (21) corresponds to the Bishop frame of framed curves in three-dimensional Euclidean space [1].*

Proposition 8. *Let $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ with $\{\nu, \eta_1, \eta_2\}$ and $(\gamma, \overline{\eta}_1, \overline{\eta}_1) : I \to G \times \Delta_G$ with adapted frame $\{\overline{\nu}, \overline{\eta}_1, \overline{\eta}_2\}$ be framed curves. Then we have*

$$\delta_G = \frac{1}{2}\langle[\nu, \eta_1], \eta_2\rangle = \frac{1}{2}\langle[\overline{\nu}, \overline{\eta}_1], \overline{\eta}_2\rangle.$$

where $\nu = \overline{\nu}$.

Proof. From Eq. (17), we get

$$\begin{aligned}\delta_G &= \frac{1}{2}\langle[\nu, \eta_1], \eta_2\rangle,\\ &= \frac{1}{2}\langle[\overline{\nu}, \cos\phi\overline{\eta}_1 + \sin\phi\overline{\eta}_2], -\sin\phi\overline{\eta}_1 + \cos\phi\overline{\eta}_2\rangle,\\ &= \frac{1}{2}\langle\cos\phi[\overline{\nu}, \overline{\eta}_1] + \sin\phi[\overline{\nu}, \overline{\eta}_2], -\sin\phi\overline{\eta}_1 + \cos\phi\overline{\eta}_2\rangle,\\ &= \frac{1}{2}(-\cos\phi\sin\phi\langle[\overline{\nu}, \overline{\eta}_1], \overline{\eta}_1\rangle + \cos^2\phi\langle[\overline{\nu}, \overline{\eta}_1], \overline{\eta}_2\rangle - \sin^2\phi\langle[\overline{\nu}, \overline{\eta}_2], \overline{\eta}_1\rangle + \cos\phi\sin\phi\langle[\overline{\nu}, \overline{\eta}_2], \overline{\eta}_2\rangle).\end{aligned}$$

Since $\langle[\overline{\nu}, \overline{\eta}_1], \overline{\eta}_1\rangle = \langle[\overline{\nu}, \overline{\eta}_2], \overline{\eta}_2\rangle = 0$ and $\langle[\overline{\nu}, \overline{\eta}_2], \overline{\eta}_1\rangle = -\langle[\overline{\nu}, \overline{\eta}_1], \overline{\eta}_2\rangle$, we have

$$\delta_G = \frac{1}{2}\langle[\overline{\nu}, \overline{\eta}_1], \overline{\eta}_2\rangle(\cos^2\phi + \sin^2\phi),$$
$$= \frac{1}{2}\langle[\overline{\nu}, \overline{\eta}_1], \overline{\eta}_2\rangle,$$

where $\nu = \overline{\nu}$. □

Remark 9. In [11], the equations of τ_G are given in some Lie group structures. We can also give the equations of δ_G using the $\{\nu, \eta_1, \eta_2\}$ orthonormal frame. Eventually, since $\{\nu, \eta_1, \eta_2\}$ is an orthonormal frame, these results coincide with the equations of τ_G.
i. If G is abelian, $\langle[\nu, \eta_1], \eta_2\rangle = \langle[\nu, \eta_2], \eta_1\rangle$, then $\delta_G = 0$.
ii. If G is $SO(3)$ three dimensional special orthogonal group with the bi-invariant metric defined by $K(X, Y) = -\frac{1}{2}trace(XY)$ for every $X, Y \in \mathfrak{so}(3)$. By considering $\mathfrak{so}(3)$ with $(\mathbb{R}^3, \wedge)$, we have

$$\delta_G = \frac{1}{2}\langle[\nu, \eta_1], \eta_2\rangle = \frac{1}{2}\langle\nu \wedge \eta_1, \eta_2\rangle = \frac{1}{2}\langle\eta_2, \eta_2\rangle = \frac{1}{2}$$

for a framed curve in $SO(3)$.
iii. If G is $SO(3)$, let Lie group homeomorphism $S^3 \cong SU(2)$, the basis $\varsigma_1, \varsigma_2, \varsigma_3$ of $su(2)$ given by

$$\varsigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \varsigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \varsigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Since $[\varsigma_1, \varsigma_2] = 2\varsigma_3, [\varsigma_2, \varsigma_3] = 2\varsigma_1, [\varsigma_3, \varsigma_1] = 2\varsigma_2$ [11], then we have

$$\delta_G = \frac{1}{2}\langle[\nu, \eta_1], \eta_2\rangle = \frac{1}{2}\langle 2\eta_2, \eta_2\rangle = 1.$$

for a framed curve in $SU(2)$.

5. Berry phase model of polarized light wave along an optical fiber in Lie groups

In 3D Riemann manifolds, an optical fiber can be thought of as a space curve. In this section, we give the Berry-phase model of the polarized light wave along an optical fiber, which is a singular curve in Lie groups. An optical fiber $(\gamma, \eta_1, \eta_2) : I \to G \times \Delta_G$ be a framed curve in three-dimensional Lie group G. On the other hand, let the electric field **E** be a not left-invariant vector field in Lie groups. Thus, the direction of **E** can be written as a linear combination of the $\{\gamma, \eta_1, \eta_2\}$ frame fields defined for the framed curves in Lie groups. Then, we denote the direction of the electric field **E** as follows

$$\dot{\mathbf{E}} = a_1\nu + a_2\eta_1 + a_3\eta_3, \tag{22}$$

where a_1, a_2, a_3 are smooth functions. Now, we investigate the direction of polarized light in three different categories: $\mathbf{E} \perp \nu$, $\mathbf{E} \perp \eta_1$ and $\mathbf{E} \perp \eta_2$.
Firstly, let us assume that

$$\langle\mathbf{E}, \nu\rangle = 0. \tag{23}$$

By differentiating Eq. (23) and by using Eq. (13) in Lie groups, we get

$$\langle\dot{\mathbf{E}}, \nu\rangle = m\langle\mathbf{E}, \eta_1\rangle + n\langle\mathbf{E}, \eta_2\rangle = a_1$$

Let's suppose that there is no loss of mechanism due to absorption in the optical fiber. So let $\langle\mathbf{E}, \mathbf{E}\rangle = c$, where c is a constant. Therefore, we have

$$\langle\dot{\mathbf{E}}, \mathbf{E}\rangle = 0,$$

and

$$a_2\langle\mathbf{E}, \eta_1\rangle + a_3\langle\mathbf{E}, \eta_2\rangle = 0.$$

If $\langle\mathbf{E}, \eta_1\rangle \neq 0$ and $\langle\mathbf{E}, \eta_2\rangle \neq 0$, we can write

$$a_2 = \lambda\langle\mathbf{E}, \eta_2\rangle, a_3 = -\lambda\langle\mathbf{E}, \eta_1\rangle.$$

Then, we have

$$\dot{\mathbf{E}} = (m\langle\mathbf{E}, \eta_1\rangle + n\langle\mathbf{E}, \eta_2\rangle)\nu - \lambda\langle\mathbf{E}, \eta_2\rangle\eta_1 + \lambda\langle\mathbf{E}, \eta_1\rangle\eta_2.$$

Since $\eta_1 \wedge \eta_2 = \nu$, we can write

$$\dot{\mathbf{E}} = (m\langle \mathbf{E}, \eta_1\rangle + n\langle \mathbf{E}, \eta_2\rangle)\nu + \lambda(\mathbf{E} \wedge \nu).$$

The expression $\mathbf{E} \wedge \nu$ represents the rotation about the unit vector ν. Let us assume that ν is parallel transported. So $\lambda = 0$ and we get

$$\dot{\mathbf{E}} = (m\langle \mathbf{E}, \eta_1\rangle + n\langle \mathbf{E}, \eta_2\rangle)\nu.$$

On the other hand, we can write

$$\mathbf{E} = \langle \mathbf{E}, \eta_1\rangle \eta_1 + \langle \mathbf{E}, \eta_2\rangle \eta_2. \tag{24}$$

By differentiating Eq. (24), we have

$$\begin{pmatrix} \dot{\omega_1} \\ \dot{\omega_2} \end{pmatrix} = \begin{pmatrix} 0 & l - \delta_G \\ -(l - \delta_G) & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

where $\omega_1 = \langle \mathbf{E}, \eta_1\rangle$ and $\omega_2 = \langle \mathbf{E}, \eta_2\rangle$. On the other hand, since $\langle \mathbf{E}, \mathbf{E}\rangle = c$ where c is a constant, $\mathbf{E}$ can be given as follows:

$$\mathbf{E} = \sin\Phi \eta_1 + \cos\Phi \eta_2. \tag{25}$$

By differentiating Eq. (25), we have

$$\dot{\mathbf{E}} = (m\sin\Phi + n\cos\Phi)\nu + (\Phi' - (l - \delta_G))(\cos\Phi\eta_1 - \sin\Phi\eta_2).$$

Then, since $\langle \mathbf{E}, \eta_1\rangle = \sin\Phi$ and $\langle \mathbf{E}, \eta_2\rangle = \cos\Phi$, we get

$$\dot{\mathbf{E}} = (m\langle \mathbf{E}, \eta_1\rangle + n\langle \mathbf{E}, \eta_2\rangle)\nu + (\Phi' - (l - \delta_G))\mathbf{E} \wedge \nu.$$

Therefore, $\Phi' = l - \delta_G$ in optical fiber. From Eq. (25), $\mathbf{E}$ is given by

$$\mathbf{E} = \sin\left(\int (l - \delta_G)ds\right)\eta_1 + \cos\left(\int (l - \delta_G)ds\right)\eta_2.$$

If we use Fermi-Walker derivative in Lie groups [30], we can write

$$\tilde{D}_\nu \mathbf{E} = D_\nu \mathbf{E} - \frac{1}{2}[\nu, \mathbf{E}] - \langle \nu, \mathbf{E}\rangle \dot{\nu} + \langle \dot{\nu}, \mathbf{E}\rangle \nu. \tag{26}$$

Since, $\langle \mathbf{E}, \nu\rangle = 0$ and $D_\nu \mathbf{E} - \frac{1}{2}[\nu, \mathbf{E}] = \dot{\mathbf{E}}$, we get

$$\tilde{D}_\nu \mathbf{E} = \dot{\mathbf{E}} + \langle -m\eta_1 - n\eta_2, \mathbf{E}\rangle \nu.$$

Therefore, since $\tilde{D}_\nu \mathbf{E} = 0$ we have

$$\dot{\mathbf{E}} = (m\langle \mathbf{E}, \eta_1\rangle + n\langle \mathbf{E}, \eta_2\rangle)\nu.$$

Then, we have the following corollary:

Corollary 10. *If* $\mathbf{E}$ *is Fermi-Walker parallel along an optical fiber in Lie groups, then the optical fiber is an* $\mathbf{E}_\nu$*-Rytov curve with the* $\langle \mathbf{E}, \nu\rangle = 0$ *condition Also,* $\mathbf{E}_\nu$*-Rytov curve is given by parametric representation*

$$\mathbf{E}_\nu = \gamma + \sin\left(\int (l - \delta_G)ds\right)\eta_1 + \cos\left(\int (l - \delta_G)ds\right)\eta_2.$$

Now, let us assume that

$$\langle \mathbf{E}, \eta_1\rangle = 0. \tag{27}$$

By differentiating Eq. (27) in Lie groups, we have

$$\langle \dot{\mathbf{E}}, \eta_1\rangle = -m\langle \mathbf{E}, \nu\rangle - (l - \delta_G)\langle \mathbf{E}, \eta_2\rangle = a_2 \tag{28}$$

By consider Eqs. (22) and (28), we get

$$a_2 = -m\langle \mathbf{E}, \nu\rangle - (l - \delta_G)\langle \mathbf{E}, \eta_2\rangle.$$

Similarly, since $\langle \dot{\mathbf{E}}, \mathbf{E}\rangle = 0$, we have

$$a_1\langle \mathbf{E}, \nu\rangle + a_3\langle \mathbf{E}, \eta_2\rangle = 0.$$

If we assume $\langle \mathbf{E}, \nu\rangle \neq 0$ and $\langle \mathbf{E}, \eta_2\rangle \neq 0$, we can write

$$a_1 = \lambda\langle \mathbf{E}, \eta_2\rangle, a_3 = -\lambda\langle \mathbf{E}, \nu\rangle.$$

Then, $\dot{\mathbf{E}}$ is given by

$$\dot{\mathbf{E}} = \lambda\langle \mathbf{E}, \eta_2\rangle\nu + (-m\langle \mathbf{E}, \nu\rangle - (l-\delta_G)\langle \mathbf{E}, \eta_2\rangle)\eta_1 + -\lambda\langle \mathbf{E}, \nu\rangle\eta_2.$$

Also, since $\eta_1 = \eta_2 \wedge \nu$, we get

$$\dot{\mathbf{E}} = (-m\langle \mathbf{E}, \nu\rangle - (l-\delta_G)\langle \mathbf{E}, \eta_2\rangle)\eta_1 + \lambda(\mathbf{E}\wedge \eta_1).$$

Let η_1 be parallel transported (i.e. $\lambda = 0$) and we have

$$\dot{\mathbf{E}} = (-m\langle \mathbf{E}, \nu\rangle - (l-\delta_G)\langle \mathbf{E}, \eta_2\rangle)\eta_1.$$

On the other hand, we can write

$$\mathbf{E} = \langle \mathbf{E}, \nu\rangle\nu + \langle \mathbf{E}, \eta_2\rangle\eta_2. \tag{29}$$

By differentiating Eq. (29), we get

$$\begin{pmatrix} \dot{\omega_3} \\ \dot{\omega_2} \end{pmatrix} = \begin{pmatrix} 0 & -n \\ n & 0 \end{pmatrix}\begin{pmatrix} \omega_3 \\ \omega_2 \end{pmatrix},$$

where $\omega_3 = \langle \mathbf{E}, \nu\rangle$ and $\omega_2 = \langle \mathbf{E}, \eta_2\rangle$. Similar to the condition $\langle \mathbf{E}, \nu\rangle$, we can write as follows:

$$\mathbf{E} = \sin\Phi\eta_2 + \cos\Phi\nu. \tag{30}$$

By differentiating Eq. (30), we get

$$\dot{\mathbf{E}} = (-m\cos\Phi - (l-\delta_G)\sin\Phi)\eta_1 + (\Phi' - n)(\cos\Phi\eta_2 - \sin\Phi\nu).$$

Since $\langle \mathbf{E}, \eta_2\rangle = \sin\Phi$ and $\langle \mathbf{E}, \nu\rangle = \cos\Phi$, we get

$$\dot{\mathbf{E}} = (-m\langle \mathbf{E}, \nu\rangle - (l-\delta_G)\langle \mathbf{E}, \eta_2\rangle)\eta_1 + (\Phi' - n)\mathbf{E}\wedge\eta_1.$$

Then, we have $\Phi' = n$ and

$$\mathbf{E} = \sin\left(\int nds\right)\eta_2 + \cos\left(\int nds\right)\nu.$$

Since the polarization vector $\mathbf{E}$ moves the parallel transport along the direction of η_1, we can replace η_1 with ν. If we use the Fermi-Walker derivative formula from Eq. (26), we have

$$\tilde{D}_{\eta_1}\mathbf{E} = \dot{\mathbf{E}} - \langle \eta_1, \mathbf{E}\rangle\dot{\eta}_1 + \langle \dot{\eta}_1, \mathbf{E}\rangle\eta_1.$$

Since, $\langle \mathbf{E}, \eta_1\rangle = 0$, we get

$$\tilde{D}_{\eta_1}\mathbf{E} = \dot{\mathbf{E}} + \langle m\nu + (l-\delta_G)\eta_2, \mathbf{E}\rangle\eta_1.$$

Therefore, since $\tilde{D}_{\eta_1}\mathbf{E} = 0$ we have

$$\dot{\mathbf{E}} = (-m\langle \mathbf{E}, \nu\rangle - (l-\delta_G)\langle \mathbf{E}, \eta_2\rangle)\eta_1.$$

Then, we have the following corollary:

Corollary 11. *If* $\mathbf{E}$ *is Fermi-Walker parallel along an optical fiber in Lie groups, then the optical fiber is an* $\mathbf{E}_{\eta_1}$*-Rytov curve with the* $\langle \mathbf{E}, \eta_1\rangle = 0$ *condition. Also,* $\mathbf{E}_{\eta_1}$*-Rytov curve is given by parametric representation*

$$\mathbf{E}_{\eta_1} = \gamma + \sin\left(\int nds\right)\eta_2 + \cos\left(\int nds\right)\nu.$$

Finally, let us suppose that

$$\langle \mathbf{E}, \eta_2 \rangle = 0. \tag{31}$$

By differentiating Eq. (31) and by using Eq. (13) in Lie groups, we have

$$\langle \dot{\mathbf{E}}, \eta_2 \rangle = -n\langle \mathbf{E}, \nu \rangle + (l - \delta_G)\langle \mathbf{E}, \eta_1 \rangle = a_3$$

By according to $\langle \dot{\mathbf{E}}, \mathbf{E} \rangle = 0$, we have

$$a_1\langle \mathbf{E}, \nu \rangle + a_2\langle \mathbf{E}, \eta_1 \rangle = 0.$$

If $\langle \mathbf{E}, \nu \rangle \neq 0$ and $\langle \mathbf{E}, \eta_1 \rangle \neq 0$, we can write

$$a_1 = \lambda\langle \mathbf{E}, \eta_1 \rangle, a_2 = -\lambda\langle \mathbf{E}, \nu \rangle.$$

Therefore, we have

$$\dot{\mathbf{E}} = \lambda\langle \mathbf{E}, \eta_1 \rangle \nu - \lambda\langle \mathbf{E}, \nu \rangle \eta_1 + (-n\langle \mathbf{E}, \nu \rangle + (l - \delta_G)\langle \mathbf{E}, \eta_1 \rangle)\eta_2.$$

Since $\nu \wedge \eta_1 = \eta_2$, we can write

$$\dot{\mathbf{E}} = (-n\langle \mathbf{E}, \nu \rangle + (l - \delta_G)\langle \mathbf{E}, \eta_1 \rangle)\eta_2 + \lambda(\mathbf{E} \wedge \eta_2).$$

$\mathbf{E} \wedge \eta_2$ represents the rotation about the unit vector η_2. Then, assume that η_2 is parallel transported. So $\lambda = 0$ and we have

$$\dot{\mathbf{E}} = (-n\langle \mathbf{E}, \nu \rangle + (l - \delta_G)\langle \mathbf{E}, \eta_1 \rangle)\eta_2.$$

Also, we can write

$$\mathbf{E} = \langle \mathbf{E}, \nu \rangle \nu + \langle \mathbf{E}, \eta_1 \rangle \eta_1. \tag{32}$$

By differentiating Eq. (32), we get

$$\begin{pmatrix} \dot{\omega_3} \\ \dot{\omega_1} \end{pmatrix} = \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix} \begin{pmatrix} \omega_3 \\ \omega_1 \end{pmatrix},$$

where $\omega_3 = \langle \mathbf{E}, \nu \rangle$ and $\omega_1 = \langle \mathbf{E}, \eta_1 \rangle$. Also, $\mathbf{E}$ can be given as follows:

$$\mathbf{E} = \sin\Phi \nu + \cos\Phi \eta_1. \tag{33}$$

By differentiating Eq. (33), we have

$$\dot{\mathbf{E}} = (-n\sin\Phi + (l - \delta_G)\cos\Phi)\eta_2 + (\Phi' + m)(\cos\Phi \nu - \sin\Phi \eta_1).$$

Since $\langle \mathbf{E}, \nu \rangle = \sin\Phi$ and $\langle \mathbf{E}, \eta_1 \rangle = \cos\Phi$, we get

$$\dot{\mathbf{E}} = (-n\langle \mathbf{E}, \nu \rangle + (l - \delta_G)\langle \mathbf{E}, \eta_1 \rangle)\eta_2 + (\Phi' + m)\mathbf{E} \wedge \eta_2.$$

Therefore, $\Phi' = -m$ in optical fiber. From Eq. (33), $\mathbf{E}$ is given by

$$\mathbf{E} = -\sin\left(\int mds\right)\nu + \cos\left(\int mds\right)\eta_1.$$

Since the polarization vector $\mathbf{E}$ moves the parallel transport along the direction of η_2, we can replace η_2 with ν. By according to Fermi-Walker derivative formula from Eq. (26), we have

$$\tilde{D}_{\eta_2}\mathbf{E} = \dot{\mathbf{E}} - \langle \eta_2, \mathbf{E} \rangle \dot{\eta}_2 + \langle \dot{\eta}_2, \mathbf{E} \rangle \eta_2.$$

Since, $\langle \mathbf{E}, \eta_2 \rangle = 0$, we get

$$\tilde{D}_{\eta_2}\mathbf{E} = \dot{\mathbf{E}} + \langle n\nu - (l - \delta_G)\eta_1, \mathbf{E} \rangle \eta_2.$$

Therefore, since $\tilde{D}_{\eta_2}\mathbf{E} = 0$ we have

$$\dot{\mathbf{E}} = (-n\langle \mathbf{E}, \nu \rangle + (l - \delta_G)\langle \mathbf{E}, \eta_1 \rangle)\eta_2.$$

Then, we have the following corollary:

Corollary 12. *If* $\mathbf{E}$ *is Fermi-Walker parallel along an optical fiber in Lie groups, then the optical fiber is an* $\mathbf{E}_{\eta_2}$*-Rytov curve with the* $\langle \mathbf{E}, \eta_2 \rangle = 0$ *condition. Also,* $\mathbf{E}_{\eta_2}$*-Rytov curve is given by parametric representation*

$$\mathbf{E}_{\eta_2} = \gamma - \sin\left(\int mds\right)\nu + \cos\left(\int mds\right)\eta_1.$$

Example 13. Let $(\gamma, \eta_1, \eta_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve [3] with

$$\begin{aligned}
\gamma(s) &= \left(\cos^3 s, \sin^3 s, \cos 2s\right), \\
\eta_1(s) &= (\sin s, \cos s, 0), \\
\eta_2(s) &= \left(\frac{4}{5}\cos s, -\frac{4}{5}\sin s, -\frac{3}{5}\right).
\end{aligned}$$

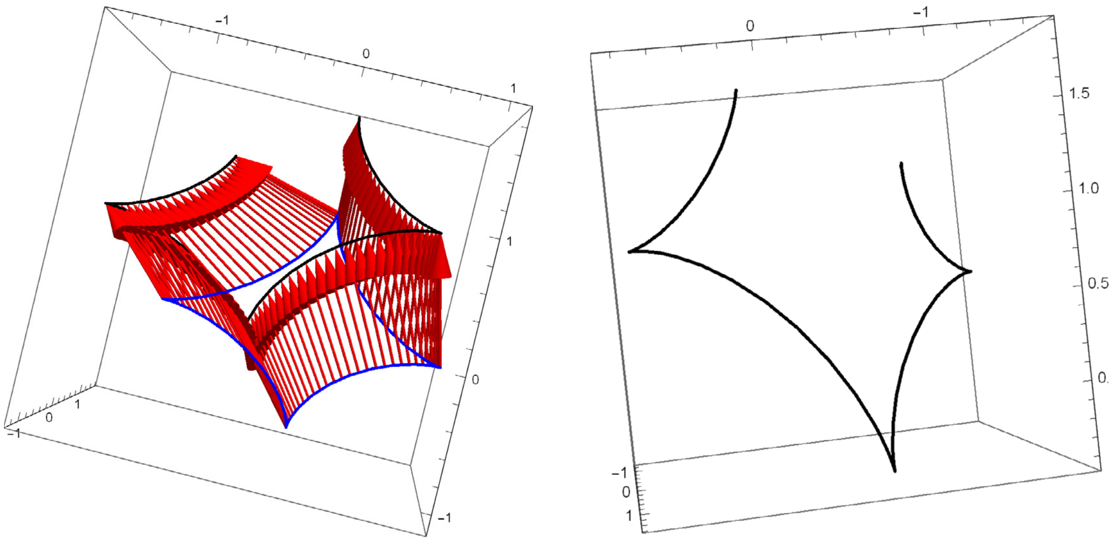
Since $\nu = \eta_1 \wedge \eta_2$, we get

$$\nu(s) = \left(-\frac{3}{5}\cos s, \frac{3}{5}\sin s, -\frac{4}{5}\right).$$

Since $(\gamma, \eta_1, \eta_2) : I \to \mathbb{R}^3 \times \Delta_2$, by according to Remark 9, $\delta_G = 0$. Then we can use the Frenet-Serret type frame formulas in three-dimensional Euclidean space. Then, framed curvatures are given by

$$\begin{aligned}
l(s) &= \langle \eta_1'(s), \eta_2(s) \rangle = \frac{4}{5}, \\
m(s) &= \langle \eta_1'(s), \nu(s) \rangle = -\frac{3}{5}, \\
n(s) &= \langle \eta_2'(s), \nu(s) \rangle = 0, \\
\alpha(s) &= \langle \gamma'(s), \nu(s) \rangle = 5\cos s \sin s.
\end{aligned}$$

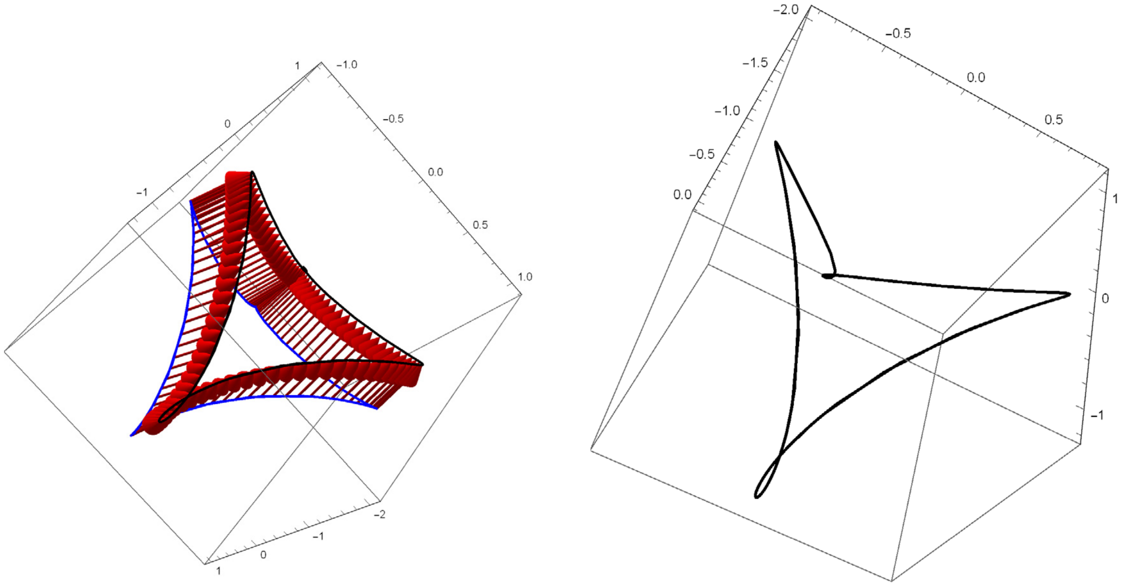
Obviously, since $n = 0$, $\{\nu, \eta_1, \eta_2\}$ is an adapted frame. Since $\alpha(0) = 0$, $\alpha(\frac{\pi}{2}) = 0$, $\alpha(\pi) = 0$ and $\alpha(\frac{3\pi}{2}) = 0$, $s = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ are singular points. In Fig. 1, the polarization vector and $\mathbf{E}_\nu$-Rytow curve in the case of $\mathbf{E} \perp \nu$, the polarization vector and $\mathbf{E}_{\eta_1}$-Rytow curve in the case of $\mathbf{E} \perp \eta_1$ in Fig. 2, and the polarization vector and $\mathbf{E}_{\eta_2}$-Rytow curve in the case of $\mathbf{E} \perp \eta_2$ in Fig. 3 are given.



(a) Polarization vectors along optical fiber γ (blue) when $\mathbf{E} \perp \nu$

(b) $\mathbf{E}_\nu$-Rytov curve

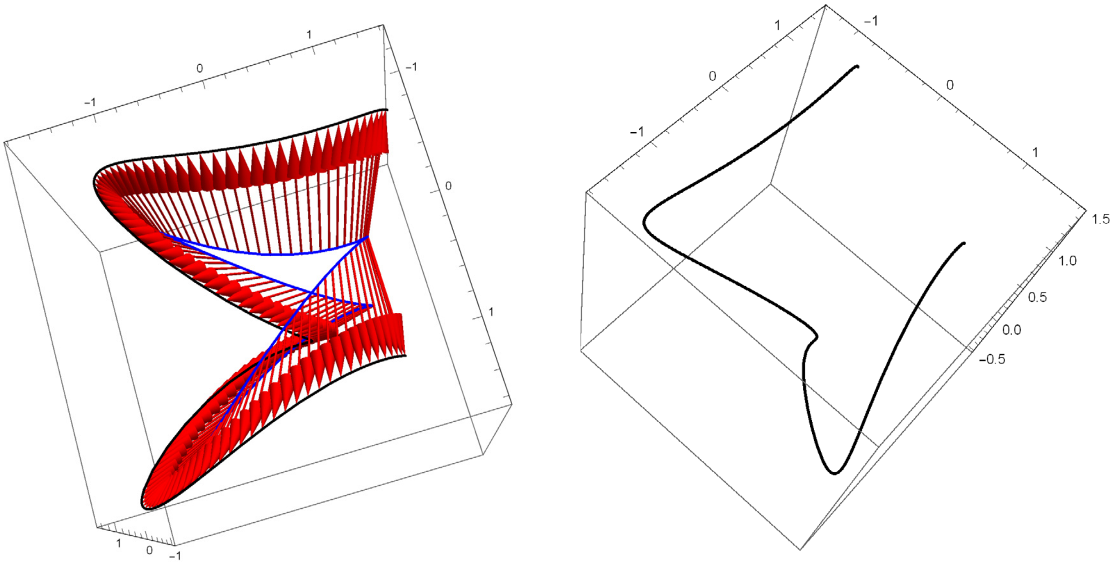
Fig. 1. The corresponding polarization vectors and Rytov curve along an optical fiber for the case $\mathbf{E} \perp \nu$.



(a) Polarization vectors along optical fiber γ (blue) when $\mathbf{E}\perp\eta_1$

(b) $\mathbf{E}_{\eta_1}$-Rytov curve

Fig. 2. The corresponding polarization vectors and Rytov curve along an optical fiber for the case $\mathbf{E}\perp\eta_1$.



(a) Polarization vectors along optical fiber γ (blue) when $\mathbf{E}\perp\eta_2$

(b) $\mathbf{E}_{\eta_2}$-Rytov curve

Fig. 3. The corresponding polarization vectors and Rytov curve along an optical fiber for the case $\mathbf{E}\perp\eta_2$.

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