



A Note on Evolution of Curves in the Minkowski Spaces

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Abstract. In this paper, we study the evolution of non-null curve in n -dimensional Minkowski Space. We express evolution equation of the Frenet frame by matrix equation. We obtain integrability conditions for the evolutions. Finally, we give examples of evolutions.

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1. Introduction

In differential geometry, there are countless study of curved spaces or shapes in which time did not play a role. Recently, contrary to what is known geometricians have made great improvement in understanding the curved spaces that involve in time. Among them, an evolving curve have in many applications such as engineering, computer vision [9, 11], computer animation [4] and even structural mechanics [13]. Also, there exists many physical applications (see [3, 5, 6]).

The evolving curve can be thought of as a family of curves parametrized by time. The time evolution of a curve generated by its corresponding flow so we shall also refer to curve evolutions as flows throughout this paper. Firstly, Kwon and Park studied inextensible flows of curves and developable surfaces in Euclidean 3-space [10]. Following them, inextensible flows of curves are studied in many different spaces. For example, Gürbüz have examined inextensible flows of spacelike, timelike and null curves in [7] and Uçum et al. have studied inextensible flows for null curves in E_2^4 [12] and Yıldız et al. have studied inextensible flows of curves according to Darboux frame in

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Euclidean 3-space [15] and they have investigated inextensible flows of curves in Euclidean n -space [14].

In [1] Abdel-All et al. obtained the evolution equation for the metric in Lemma 1. Also in Theorem 1, the authors gave the necessary and sufficient conditions for inextensible flow of curve. They obtained evolution equation of the orthonormal frame in Theorem 2. Furthermore in Theorem 3, an integrability condition is given for inextensible curve flow by using the obtained evolution matrix.

Our aim is to study concept of evolution as in [1] for non-null curves in the Minkowski space, E_1^n . We give necessary and sufficient conditions for inextensible flows of non-null curves in E_1^n . We express evolution equation of the Frenet frame by matrix equation. Further, we obtain integrability condition (zero curvature condition) for the considering model. Finally, we give examples of curve evolution and curvature evolution.

2. Preliminaries and Notations

Let E_1^n be the n -dimensional pseudo-Euclidean space with index 1 endowed with the indefinite inner product given by

$$\langle X, Y \rangle = -x_1y_1 + \sum_{i=2}^n x_iy_i,$$

where $X = (x_1, x_2, \dots, x_n), Y = (y_1, y_2, \dots, y_n)$ is the usual coordinate system. An arbitrary vector $X = (x_1, x_2, \dots, x_n)$ in E_1^n can have one of three Lorentzian causal characters; it can be spacelike if $\langle X, X \rangle > 0$ or $X = 0$, timelike if $\langle X, X \rangle < 0$ and null (lightlike) if $\langle X, X \rangle = 0$ and $X \neq 0$. The category into which a given tangent vector falls is called its causal character. These definitions can be generalized for curves as follows. A curve α in E_1^n is said to be spacelike if all of its velocity vectors α' are spacelike, similarly for timelike and null [2].

Let $\alpha : I \subset R \rightarrow E_1^n$ be non-null curve in E_1^n . A non-null curve $\alpha(s)$ is said to be a unit speed curve if $\langle \alpha'(s), \alpha'(s) \rangle = \varepsilon_0$, (ε_0 being $+1$ or -1 according to α is spacelike or timelike respectively). Let $\{V_1, V_2, \dots, V_n\}$ be the moving Frenet frame along the unit speed curve α , where $V_i (i = 1, 2, \dots, n)$ denote i^{th} Frenet vector fields and $k_i (i = 1, 2, \dots, n - 1)$ denotes the i th curvature function of the curve. Then the Frenet formulas are given as

$$\begin{aligned} V_1' &= k_1 V_2, \\ V_i' &= -\varepsilon_{i-2} \varepsilon_{i-1} k_{i-1} V_{i-1} + k_i V_{i+1}, \quad 1 < i < n, \\ V_n' &= -\varepsilon_{n-2} \varepsilon_{n-1} k_{n-1} V_{n-1}, \end{aligned}$$

where $\langle V_i, V_i \rangle = \varepsilon_{i-1} = \mp 1$ [8]. Moreover, these formulas can be stated as the follows matrix

$$V_s = QV \tag{2.1}$$

where

$$V = [V_1, V_2, \dots, V_n]^t, Q = \begin{bmatrix} 0 & k_1 & 0 & \dots & 0 & 0 \\ -\varepsilon_0 \varepsilon_1 k_1 & 0 & k_2 & \dots & 0 & 0 \\ 0 & -\varepsilon_1 \varepsilon_2 k_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & k_{n-1} \\ 0 & 0 & 0 & \dots & -\varepsilon_{n-2} \varepsilon_{n-1} k_{n-1} & 0 \end{bmatrix} \tag{2.2}$$

Unless otherwise stated we assume that.

$$\alpha : [0, l] \times [0, w) \longrightarrow E_1^n$$

is a one parameter family of smooth non-null curves in E_1^n , where l is the arclength of the initial curve. There u is the curve parametrization variable, $0 \leq u \leq l$ and t is time parameter. If the metric on the non-null curve α is given by $v(u, t) = \langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \rangle$, then the arclength variation of $\alpha(u, t)$ is

$$s(u, t) = \int_0^u \left\| \frac{\partial \alpha}{\partial u} \right\| du = \int_0^u \sqrt{|v(u, t)|} du.$$

The operator $\frac{\partial}{\partial s}$ is given by

$$\frac{\partial}{\partial s} = \frac{1}{\sqrt{|v|}} \frac{\partial}{\partial u}. \tag{2.3}$$

i.e., the arclength parameter is $ds = \sqrt{|v|} du$.

3. Evolution of Curves by Flow in E_1^n

Definition 1. Let α be a differentiable non-null curve and $\{V_1, V_2, \dots, V_n\}$ be the Frenet frame of α in E_1^n . Any flow of the non-null curve can be expressed as follows

$$\frac{\partial \alpha}{\partial t} = \sum_{i=1}^n f_i V_i, \tag{3.1}$$

where, f_i is the i th scalar speed of the non-null curve α along the V_i .

In E_1^n , the requirement that the non-null curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial \sqrt{|v|}}{\partial t} du = 0, \quad u \in [0, l]. \tag{3.2}$$

where $u \in [0, l]$ [10].

Definition 2. Let α be a non-null curve in E_1^n . A non-null curve evolution $\alpha(u, t)$ and its flow $\frac{\partial \alpha}{\partial t}$ are said to be inextensible if

$$\frac{\partial}{\partial t} \left\| \frac{\partial \alpha}{\partial u} \right\| = 0.$$

Before deriving the necessary and sufficient condition for inelastic non-null curve flow, we need the following lemma.

Lemma 1. *Let $\{V_1, V_2, \dots, V_n\}$ be the Frenet frame of non-null curve α and $\frac{\partial \alpha}{\partial t} = \sum_{i=1}^n f_i V_i$ be a smooth flow of α in E_1^n . Then, evolution equation of v is*

$$\frac{\partial v}{\partial t} = v_t = 2|v| \left(\varepsilon_0 \frac{\partial f_1}{\partial s} - \varepsilon_1 f_2 k_1 \right). \tag{3.3}$$

Proof. As $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ are commutative and $v = \langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \rangle$, we have

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t} \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \right\rangle \\ &= 2 \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial}{\partial u} \left(\sum_{i=1}^n f_i V_i \right) \right\rangle \\ &= 2|v| \left\langle V_1, \sum_{i=1}^n \frac{\partial f_i}{\partial s} V_i + \sum_{i=1}^n f_i \frac{\partial V_i}{\partial s} \right\rangle \\ &= 2|v| \left\langle V_1, \lambda V_1 + \sum_{i=2}^n A_i V_i \right\rangle, \end{aligned}$$

where

$$\begin{aligned} \lambda &= \left(\frac{\partial f_1}{\partial s} - \varepsilon_0 \varepsilon_1 k_1 f_2 \right), \\ A_i &= \frac{\partial f_i}{\partial s} + k_{i-1} f_{i-1} - \varepsilon_{i-1} \varepsilon_i k_i f_{i+1}; \quad i = 2, 3, \dots, n; \quad k_n = 0. \end{aligned} \tag{3.4}$$

Then,

$$\frac{\partial v}{\partial t} = 2\varepsilon_0 |v| \lambda.$$

□

Theorem 1. *The flow of the curve is inextensible if and only if*

$$\frac{\partial f_1}{\partial s} = \varepsilon_0 \varepsilon_1 f_2 k_1. \tag{3.5}$$

Proof. Assume that the non-null curve flow is inextensible. From Eqs. (3.2) and (3.3) it follows that

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{|v|_t}{2\sqrt{|v|}} du = 0, \quad u \in [0, l].$$

This clearly forces

$$\varepsilon_0 \frac{\partial f_1}{\partial s} - \varepsilon_1 f_2 k_1 = 0 \Rightarrow \frac{\partial f_1}{\partial s} = \varepsilon_0 \varepsilon_1 f_2 k_1.$$

On the contrary, assume that $\frac{\partial f_1}{\partial s} = \varepsilon_0 \varepsilon_1 f_2 k_1$. By applying $\frac{\partial f_1}{\partial s}$ into (3.3), we get $|v|_t = 0$ then $s_t = 0$. This means that the flow is inextensible. □

Theorem 2. *Let the flow of $\alpha(u, t)$ be $\frac{\partial \alpha}{\partial t} = \sum_{i=1}^n f_i V_i$. Then, the following statements hold:*

- (i) *Derivatives of the elements of Frenet frame with respect to time parameter t can be given as*

$$V_t = MV$$

where

$$M = \begin{bmatrix} 0 & M_{12} & \cdots & M_{1n} \\ -\varepsilon_0 \varepsilon_1 M_{12} & 0 & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\varepsilon_0 \varepsilon_{n-1} M_{1n} & -\varepsilon_1 \varepsilon_{n-1} M_{2n} & \cdots & 0 \end{bmatrix}, \tag{3.6}$$

$$M_{1i} = f_{i,s} + k_{i-1} f_{i-1} - \varepsilon_{i-1} \varepsilon_i k_i f_{i+1}; \quad i = 2, 3, \dots, n.$$

$$M_{\alpha\beta} = \frac{1}{k_{\alpha-1}} (M_{(\alpha-1)\beta,s} - \varepsilon_{\alpha-3} \varepsilon_{\alpha-2} k_{\alpha-2} M_{(\alpha-2)\beta} + k_{\beta-1} M_{(\alpha-1)(\beta-1)} - \varepsilon_{\beta-1} \varepsilon_{\beta} k_{\beta} M_{(\alpha-1)(\beta+1)}),$$

$$\alpha = 2, \dots, n-1; \quad \beta = 3, \dots, n; \quad \alpha < \beta; \quad k_0 = k_n = 0.$$

- (ii) *The evolutions of the curvatures are*

$$\begin{aligned} k_{1,t} &= M_{12,s} - \lambda k_1 - \varepsilon_1 \varepsilon_2 k_2 M_{13} \\ k_{\alpha,t} &= M_{\alpha(\alpha+1),s} - \lambda k_{\alpha} + \varepsilon_{(\alpha-2)} \varepsilon_{(\alpha-1)} k_{(\alpha-1)} M_{(\alpha-1)(\alpha+1)} \\ &\quad - \varepsilon_{\alpha} \varepsilon_{(\alpha+1)} k_{(\alpha+1)} M_{\alpha(\alpha+2)} \end{aligned} \tag{3.7}$$

Proof. Let α be the non-null curve. By taking derivative of (3.1) with respect to u , we get

$$\alpha_{tu} = \sqrt{|v|} \alpha_{ts} = \sqrt{|v|} \left(\lambda V_1 + \sum_{i=1}^n A_i V_i \right). \tag{3.8}$$

Since $\alpha_u = \sqrt{|v|} \alpha_s = \sqrt{|v|} V_1$, by taking derivative of α_u with respect to t , we have

$$\alpha_{ut} = \sqrt{|v|} \left(\frac{|v|_t}{2|v|} V_1 + V_{1,t} \right) \tag{3.9}$$

By using (3.4), (3.8), (3.9) and $\alpha_{tu} = \alpha_{ut}$, we have

$$V_{1,t} = \sum_{i=2}^n A_i V_i \tag{3.10}$$

By taking derivative of (3.10) with respect to u , then we get

$$\begin{aligned} V_{1,tu} &= \sqrt{|v|} \left((-\varepsilon_0 \varepsilon_1 k_1 A_2) V_1 + (A_{2,s} - \varepsilon_1 \varepsilon_2 k_2 A_3) V_2 \right. \\ &\quad \left. + \sum_{i=3}^n (k_{i-1} A_{i-1} + A_{i,s} - \varepsilon_{i-1} \varepsilon_i k_i A_{i+1}) V_i \right) \end{aligned} \tag{3.11}$$

and by taking derivative of $V_{1,u}$ with respect to t , we have

$$V_{1,ut} = \sqrt{|v|} \left(\left(\frac{|v|_t}{2|v|} k_1 + k_{1,t} \right) V_2 + k_1 V_{2,t} \right). \tag{3.12}$$

By using (3.11), (3.12) and $V_{1,tu} = V_{1,ut}$, we get

$$\begin{aligned} k_{1,t} &= A_{2,s} - k_1 \lambda - \varepsilon_1 \varepsilon_2 k_2 A_3 \\ V_{2,t} &= -\varepsilon_0 \varepsilon_1 A_2 V_1 + \sum_{i=3}^n B_i V_i \\ B_i &= \frac{1}{k_1} (A_{i,s} + k_{i-1} A_{i-1} - \varepsilon_{i-1} \varepsilon_i k_i A_{i+1}); \quad i = 3, \dots, n. \end{aligned} \tag{3.13}$$

Next, by taking derivative of (3.13) with respect to u , then we get

$$\begin{aligned} V_{2,tu} &= \sqrt{|v|} ((-\varepsilon_0 \varepsilon_1 A_{2,s}) V_1 + (-\varepsilon_0 \varepsilon_1 k_1 A_2 - \varepsilon_1 \varepsilon_2 k_2 B_3) V_2 \\ &\quad + (B_{3,s} - \varepsilon_2 \varepsilon_3 k_3 B_4) V_3 + \dots \\ &\quad + (k_{n-2} B_{n-2} + B_{n-1,s} - \varepsilon_{n-2} \varepsilon_{n-1} k_{n-1} B_n) V_{n-1} \\ &\quad + (k_{n-1} B_{n-1} + B_{n,s}) V_n). \end{aligned} \tag{3.14}$$

and by taking derivative of $V_{2,u}$ with respect to t , we have

$$\begin{aligned} V_{2,ut} &= \sqrt{|v|} \left(-(\varepsilon_0 \varepsilon_1 k_1 \lambda + \varepsilon_0 \varepsilon_1 k_{1,t}) V_1 - (\varepsilon_0 \varepsilon_1 k_1 A_2) V_2 \right. \\ &\quad + (k_2 \lambda + k_{2,t} - \varepsilon_0 \varepsilon_1 k_1 A_3) V_3 \\ &\quad \left. + k_2 V_3 - \varepsilon_0 \varepsilon_1 k_1 \sum_{i=4}^n A_i V_i \right). \end{aligned} \tag{3.15}$$

By using (3.14), (3.15) and $V_{2,tu} = V_{2,ut}$, we get

$$\begin{aligned} k_{2,t} &= B_{3,s} - k_2 \lambda - \varepsilon_2 \varepsilon_3 k_3 B_4 + \varepsilon_0 \varepsilon_1 k_1 A_3 \\ V_{3,t} &= -\varepsilon_0 \varepsilon_1 A_3 V_1 - \varepsilon_1 \varepsilon_2 B_3 V_2 + \sum_{i=4}^n C_i V_i \\ C_i &= \frac{1}{k_2} (B_{i,s} + k_{i-1} B_{i-1} - \varepsilon_{i-1} \varepsilon_i k_i B_{i+1} + \varepsilon_0 \varepsilon_1 k_1 A_i); \quad i = 4, \dots, n. \end{aligned} \tag{3.16}$$

Then, by taking derivative of $V_{3,t}$ with respect to u , we have

$$\begin{aligned} V_{3,tu} &= \sqrt{|v|} ((-\varepsilon_0 \varepsilon_2 A_{3,s} + \varepsilon_0 \varepsilon_2 k_1 B_3) V_1 - (\varepsilon_0 \varepsilon_2 k_1 A_3 - \varepsilon_1 \varepsilon_2 B_{3,s}) V_2 \\ &\quad - (\varepsilon_1 \varepsilon_2 k_2 B_3 + \varepsilon_2 \varepsilon_3 k_3 C_4) V_3 + (C_{4,s} - \varepsilon_3 \varepsilon_4 k_4 C_5) V_4 \\ &\quad + (k_4 C_4 + C_{5,s} - \varepsilon_4 \varepsilon_5 k_5 C_6) V_5 + \dots \\ &\quad + (k_{n-2} C_{n-2} + C_{n-1,s} - \varepsilon_{n-2} \varepsilon_{n-1} k_{n-1} C_n) V_{n-1} \\ &\quad + (k_{n-1} C_{n-1} + C_{n,s}) V_n) \end{aligned} \tag{3.17}$$

and by taking derivative of $V_{3,u}$ with respect to t , we have

$$\begin{aligned}
 V_{3,ut} = \sqrt{|v|} & \left(-(\varepsilon_0\varepsilon_2A_2)V_1 + (-\varepsilon_1\varepsilon_2B_{3,s} + \varepsilon_1\varepsilon_3k_3B_4 - \varepsilon_0\varepsilon_2k_1A_3)V_2 \right. \\
 & - (\varepsilon_1\varepsilon_2k_2B_3)V_3 + (-\varepsilon_1\varepsilon_2k_2B_4 + k_{3,t})V_4 \\
 & \left. + k_3V_{4,t} - \varepsilon_1\varepsilon_2k_2 \sum_{i=5}^n B_iV_i \right). \tag{3.18}
 \end{aligned}$$

Since $V_{3,tu} = V_{3,ut}$ and from (3.17), (3.18), we get

$$\begin{aligned}
 k_{3,t} &= C_{4,s} - \varepsilon_0k_3\lambda + \varepsilon_1\varepsilon_2k_2B_4 - \varepsilon_3\varepsilon_4k_4C_5 \\
 V_{4,t} &= -\varepsilon_0\varepsilon_3A_4V_1 - \varepsilon_1\varepsilon_3B_4V_2 - \varepsilon_2\varepsilon_3C_4V_3 + \sum_{i=5}^n D_iV_i \tag{3.19}
 \end{aligned}$$

$$D_i = \frac{1}{k_3} (C_{i,s} + k_{i-1}C_{i-1} - \varepsilon_{i-1}\varepsilon_i k_i C_{i+1} + \varepsilon_1\varepsilon_2k_2B_i); \quad i = 4, \dots, n.$$

By using the mathematical induction, we obtain

$$V_t = M \cdot V$$

where

$$M = \begin{bmatrix} 0 & A_2 & A_3 & A_4 & \cdots & A_n \\ -\varepsilon_0\varepsilon_1A_2 & 0 & B_3 & B_4 & \cdots & B_n \\ -\varepsilon_0\varepsilon_2A_3 & -\varepsilon_1\varepsilon_2B_3 & 0 & C_4 & \cdots & C_n \\ -\varepsilon_0\varepsilon_3A_4 & -\varepsilon_1\varepsilon_3B_4 & -\varepsilon_2\varepsilon_3C_4 & 0 & \cdots & D_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\varepsilon_0\varepsilon_{n-1}A_n & -\varepsilon_1\varepsilon_{n-1}B_n & -\varepsilon_2\varepsilon_{n-1}C_n & -\varepsilon_3\varepsilon_{n-1}D_n & \cdots & 0 \end{bmatrix}$$

and A_i, B_i, C_i and D_i can be written as follows

$$\begin{aligned}
 A_i &= M_{1i}, & 1 < i \leq n \\
 B_i &= M_{2i}, & 2 < i \leq n \\
 C_i &= M_{3i}, & 3 < i \leq n \\
 D_i &= M_{4i}, & 4 < i \leq n.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 M_{1i} &= A_i = f_{i,s} + k_{i-1}f_{i-1} - \varepsilon_{i-1}\varepsilon_i k_i f_{i+1}, \quad i = 2, 3, \dots, n. \\
 M_{\alpha\beta} &= \frac{1}{k_{\alpha-1}} (M_{(\alpha-1)\beta,s} - \varepsilon_{\alpha-3}\varepsilon_{\alpha-2}k_{\alpha-2}M_{(\alpha-2)\beta} + k_{\beta-1}M_{(\alpha-1)(\beta-1)} \\
 &\quad - \varepsilon_{\beta-1}\varepsilon_{\beta}k_{\beta}M_{(\alpha-1)(\beta+1)}), \\
 \alpha &= 2, \dots, n-1; \quad \beta = 3, \dots, n; \quad \alpha < \beta; \quad k_0 = k_n = 0
 \end{aligned}$$

and

$$M = \begin{bmatrix} 0 & M_{12} & \cdots & M_{1(n-1)} & M_{1n} \\ -\varepsilon_0\varepsilon_1M_{12} & 0 & \cdots & M_{2(n-1)} & M_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\varepsilon_0\varepsilon_{n-2}M_{1(n-1)} & -\varepsilon_1\varepsilon_{n-2}M_{2(n-1)} & \cdots & 0 & M_{(n-1)n} \\ -\varepsilon_0\varepsilon_{n-1}M_{1n} & -\varepsilon_1\varepsilon_{n-1}M_{2n} & \cdots & -\varepsilon_{n-2}\varepsilon_{n-1}M_{(n-1)n} & 0 \end{bmatrix}.$$

From first equation of (3.13), (3.16) and (3.19), we get

$$\begin{aligned} k_{1,t} &= M_{12,s} - \lambda k_1 - \varepsilon_1 \varepsilon_2 k_2 M_{13}, \\ k_{\alpha,t} &= M_{\alpha(\alpha+1),s} - \lambda k_\alpha + \varepsilon_{(\alpha-2)} \varepsilon_{(\alpha-1)} k_{(\alpha-1)} M_{(\alpha-1)(\alpha+1)} \\ &\quad - \varepsilon_\alpha \varepsilon_{(\alpha+1)} k_{(\alpha+1)} M_{\alpha(\alpha+2)}. \end{aligned}$$

□

Lemma 2. *If the flow of $\alpha(u, t)$ is inextensible, then partial differential equations (3.7) are*

$$\begin{aligned} k_{1,t} &= M_{12,s} - \varepsilon_1 \varepsilon_2 k_2 M_{13} \\ k_{\alpha,t} &= M_{\alpha(\alpha+1),s} + \varepsilon_{(\alpha-2)} \varepsilon_{(\alpha-1)} k_{(\alpha-1)} M_{(\alpha-1)(\alpha+1)} - \varepsilon_\alpha \varepsilon_{(\alpha+1)} k_{(\alpha+1)} M_{\alpha(\alpha+2)} \end{aligned}$$

Proof. If the flow of $\alpha(u, t)$ be inextensible, then $|v|_t = 0$, moreover $\lambda = 0$. By applying $\lambda = 0$ into (3.7), the lemma holds. □

Theorem 3. *The flow of $\alpha(u, t)$ is inextensible if and only if the following condition (zero curvature condition) holds*

$$Q_t - M_s + [Q, M] = 0 \tag{3.20}$$

where $[Q, M] = QM - MQ$ is the Lie bracket.

Proof. In order to prove the theorem, there is need some calculations. Considering the Eqs. (2.2) and (3.6). By taking derivative of $V_u = \sqrt{|v|}QV$ with respect to t , we obtain

$$V_{ut} = \sqrt{|v|} \left(\frac{|v|_t}{2|v|} Q + Q_t + QM \right) V \tag{3.21}$$

and taking derivative of V_t with respect to u , we have

$$V_{tu} = \sqrt{|v|} (M_s + MQ) V. \tag{3.22}$$

From (3.21) and (3.22), we obtain

$$V_{ut} - V_{tu} = \sqrt{|v|} \left(\frac{|v|_t}{2|v|} Q + Q_t - M_s + [Q, M] \right) V.$$

First, if the flow is an inextensible, then $|v|_t = 0$ and $\partial/\partial u$ and $\partial/\partial t$ are commutative, hence

$$Q_t - M_s + [Q, M] = 0.$$

Conversely, suppose the integrability condition is satisfied, i.e.,

$$Q_t - M_s + [Q, M] = 0.$$

From (2.2) and (3.6), we get

$$[Q, M] = \begin{bmatrix} 0 & \varepsilon_1 \varepsilon_2 k_2 M_{13} & \cdots & M_{1n,s} \\ -\varepsilon_0 \varepsilon_1 k_2 M_{13} & 0 & \cdots & M_{2n,s} \\ \vdots & \vdots & \vdots & \vdots \\ -\varepsilon_0 \varepsilon_{n-2} M_{1(n-1),s} & -\varepsilon_1 \varepsilon_{n-2} M_{2(n-1),s} & \cdots & -\varepsilon_{n-3} \varepsilon_{n-2} k_{n-2} M_{(n-2)n} \\ -\varepsilon_0 \varepsilon_{n-1} M_{1n,s} & -\varepsilon_1 \varepsilon_{n-1} M_{2n,s} & \cdots & 0 \end{bmatrix}. \tag{3.23}$$

By taking derivative of Q with respect to t and derivative of M with respect to s and using (3.7), we get $M_s - Q_t$ as follows

$$\begin{bmatrix} 0 & \lambda k_1 + \varepsilon_1 \varepsilon_2 k_2 M_{13} & \cdots & M_{1n,s} \\ -\varepsilon_0 \varepsilon_1 \lambda k_1 - \varepsilon_0 \varepsilon_2 k_2 M_{13} & 0 & \cdots & M_{2n,s} \\ -\varepsilon_0 \varepsilon_2 M_{13,s} & -\varepsilon_1 \varepsilon_2 \lambda k_2 + \varepsilon_0 \varepsilon_2 k_1 M_{13} - \varepsilon_1 \varepsilon_3 k_3 M_{24} & \cdots & M_{3n,s} \\ \vdots & \vdots & \vdots & \vdots \\ -\varepsilon_0 \varepsilon_{n-2} M_{1(n-1),s} & -\varepsilon_1 \varepsilon_{n-2} M_{2(n-1),s} & \cdots & \lambda k_{n-1} - \varepsilon_{n-3} \varepsilon_{n-2} k_{n-2} M_{(n-2)n} \\ -\varepsilon_0 \varepsilon_{n-1} M_{1n,s} & -\varepsilon_1 \varepsilon_{n-1} M_{2n,s} & \cdots & 0 \end{bmatrix} \quad (3.24)$$

From the Eqs. (3.23) and (3.24) we obtain

$$\begin{bmatrix} 0 & -\lambda k_1 & 0 & \cdots & 0 \\ \varepsilon_0 \varepsilon_1 \lambda k_1 & 0 & -\lambda k_2 & \cdots & 0 \\ 0 & \varepsilon_1 \varepsilon_2 \lambda k_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda k_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = 0.$$

From last equation it is easily seen that $\lambda = 0$ i.e., $v = \text{constant}$, so the flow is an inextensible. \square

Theorem 4. *Let the curve flow be inextensible. If the Q and M are abelian, then*

$$M_{(\alpha-1)(\alpha+1)} = 0; \quad \alpha = 2, 3, \dots, n - 1.$$

Proof. Let Q and M be abelian, so $[Q, M] = 0$, then (3.20) as follows

$$M_s - Q_t = 0. \quad (3.25)$$

Since the flow is inextensible, then

$$M_s - Q_t = \begin{bmatrix} 0 & \varepsilon_1 \varepsilon_2 k_2 M_{13} & \cdots & M_{1n,s} \\ -\varepsilon_0 \varepsilon_2 k_2 M_{13} & 0 & \cdots & M_{2n,s} \\ -\varepsilon_0 \varepsilon_2 M_{13,s} & \varepsilon_0 \varepsilon_2 k_1 M_{13} - \varepsilon_1 \varepsilon_3 k_3 M_{24} & \cdots & M_{3n,s} \\ \vdots & \vdots & \vdots & \vdots \\ -\varepsilon_0 \varepsilon_{n-2} M_{1(n-1),s} & -\varepsilon_1 \varepsilon_{n-2} M_{2(n-1),s} & \cdots & -\varepsilon_{n-3} \varepsilon_{n-2} k_{n-2} M_{(n-2)n} \\ -\varepsilon_0 \varepsilon_{n-1} M_{1n,s} & -\varepsilon_1 \varepsilon_{n-1} M_{2n,s} & \cdots & 0 \end{bmatrix} \quad (3.26)$$

By using (3.25) and (3.26), we have

$$M_{(\alpha-1)(\alpha+1)} = 0, \quad \alpha = 2, 3, \dots, n - 1. \quad \square$$

Example 1. If $f_1 = \sin 2s$ and $f_2 = \cos s$, then evolution equation of $k_{1,t}(s, t)$ is

$$k_{1,t} = -\cos s + \sin 2s k_{1,s} + 2 \cos 2s k_1.$$

Then solution of above PDE gives

$$k_1(s, t) = \frac{\sin s + (t + \frac{1}{2} \ln(\csc 2s - \cot 2s))}{\sin 2s}.$$

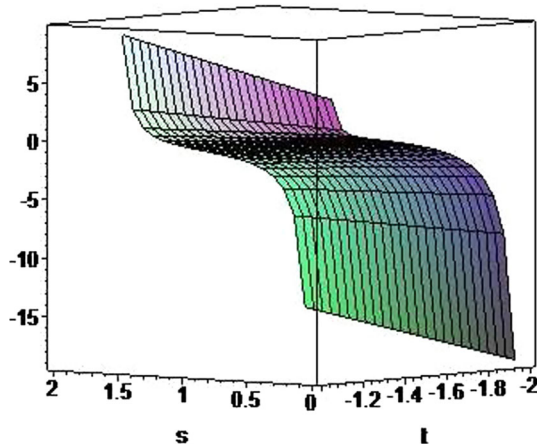


FIGURE 1. The time evolution for the curvature

The graph of curvature $k_{1,t}(s, t)$ of the curve in domain

$$D : \begin{cases} 0 \leq s \leq 2 \\ -2 \leq t \leq -1 \end{cases}$$

is given Fig. 1.

Example 2. If $f_1 = s$ and $f_2 = s$, then evolution equation of $k_{1,t}(s, t)$ is

$$k_{1,t} = sk_{1,s} + k_1.$$

Then solution of above PDE gives

$$k_1(s, t) = \frac{(t + \ln(s))^5}{s}.$$

The graph of curvature $k_{1,t}(s, t)$ of the curve in domain

$$D : \begin{cases} 2 \leq s \leq 10 \\ 2 \leq t \leq 10 \end{cases}$$

is given Fig. 2.

Example 3. If $f_1 = s^2$ and $f_2 = 0$, then evolution equation of $k_{1,t}(s, t)$ is

$$k_{1,t} = s^2k_{1,s} + 2sk_1.$$

Then solution of above PDE gives

$$k_1(s, t) = \frac{(ts - 1)^9}{s^{11}}.$$

Consider (3.1) and by using $k_{1,t}(s, t)$, we obtain PDE

$$\alpha(s, t) = \left(\frac{ts - 1}{s}\right)^9.$$

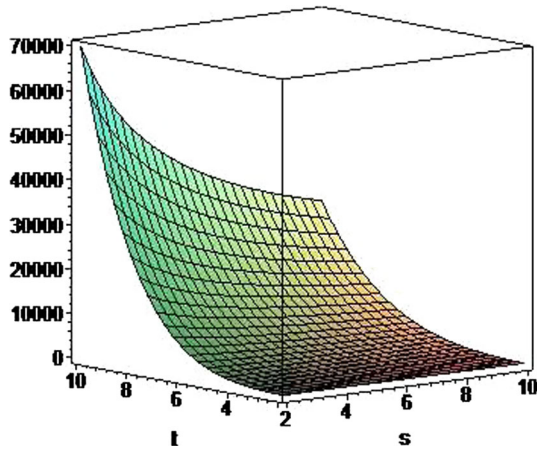


FIGURE 2. The time evolution for the curvature

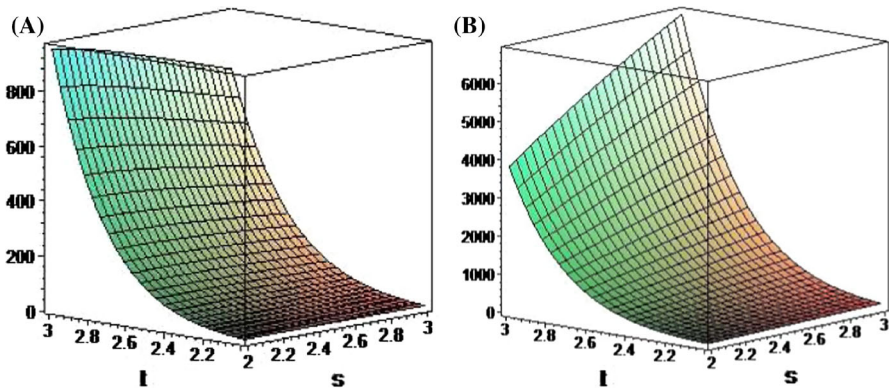


FIGURE 3. a The time evolution for the curvature. b The evolution of the curve

The graph of $k_{1,t}(s, t)$ and $\alpha(s, t)$ in domain

$$D : \begin{cases} 2 \leq s \leq 3 \\ 2 \leq t \leq 3 \end{cases}$$

are given Fig. 3a, b.

References

- [1] Abdel-All, N., Mohamed, S., Al-Dossary, M.: Evolution of generalized space curve as a function of its local geometry. *Appl. Math.* **5**, 2381–2392 (2014). doi:[10.4236/am.2014.515230](https://doi.org/10.4236/am.2014.515230)
- [2] Barros, M.: General helices and a theorem of Lancert. *Proc. AMS* **125**, 1503–1509 (1997)

- [3] Chirikjian, G., Burdick, J.: A modal approach to hyper-redundant manipulator kinematics. *IEEE Trans. Robot. Autom.* **10**, 343–354 (1994)
- [4] Desbrun, M., Cani, M.P.: Active implicit surface for animation. In: *Proceedings of Graphics Interface*. pp. 143–150 (1998)
- [5] Gage, M., Hamilton, R.S.: The heat equation shrinking convex plane curves. *J. Differ. Geom.* **23**, 69–96 (1986)
- [6] Grayson, M.: The heat equation shrinks embedded plane curves to round points. *J. Differ. Geom.* **26**, 285–314 (1987)
- [7] Gürbüzü, N.: Inextensible flows of spacelike, timelike and null curves. *Int. J. Contemp. Math. Sci.* **4**, 1599–1604 (2009)
- [8] İlarıslan, K.: Some special curves on non-Euclidean manifolds. Ph.D. Thesis, Ankara University, Graduate School of Natural and Applied Sciences (2002)
- [9] Kass, M., Witkin, A., Terzopoulos, D.: Snakes: active contour models. *Int. J. Comput. Vision.* **1**(4), 321–331 (1988)
- [10] Kwon, D.Y., Park, F.C., Chi, D.P.: Inextensible flows of curves and developable surfaces. *Appl. Math. Lett.* **18**, 1156–1162 (2005)
- [11] Lu, H.Q., Todhunter, J.S., Sze, T.W.: Congruence conditions for nonplanar developable surfaces and their application to surface recognition. *CVGIP Image Underst.* **56**, 265–285 (1993)
- [12] Uçum, A., Erdem, H.A., İlarıslan, K.: Inextensible flows of partially null and pseudo null curves in semi-Euclidean 4-space with index 2. *Novi Sad J. Math.* **46**(1), 115–129 (2016)
- [13] Unger, D.J.: Developable surfaces in elastoplastic fracture mechanics. *Int. J. Fract.* **50**, 33–38 (1991). doi:[10.1007/BF00032160](https://doi.org/10.1007/BF00032160)
- [14] Yıldız, Ö.G., Tosun, M., Karakuş, S.Ö.: A note on inextensible flows of curves in E^n . *Int. Electron. J. Geom.* **6**(2), 118–124 (2013)
- [15] Yıldız, Ö.G., Ersoy, S., Masal, M.: A note on inextensible flows of curves on oriented surface. *CUBO Math. J.* **16**(3), 11–19 (2014)

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