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Spinor Frenet Equations in Three Dimensional Lie Groups

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Abstract. In this paper, we study spinor Frenet equations in three dimensional Lie groups with a bi-invariant metric. Also, we obtain spinor Frenet equations for some special cases of three dimensional Lie groups.

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1. Introduction

In differential geometry, curves can be thought as geometric sets of points of loci. Theory of curves is an important workframe in differential geometry. Geometric properties of a curve have been studied for a long time in Euclidean space and other spaces. One of the most important tools is the Frenet frame to analyze a curve, a moving frame that provides an orthonormal coordinate system at each point of the curve. We show this orthonormal system with $\{T, N, B\}$ in Euclidean 3-space, which are called tangent vector field, principal normal vector field and binormal vector field, respectively. For any curve, we can calculate the curvature and the torsion along the curve by using Frenet vectors. They are so useful for characterizations of special curves like general helices, slant helices, Mannheim curves, Bertrand curves etc. Characterizations of these curves have been studied in various spaces and we see the applications of special curves in nature, mechanics, computer aided design and computer graphics etc.

Spinors were discovered by Cartan [2]. Then were applied for studying the properties of the intrinsic angular momentum of the electron and other fermions by quantum mechanics. In three dimensions, spinors are used to describe the spin of the non-relativistic electron and other spin-1/2 particles. And also they are used to describe the state of relativistic many-particle systems in quantum field theory.

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Recently, Castillo and Barrales studied spinor formulation of the Frenet equations of a curve and obtained spinor equivalence of the Frenet equations for a curve [5]. Moreover Kişi et al. and Ünal et al. defined spinor formulations of the Frenet equations of a curve according to Darboux Frame and Bishop Frame, respectively (see [9,11]). Also, Erişir et al. expressed the theory of curves including a wide section of Lorentzian geometry in terms of spinors with two hyperbolic components [7].

Çiftçi has defined general helices and gave a generalization of Lancert's theorem in three dimesional Lie groups with a bi-invariant metric [4]. Yoon has examined general helices of $AW(k)-Type$ in Lie groups [12]. Then, in [10] Okuyucu et al. defined slant helices in three dimensional Lie groups with a bi-invariant metric. Gök et al. studied Mannheim partner curves in three dimensional Lie groups [8].

In this paper, we define spinor formulation of the Frenet equations for a curve in three dimensional Lie groups. Also we give this formulation for special cases of three dimensional Lie groups. This study may be useful for curves theory and physical applications of spinors in Lie groups.

2. Preliminaries

Let G be a Lie group with a bi-invariant metric $\langle\ ,\rangle$ and D be the Levi-Civita connection of G. If $\mathfrak{g}$ denotes the Lie algebra of G then we know that $\mathfrak{g}$ is isomorphic to T_eG where e is the neutral element of G. If $\langle\ ,\rangle$ is a bi-invariant metric on G then we have

$$\langle X,[Y,Z]\rangle = \langle [X,Y],Z\rangle$$

and

$$D_X Y = \frac{1}{2}[X,Y] \tag{2.1}$$

for all $X,Y,Z\in\mathfrak{g}$.

Let $\alpha\colon I\subset\mathbb{R}\to G$ be an arc-lenghted curve and $\{X_1,X_2,\ldots,X_n\}$ be an orthonormal basis of $\mathfrak{g}$. In this case, any two vector fields W and Z can be written as $W=\sum_{i=1}^{n} w_iX_i$ and $Z=\sum_{i=1}^{n} z_iX_i$ along the curve α , where $w_i\colon I\to\mathbb{R}$ and $z_i\colon I\to\mathbb{R}$ are smooth functions. Also the Lie bracket of two vector fields W and Z is given by

$$[W,Z]=\sum_{i=1}^{n} w_iz_i\,[X_i,X_j]$$

and the covariant derivative of W along the curve α, $D_{\alpha^{|}}W$, is given as follows

$$D_{\alpha^{|}}W=\dot{W}+\frac{1}{2}[T,W] \tag{2.2}$$

where $T=\alpha'$ and $\dot{W}=\sum_{i=1}^{n}\dot{w}_iX_i$ or $\dot{W}=\sum_{i=1}^{n}\frac{dw}{dt}X_i$. Note that if W is a left-invariant vector field to the curve α then $\dot{W}=0$ (see [3] for details).

Let $(T,N,B,\varkappa,\tau)$ denote the Frenet apparatus of the curve α in G, then it is easy to see that $\varkappa=\|\dot{T}\|$.

Definition 2.1. Let $\alpha\colon I\subset\mathbb{R}\to G$ be a parametrized curve with the Frenet apparatus $(T,N,B,\varkappa,\tau)$ then the followings hold:

$$\tau_G=\frac{1}{2}\left\langle [T,N]\,,B\right\rangle, \tag{2.3}$$

and

$$\tau_G=\frac{1}{2\varkappa^2\tau}\langle \ddot{T},[T,\dot{T}]\rangle+\frac{1}{4\varkappa^2\tau}\|[T,\dot{T}]\|^2$$

(see [4]).

Proposition 2.2. *Let $\alpha\colon I\subset\mathbb{R}\to G$ be an arc length parametrized curve with the Frenet apparatus $\{T,N,B\}$. Then the following equalities*

$$\begin{aligned}[T,N]&=\left\langle [T,N]\,,B\right\rangle B=2\tau_G B\\ [T,B]&=\left\langle [T,B]\,,N\right\rangle N=-2\tau_G N\end{aligned}$$

hold (*see* [10]).

In the following we give some properties about spinors:
We can represent a spinor

$$\phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

with

$$a+ib=\phi^t\sigma\phi,\quad c=-\widehat{\phi}^t\sigma\phi, \tag{2.4}$$

where a,b,c are vectors in $\mathbb{R}^3$ and the superscript t denotes transposition and $\widehat{\phi}$ is the mate (or conjugate [2])

$$\widehat{\phi}\equiv-\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \overline{\phi}=\begin{pmatrix}-\overline{\phi}_2\\ \overline{\phi}_1\end{pmatrix}$$

where the bar indicates complex conjugation. In addition, $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ is a vector whose cartesian components are the complex symmetrics 2×2 matrices

$$\sigma_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma_2=\begin{pmatrix}i&0\\0&i\end{pmatrix},\quad \sigma_3=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}. \tag{2.5}$$

Then, the vectors a,b and c are clearly given by

$$\begin{aligned}a+ib&=(\phi_1^2-\phi_2^2,i(\phi_1^2+\phi_2^2),-2\phi_1\phi_2)\\ c&=(\phi_1\overline{\phi}_2+\overline{\phi}_1\phi_2,i(\overline{\phi}_2\phi_1-\overline{\phi}_1\phi_2),|\phi_1|^2-|\phi_2|^2)\end{aligned}$$

where the vector $a+ib$ is an isotropic vector. From the above equations

$$\begin{aligned}\langle a+ib,c\rangle&=\langle(\phi_1^2-\phi_2^2,i(\phi_1^2+\phi_2^2),-2\phi_1\phi_2),\\ &(\phi_1\overline{\phi}_2+\overline{\phi}_1\phi_2,i(\overline{\phi}_2\phi_1-\overline{\phi}_1\phi_2),|\phi_1|^2-|\phi_2|^2)\rangle=0\end{aligned}$$

and we know that

$$\langle a+ib,c\rangle=\langle a,c\rangle+i\,\langle b,c\rangle\,.$$

If we combine the last two equations, it can be seen easily that the vectors a, b and c are mutually orthogonal vectors and $|a| = |b| = |c| = \overline{\phi}^t \phi$. Moreover $\langle a \times b, c \rangle = \det(a, b, c) > 0$.

Conversely, if the vectors a, b and c one another orthogonal vectors of same magnitude $(\langle a \times b, c \rangle > 0)$, then there is a spinor defined up to sign such that the Eq. (2.4) holds.

As it is mentioned above, for two arbitrary ψ and ϕ, following equalities exist

$$\overline{\psi^t \sigma \phi} = -\widehat{\psi}^t \sigma \widehat{\phi}$$
$$\widehat{a\psi + b\phi} = \overline{a}\widehat{\psi} + \overline{b}\widehat{\phi}$$
$$\widehat{\widehat{\phi}} = -\phi$$

where a and b are complex numbers [2,5].

The relations between spinors and orthonormal bases given in Eq. (2.4). The spinors ϕ and $-\phi$ correspond to the same ordered orthonormal bases $\{a, b, c\}$, with $|a| = |b| = |c|$ and $\langle a \times b, c \rangle > 0$. Moreover we can define different spinors by using the ordered triads $\{a, b, c\}, \{b, c, a\}$ and $\{c, a, b\}$.

The equation $\psi^t \sigma \phi = \phi^t \sigma \psi$ is satisfied for any pair of ψ and ϕ, since the matrices σ_i given in Eq. (2.5) are symmetric. For any non zero spinor ϕ, ϕ and $\widehat{\phi}$ are linearly independent.

3. Spinor Frenet Equations in a Three Dimensional Lie Groups

In this section we define the spinor Frenet equations for a curve in G with a bi-invariant metric $\langle\ ,\rangle$. Also we give the spinor Frenet equations in the special cases of G.

Proposition 3.1. *Let $\alpha\colon I \subset \mathbb{R} \to G$ be a curve with arc-length parameter s. Then by Eq. (2.2) and Proposition 2.2 the following equations for Frenet vectors of α hold*

$$\begin{bmatrix} \frac{dT}{ds} \\ \frac{dN}{ds} \\ \frac{dB}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \varkappa & 0 \\ -\varkappa & 0 & (\tau - \tau_G) \\ 0 & -(\tau - \tau_G) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where $\{T, N, B\}$ is Frenet frame, $\tau_G = \frac{1}{2}\langle [T, N], B\rangle$ and $\varkappa$, τ are curvatures of α in G, respectively [1].

According to the results presented in Sect. 2, there exists a spinor φ, such that

$$N + iB = \varphi^t \sigma \varphi, \quad T = -\widehat{\varphi}^t \sigma \varphi \tag{3.1}$$

with $\overline{\varphi^t}\varphi = 1$. Therefore the spinor φ corresponds with $\{N, B, T\}$. And the Frenet equations for each point of α must correspond to some expression for $\frac{d\varphi}{ds}$.

As $\{\varphi, \widehat{\varphi}\}$ is a basis for the two component spinors ($\varphi \neq 0$), there are two functions g and h, such that

$$\frac{d\varphi}{ds} = g\varphi + h\widehat{\varphi}, \tag{3.2}$$

where g and h are possibly complex-valued functions.

Differentiating the first equation in the Eq. (3.1) with respect to s and by using the Frenet equations, we get

$$\frac{dN}{ds} + i\frac{dB}{ds} = \frac{d}{ds}\left(\varphi^t \sigma \varphi\right)$$
$$-\varkappa T - i\left(\tau - \tau_G\right)\{N + iB\} = \left(\frac{d\varphi}{ds}\right)^t \sigma\varphi + \varphi^t\sigma\left(\frac{d\varphi}{ds}\right). \tag{3.3}$$

By considering the equations (3.1)–(3.3), we get

$$-\varkappa T - i\left(\tau - \tau_G\right)\{N + iB\} = 2g(N + iB) - 2h(T).$$

From the last equation we get

$$g = -i\frac{(\tau - \tau_G)}{2} \quad \text{and} \quad h = \frac{\varkappa}{2}.$$

Thus, we have proved the following theorem.

Theorem 3.2. *Let α be an arc-lenghted regular curve with the Frenet vector fields $\{T, N, B\}$ in G. If its two-component spinors φ represents the triad $\{N, B, T\}$, then the Frenet equations are equivalent to the single spinor' equation*

$$\frac{d\varphi}{ds} = -i\frac{(\tau - \tau_G)}{2}\varphi + \frac{\varkappa}{2}\widehat{\varphi},$$

where $\tau_G = \frac{1}{2}\left\langle [T, N], B\right\rangle$ and $\varkappa$, τ are curvatures of the curve α.

Now, we introduce this equation for special cases of three dimensional Lie group. In the following remark, we note that three dimensional Lie groups admitting bi-invariant metrics are S^3, SO^3 and Abelian Lie groups using the same notation as in [4,6] as follows:

Remark 3.3. Let G be a Lie group with a bi-invariant metric $\langle , \rangle$. Then the following equalities can be given in different Lie groups:

(i) If G is abelian group then $\tau_G = 0$.
(ii) If G is SO^3 then $\tau_G = \frac{1}{2}$.
(iii) If G is S^3 then $\tau_G = 1$ (see for details [4,6]).

Corollary 3.4. *Let α be an arc-lenghted regular curve with the Frenet vector fields $\{T, N, B\}$ in the Abelian Lie group G. If its two-component spinors φ represents the triad $\{N, B, T\}$, then the Frenet equations are equivalent to the single spinor equation*

$$\frac{d\varphi}{ds} = -i\frac{\tau}{2}\varphi + \frac{\varkappa}{2}\widehat{\varphi},$$

where $\varkappa$ and τ are curvatures of α.

Proof. If G is an Abellian Lie group, by using the above remark and the Theorem 3.2 we get the result. □

From the above corollary we can see easily that this study is a generalization of the spinor equation of the Frenet equations for a curve defined by del Castillo and Barrales [5] in Euclidean 3-space. Moreover, with a similar proof, we have the following two corollaries.

Corollary 3.5. *Let α be an arc-lenghted regular curve with the Frenet vector fields $\{T, N, B\}$ in the Lie group SO^3. If its two-component spinors φ represents $\{N, B, T\}$, then the Frenet equations are equivalent to the single spinor equation*

$$\frac{d\varphi}{ds} = -i\frac{\left(\tau - \frac{1}{2}\right)}{2}\varphi + \frac{\varkappa}{2}\widehat{\varphi},$$

where $\varkappa$ and τ are the curvatures of α.

Corollary 3.6. *Let α be an arc-lenghted regular curve with the Frenet vector fields $\{T, N, B\}$ in the Lie group S^3. If its two-component spinors φ represents $\{N, B, T\}$, then the Frenet equations are equivalent to the single spinor equation*

$$\frac{d\varphi}{ds} = -i\frac{(\tau - 1)}{2}\varphi + \frac{\varkappa}{2}\widehat{\varphi},$$

where $\varkappa$ and τ are the curvatures of α.

4. Applications

If we consider the curve α is a general helix in a three dimensional Lie group G, the harmonic curvature function $H = \frac{\tau - \tau_G}{\varkappa}$ is a constant of α (see [4,10]) and so we give the following corollary without proof.

Corollary 4.1. *Let α be an arc-lenghted regular curve with the Frenet vectors $\{T, N, B\}$ in G. If its two-component spinors φ represents the triad $\{N, B, T\}$ and the curve α is helix, then the Frenet equations are equivalent to the spinor equation*

$$\frac{d\varphi}{ds} = \frac{\varkappa}{2}\left(\widehat{\varphi} - ic\varphi\right)$$

where $H = \frac{\tau - \tau_G}{\varkappa}$ is a real constant.

Example 4.2. The helix

$$\alpha(s) = \left(a\cos\left(\frac{s}{a^2+b^2}\right), a\sin\left(\frac{s}{a^2+b^2}\right), \frac{b}{a^2+b^2}s\right)$$

is parametrized by arc-lengthed in $\mathbb{E}^3$. Then,

$$\varkappa(s) = \frac{a}{a^2+b^2}$$

and

$$\tau(s) = \frac{b}{a^2+b^2}$$

where $\varkappa$ and τ are the curvature functions of α. From Eqs. (2.1) and (2.3), we get $\tau_G = 0$. Thus, we obtain spinor equation

$$\frac{d\varphi}{ds} = -i\frac{b}{2\left(a^2+b^2\right)}\varphi + \frac{a}{2\left(a^2+b^2\right)}\widehat{\varphi}$$

where φ is two-component spinor.

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