Received: 21 December 2020 | Revised: 22 April 2021 | Accepted: 21 May 2021

DOI: 10.1002/mma.7590

SPECIAL ISSUE PAPER

WILEY

Tubular surfaces associated with framed base curves in Euclidean 3-space

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Communicated by: M. Tosun

The aim of this paper is to examine tubular surfaces with framed base curves. We build the structure of the tubular surface which has singular points. In the second section, we give a brief exposition of framed base curves and framed surfaces, respectively. In the third section, our main results are stated and proved. Moreover, in this section, the normal of the tubular surface and its mean and Gauss curvatures are found, and the characterizations of the parameter curves on the surface are given. Finally, we have expressed the tubular surface with a framed base curve with examples.

KEYWORDS

asymptotic curve, frame base curve, Gaussian and mean curvatures, geodesic curve, line of curvature, tubular surfaces

MSC CLASSIFICATION

14B05; 53A05; 57R45

1 | INTRODUCTION

The subject of surfaces in differential geometry is a very important research subject, and it also has application areas such as physics, computer sciences, and design. In the studies about the surface, the moving frame of a regular space curve is generally taken into consideration. However, the moving frame of a regular space curve could not be defined at points where the first derivative of the curve is zero. In other words, the Frenet frame could not be defined at the singular points of a regular space curve. To solve this problem, a Frenet-type frame is defined along the regular curve under a certain condition. Basic concepts related to Frenet-type framed base curve and the existence and uniqueness conditions of the framed curve have been explored extensively in studies.[1–4] Framed surfaces were investigated using curves with singular points. A framed surface is a generalization of regular surfaces and frontals at least locally; see previous studies.[5–9] Tubular surfaces are special versions of canal surfaces which described as the envelope of the moving sphere of a variable radius. If the radius of the sphere forming the canal surface is constant, these canal surfaces are called the tubular surface. These surfaces, where their analytical and geometric properties are also examined, have added a different perspective to surface theory in Euclidean or different spaces.[10–16]

Our aim in this study is to investigate the geometric properties of a tubular surface associated with a framed base curve. Also, the characterizations of the parameter curves of the tubular surface are examined. All of these studies were done considering curves without a singular point. However, our study refers to curves with singular points, unlike others to make a difference to the previous research and to create new research resources.

2 | FRAMED CURVES AND SURFACES IN EUCLIDEAN SPACE

Let $\gamma : I \to \mathbb{R}^3$ be a smooth space curve with singular points in Euclidean 3-space. We will give the definition of the framed curve to investigate the properties of this curve. Let's define Δ_2, which is a three-dimensional smooth manifold, as follows:

$$\Delta_2 = \left\{ \mu = \left(\mu_{1,}\mu_2\right) \in \mathbb{R}^3 \times \mathbb{R}^3 : \left\langle \mu_{i,}\mu_j \right\rangle = \delta_{ij},\ \ i,j = 1,2 \right\}.$$

It is now easy to give the following framed curve definition with the help of γ and Δ_2.

Definition 1. $(\gamma, \mu) : I \to \mathbb{R}^3 \times \Delta_2$ is called a framed curve if $\langle \gamma', \mu_1 \rangle = \langle \gamma', \mu_2 \rangle = 0$ for each $z \in I$. Also, $\gamma : I \to \mathbb{R}^3$ is a framed base curve if there exist $\mu : I \to \Delta_2$ such that (γ, μ) is a framed curve.[1]

Let $\mu = \left(\mu_{1,}\mu_2\right) \in \Delta_2$; then, we define the unit vector ν such that $\nu = \mu_1 \wedge \mu_2$. Moreover, let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed curve. Then, we can write the Frenet–Serret-type formula of the framed curve as follows:

$$\begin{aligned} {\mu_1}'(z) &= l(z)\,\mu_2(z) + m(z)\,\nu(z), \\ {\mu_2}'(z) &= -l(z)\,\mu_1(z) + n(z)\,\nu(z), \\ \nu'(z) &= -m(z)\,\mu_1(z) - n(z)\,\mu_2(z). \end{aligned}$$

Also, there exists a smooth mapping $\alpha : I \to \mathbb{R}$ such that

$$\gamma'(z) = \alpha(z)\,\nu(z),$$

and $\alpha(z_0) = 0$ if and only if z_0 is a singular point of the curve γ. In addition, the functions $(l(z), m(z), n(z), \alpha(z))$ are called the curvature of framed curve γ. If $m(z) = n(z) = 0$, then $\nu'(z) = 0$. In this article, the case $\nu'(z) \neq 0$ is supposed.

If the curve γ has a singular point, the Frenet frame is not constructed along the curve γ. But under a certain condition, we can define Frenet-type frame along γ.

Definition 2. $\gamma : I \to \mathbb{R}^3$ is called a Frenet-type framed base curve if there exist a regular curve $\mathcal{T} : I \to S^2$ and a smooth function $\alpha : I \to \mathbb{R}$ such that $\gamma'(z) = \alpha(z)\,\mathcal{T}(z)$ for each $z \in I$. Then, we call $\alpha(z)$ as a speed function of $\gamma(z)$, and $\mathcal{T}(z)$ is a unit tangent vector.[17]

$\mathcal{T}(z)$, $\mathcal{N}(z) = \frac{\mathcal{T}'(z)}{\|\mathcal{T}'(z)\|}$, and $\mathcal{B}(z) = \mathcal{T}(z) \wedge \mathcal{N}(z)$ are unit tangent, unit principal normal, and unit binormal vectors of the curve $\gamma(z)$ with singular point, respectively. Then, there exists an orthonormal frame $\left\{\mathcal{T}(z), \mathcal{N}(z), \mathcal{B}(z)\right\}$. This frame is called the Frenet-type frame along γ, and we can give the Frenet-type framed formula as follows:

$$\left\{\mathcal{T}'(z), \mathcal{N}'(z), \mathcal{B}'(z)\right\} = \left\{\kappa(z)\,\mathcal{N}(z), -\kappa(z)\,\mathcal{T} + \tau(z)\,\mathcal{B}(z), -\tau(z)\,\mathcal{N}(z)\right\},$$

where $\kappa(z) = \|\mathcal{T}'(z)\|$ and $\tau(z) = \frac{\det\left(\mathcal{T}(z), \mathcal{T}'(z), \mathcal{T}''(z)\right)}{\|\mathcal{T}'(z)\|^2}$ are the curvature and the torsion of Frenet-type framed base curve γ, respectively. From here, we can say that γ is a framed base curve. Moreover, $\left(\gamma, \mathcal{N}, \mathcal{B}\right)$ is a framed curve with the curvature $(\tau(z), -\kappa(z), 0, \alpha(z))$.[17]

If $\psi_z(z,s).n(z,s) = 0$, $\psi_s(z,s).n(z,s) = 0$ for each $(z,s) \in U$, $(\psi, n) : U \to \mathbb{R}^3 \times S^2$ has a Legendre immersion. Moreover, if there exists $n : U \to S^2$ so that (ψ, n) is a Legendre immersion, $\psi : U \to \mathbb{R}^3$ is a front.

Let's assume that $\psi : U \to \mathbb{R}^3$ is a regular surface. In this case, $(\psi, n) : U \to \mathbb{R}^3 \times S^2$ is a Legendre immersion with $n = \frac{\psi_z \wedge \psi_s}{\|\psi_z \wedge \psi_s\|}$. If (ψ, n, r) is defined to be a framed surface, there is a smooth mapping $r : U \to S^2$, where $r = \frac{\psi_z}{\|\psi_z\|}$ or $r = \frac{\psi_s}{\|\psi_s\|}$.

By the help of these descriptions, we can construct the moving frame

$$\{n(z,s), r(z,s), d(z,s)\}$$

along $\psi(z,s)$, where $d(z,s) = n(z,s) \wedge r(z,s)$. Thanks to this frame, the following equations can be given:

$$\begin{bmatrix} \psi_z \\ \psi_s \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix},$$
$$\begin{bmatrix} n_z \\ r_z \\ d_z \end{bmatrix} = \begin{bmatrix} 0 & e_1 & f_1 \\ -e_1 & 0 & g_1 \\ -f_1 & -g_1 & 0 \end{bmatrix} \begin{bmatrix} n \\ r \\ d \end{bmatrix}, \tag{1}$$
$$\begin{bmatrix} n_s \\ r_s \\ d_s \end{bmatrix} = \begin{bmatrix} 0 & e_2 & f_2 \\ -e_2 & 0 & g_2 \\ -f_2 & -g_2 & 0 \end{bmatrix} \begin{bmatrix} n \\ r \\ d \end{bmatrix}.$$

Here $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R},\ i = 1, 2$, are smooth functions called basic invariants of the framed surface.[5]

Definition 3. We define a smooth mapping $D_F = (J_F, H_F, K_F) : U \to \mathbb{R}^3$ by

$$J_F = \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, K_F = \det \begin{bmatrix} e_1 & f_1 \\ e_2 & f_2 \end{bmatrix}, \tag{2}$$
$$H_F = -\frac{1}{2}\left\{ \det \begin{bmatrix} a_1 & f_1 \\ a_2 & f_2 \end{bmatrix} - \det \begin{bmatrix} a_1 & e_1 \\ a_2 & e_2 \end{bmatrix} \right\},$$

where $D_F = (J_F, H_F, K_F)$ is a curvature of the framed surface.[5]

Proposition 1. *Let $(\psi, n, r) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface and $s \in U$. Then, the surface $(\psi, n) : U \to \mathbb{R}^3 \times S^2$ is a Legendre immersion at the point s if and only if $D_F(s) \neq 0$. We denote $\Delta = \{(n, r) \in S^2 \times S^2 | \langle n, r \rangle = 0\}$.*

Let $\psi : U \to \mathbb{R}^3$ be a regular surface. So there exists $(n, r) : U \to \Delta$ so that (ψ, n, r) is a framed surface. We can express the coefficients of the fundamental forms, which can be calculated with the partial derivatives of the surface according to the z and s parameters, with the help of the basic invariants of the framed surface as follows:

$$E = a_1^2 + b_1^2,\ F = a_1 b_1 + a_2 b_2,\ G = a_2^2 + b_2^2, \tag{3}$$
$$L = -a_1 e_1 - b_1 f_1,\ M = -a_1 e_2 - b_1 f_2,\ N = -a_2 e_2 - b_2 f_2.$$

The Gauss and mean curvatures of a regular surface are expressed as

$$K = \frac{LN - M^2}{EG - F^2} \text{ and } H = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2} \tag{4}$$

in fundamental forms. On the other hand, they are also expressed as follows:

$$K = \frac{K_F}{J_F} \text{ and } H = \frac{H_F}{J_F} \tag{5}$$

in terms of curvatures of the framed surface (ψ, n, r).

3 | PROPERTIES OF TUBULAR SURFACES ASSOCIATED WITH FRENET-TYPE FRAMED BASE CURVES

In this section, we investigate the tubular framed base (TFB) surfaces associated with Frenet-type framed base curves and characterize the parametric curves on these surfaces. The canal surfaces are defined as the envelope of the moving sphere of variable radius. If the radius is taken as fixed, the canal surface is called the tubular surface. We do not restrict ourselves that $\gamma : I \to \mathbb{R}^3$ is a regular curve with the linearly independent condition, so that γ may have singular points. The parametric equation of the TFB surface with Frenet-type framed base curve which obtained $\{\mathcal{T}(z), \mathcal{N}(z), \mathcal{B}(z)\}$ along γ such that $\gamma'(z) = \alpha(z)\mathcal{T}(z)$ is given as follows:

$$\psi(z,s) = \gamma(z) + \rho\left(\cos s\mathcal{N} + \sin s\mathcal{B}\right), \tag{6}$$

where $s \in [0, 2\pi)$, ρ and $\gamma(z)$ are a positive radius and center curve of the tubular surface with the Frenet-type framed base curve, respectively.

Let $\psi : U \to \mathbb{R}^3$ be a regular surface. Then, there exists a Legendre immersion $(\psi, n) : U \to \mathbb{R}^3 \times S^2$, where $n = \frac{\psi_z \wedge \psi_s}{\|\psi_z \wedge \psi_s\|}$. There exists a smooth mapping $r : U \to S^2$ such that (ψ, n, r) is a TFB surface. In this study, we can take $r = \frac{\psi_s}{\|\psi_s\|}$. The framed base surface is a frontal. In other respect, the frontal is a framed base surface. In this article, we consider TFB surfaces as singular surfaces. The moving frame $\{n(z,s), r(z,s), d(z,s)\}$ along $\psi(z,s)$ is found as

$$\begin{aligned} &\{(0, \cos s, \sin s), (0, -\sin s, \cos s), (1,0,0)\}, \quad \text{if } \alpha(z) < \rho\kappa(z)\cos s, \\ &\{(0, -\cos s, -\sin s), (0, -\sin s, \cos s), (-1,0,0)\}, \quad \text{if } \alpha(z) > \rho\kappa(z)\cos s, \end{aligned}$$

or

$$\{n(z,s), r(z,s), d(z,s)\} = \{\mp(0, \cos s, \sin s), (0, -\sin s, \cos s), \mp(1,0,0)\}, \tag{7}$$

where $\alpha(z) \neq \rho\kappa(z)\cos s$. For abbreviation, we will not write z parameter anymore.

If we take the partial derivative of Equations (6) and (7) in terms of the parameters z and s and the obtained derivative equations is also substituted into Equation (1), we get the smooth basic invariants of a TFB surface as

$$\begin{aligned} &a_1 = \rho\tau, \quad a_2 = \rho, \quad b_1 = \mp(\alpha - \rho\kappa\cos s), \quad b_2 = 0, \\ &e_1 = \mp\tau, \quad e_2 = \mp 1, \quad f_1 = -\kappa\cos s, \quad f_2 = 0. \end{aligned}$$

If these basic invariants of the TFB surface are substituted into Equation (2), we obtain the curvatures of the TFB surface as

$$\begin{aligned} J_F &= \pm\rho(\alpha - \rho\kappa\cos s), \\ K_F &= \pm\kappa\cos s, \\ H_F &= \frac{-\rho\kappa\cos s \pm (\alpha - \rho\kappa\cos s)}{2}, \end{aligned}$$

where $\alpha - \rho\kappa\cos s \neq 0$ and $J_F \neq 0$. Thus, the curvature of the TFB surface $D_F = (J_F, K_F, H_F)$ is nonzero. By the help of curvatures, we can express the following result.

Corollary 1. *Let $(\psi, n, r) : U \to \mathbb{R}^3 \times \Delta$ be a framed TFB surface with $p \in U$.*

1. *(ψ, n, r) is an immersion (a regular surface) around p.*
2. *$(\psi, n) : U \to \mathbb{R}^3 \times S^2$ is a Legendre immersion around p.*

We can express the coefficients of the fundamental form with the help of the basic invariants of a framed surface (ψ, n, r) as follows.

$$\begin{aligned} E &= {a_1}^2 + {b_1}^2 = (\alpha - \rho\kappa\cos s)^2 + \rho^2\tau^2, \\ F &= a_1 b_1 + a_2 b_2 = \rho^2\tau, \\ G &= {a_2}^2 + {b_2}^2 = \rho^2, \end{aligned}$$

and

$$\begin{aligned} L &= -a_1 e_1 - b_1 f_1 = \pm\left(-\alpha\kappa\cos s + \rho\left(\kappa^2\cos s^2 + \tau^2\right)\right), \\ M &= -a_1 e_2 - b_1 f_2 = \pm\rho\tau, \\ N &= -a_2 e_2 - b_2 f_2 = \pm\rho. \end{aligned}$$

Moreover, we can express the Gauss and mean curvatures with the help of the curvature of the framed surface (ψ, n, r) as follows:

$$K = \frac{K_F}{J_F} = \frac{\kappa \cos s}{\rho (\alpha - \rho\kappa \cos s)},$$
$$H = \frac{H_F}{J_F} = \frac{-\rho\kappa \cos s \pm (\alpha - \rho\kappa \cos s)}{\pm 2\rho (\alpha - \rho\kappa \cos s)}.$$

Corollary 2. *Let (ψ, n, r) be a TFB surface. Then, (ψ, n, r) is a developable surface if and only if $\kappa \cos s = 0$.*

Corollary 3. *Let (ψ, n, r) be a TFB surface.*

1. *For $\alpha > \rho\kappa \cos s$, (ψ, n, r) is a minimal surface iff $\rho = \frac{\alpha}{2\kappa \cos s}$.*
2. *For $\alpha < \rho\kappa \cos s$, (ψ, n, r) is not a minimal surface.*

Let's give some theorems about the geometric interpretation of parametric curves of the TFB surfaces.

Theorem 1. *Let (ψ, n, r) be a TFB surface, then the following statements are satisfied.*

1. *The z-parameter curves of the TFB surface is a geodesic curve iff*

$$\rho = \frac{\alpha\kappa \sin s - \alpha'}{\kappa \sin s (\tau + \kappa \cos s) - \kappa' \cos s + \tau'}.$$

2. *The s-parameter curves of the TFB surface is geodesic curve.*

Proof. If the acceleration vector of a curve is perpendicular to the surface and therefore parallel to the normal vector of the surface, this curve is called the geodesic curve. We can prove the theorem in terms of this definition. The second-order partial derivatives of TFB surface are found in terms of $\mathcal{T}(z), \mathcal{N}(z), \mathcal{B}(z)$ as

$$\begin{aligned}
\psi_{zz} &= \left(\alpha' - \rho\kappa' \cos s + \rho\kappa\tau \sin s\right) \mathcal{T} \\
&\quad + \left(\kappa (\alpha - \rho\kappa \cos s) - \rho\tau' \sin s - \rho\tau^2 \cos s\right) \mathcal{N} \\
&\quad + \left(\rho\tau' \cos s - \rho\tau^2 \sin s\right) \mathcal{B}, \\
\psi_{zs} &= (\rho\kappa \sin s)\mathcal{T} - \rho\tau \left(\cos s\mathcal{N} + \sin s\mathcal{B}\right), \\
\psi_{ss} &= -\rho \left(\cos s\mathcal{N} + \sin s\mathcal{B}\right).
\end{aligned} \tag{8}$$

- Considering the normal vector of the TFB surface and Equation (8), we get

$$\begin{aligned}
n \wedge \psi_{zz} &= \mp \left(\kappa \sin s (-\alpha + \rho\kappa \cos s) + \rho\tau'\right) \mathcal{T} \\
&\quad \pm \left(\alpha' + \rho\left(-\kappa' \cos s + \kappa\tau \sin s\right)\right) \left(-\sin s\mathcal{N} + \cos s\mathcal{B}\right).
\end{aligned}$$

Since $\mathcal{T}$, $\mathcal{N}$, and $\mathcal{B}$ are linearly independent, there exist $z \wedge \psi_{zz} = 0$; from here, we can write

$$\begin{aligned}
\mp \left(\kappa \sin s (-\alpha + \rho\kappa \cos s) + \rho\tau'\right) &= 0, \\
\mp \left(\alpha' + \rho\left(-\kappa' \cos s + \kappa\tau \sin s\right)\right) \sin s &= 0, \\
\pm \left(\alpha' + \rho\left(-\kappa' \cos s + \kappa\tau \sin s\right)\right) \cos s &= 0.
\end{aligned}$$

If the necessary operations are performed in these last equations, we get

$$\rho = \frac{\alpha' - \alpha\kappa \sin s}{-\kappa' \cos s + \kappa (\tau + \kappa \cos s) \sin s + \tau'}.$$

- From Equations (7) and (8), we have $n \wedge \psi_{ss} = 0$. In that case, the s-parameter curves are geodesic curves. □

Theorem 2. *Let (ψ, n, r) be a TFB surface; then, the following statements are satisfied;*

- *The z-parameter curves of (ψ, n, r) are asymptotic curves, iff*

$$\rho = \frac{\alpha\kappa\cos s}{\kappa^2\cos s^2 + \tau^2}.$$

- *The s-parameter curves of (ψ, n, r) are not asymptotic curves.*

Proof. If the normal curvature of the parameter curves is zero everywhere, then the curves on the surface are called the asymptotic curves. Therefore, $\langle\psi_{zz}, n\rangle = 0$ and $\langle\psi_{ss}, n\rangle = 0$ must be provided for the z-parameter and s-parameter curves.

- From Equations (7) and (8), we know that

$$\langle\psi_{zz}, n\rangle = \mp\left(\kappa\cos s\left(-\alpha + \rho\kappa\cos s\right) + \rho\tau^2\right).$$

  From here, $\langle\psi_{zz}, n\rangle = 0$, if and only if

$$\rho = \frac{\alpha\kappa\cos s}{\kappa^2\cos s^2 + \tau^2}.$$

  So z-parameter curves of the TFB surface are asymptotic curves iff

$$\rho = \frac{\alpha\kappa\cos s}{\kappa^2\cos s^2 + \tau^2}.$$

- From Equations (7) and (8), we know that

$$\langle\psi_{ss}, n\rangle = \mp\rho.$$

  Since $\rho \neq 0$, $\langle\psi_{ss}, n\rangle \neq 0$. As a result, we call that s-parameter curves of the TFB surface are not asymptotic curve.

□

Theorem 3. *Let (ψ, n, r) be a TFB surface, the z-parameter and s-parameter curves of (ψ, n, r) are lines of curvature iff $\tau = 0$.*

Proof. If the parameter curves of a surface are lines of curvature, then $F = M = 0$. In that case, by considering the equations

$$F = \rho^2\tau \text{ and } M = \pm\rho\tau,$$

$\rho \neq 0$ and $\tau = 0$ are obtained. So the z-parameter and s-parameter curves of the TFB surface are lines of curvature.

□

Example 1. Let's sketch the graph of the tubular surfaces with the framed base curve defined by the parametric equation

$$\gamma(z) = \left(\cos z + (z+1)\sin z, \sin z - (z+1)\cos z, z + \frac{z^2}{2}\right).$$

The curve γ has a singular point at $z = -1$, so this curve is not a Frenet curve. The curve γ is a Frenet-type framed base curve with the mapping $(\mathcal{T}, \alpha) : \mathbb{R} \to S^2 \times \mathbb{R}$, such that

$$\mathcal{T}(z) = \frac{(\cos z, \sin z, 1)}{\sqrt{2}}, \alpha(z) = \sqrt{2}(z+1).$$

The normal and binormal vectors of the curve γ are obtained as

$$\mathcal{N}(z) = (-\sin z, \cos z, 0) \text{ and } \mathcal{B}(z) = \frac{(-\cos z, -\sin z, 1)}{\sqrt{2}}.$$

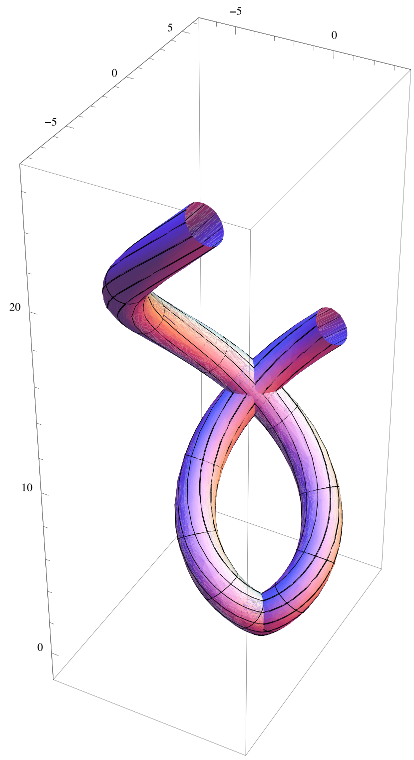

FIGURE 1 A TFB surface $\psi(z, s)$ with $z \in (-2\pi, 2\pi)$ and $s \in (-5, 5)$ [Colour figure can be viewed at wileyonlinelibrary.com]

Now, let's draw the graphs of the TFB surface with framed base curve whose equation is

$$\psi(z,s) = \left((1+z-\cos s)\sin z + \left(1 - \frac{\sin s}{\sqrt{2}}\right)\cos z, \right.$$
$$\left. (-1-z+\cos s)\cos z + \left(1 - \frac{\sin s}{\sqrt{2}}\right)\sin z, \frac{1}{2}z(2+z) + \frac{\sin s}{\sqrt{2}} \right),$$

where $\rho = 1$ (Figure 1).

Example 2. Let $\gamma : [0, 2\pi) \to \mathbb{R}^3$ be an astroid curve defined by the parametric equation

$$\gamma(z) = \left(\cos z^3, \sin z^3, \cos(2z)\right).$$

Then, the unit tangent vector and the speed function of the curve are

$$\mathcal{T}(z) = \frac{1}{5}(-3\cos z, 3\sin z, -4)$$

and

$$\alpha(z) = 5\sin z\cos z,$$

respectively, since the curve γ has the singular points at $z = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, the curve γ is not a Frenet curve. After the necessary calculations, we have the normal and binormal vectors of the curve γ as

$$\mathcal{N}(z) = (\sin z, \cos z, 0)$$

and

$$\mathcal{B}(z) = \frac{1}{5}(4\cos z, -4\sin z, -3).$$

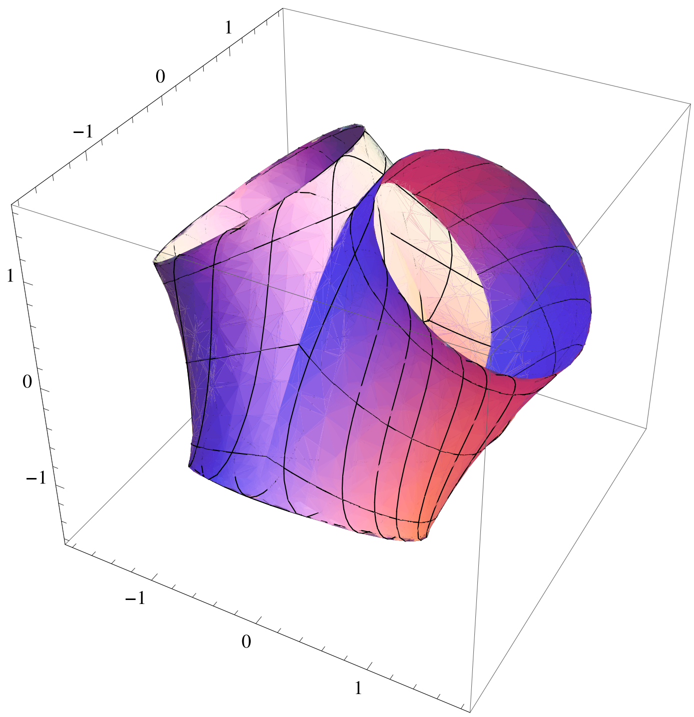

FIGURE 2 A TFB surface $\psi(z, s)$ with $z \in [0, 2\pi)$ and $s \in (-4, 4)$ [Colour figure can be viewed at wileyonlinelibrary.com]

Now, let's draw the graphs of the TFB surface with framed base curve whose equations are

$$\psi(z, s) = \Big(\cos z^3 + \sin z \cos s + \frac{4}{5} \cos z \sin s, \\ \cos s \cos z + \sin z^3 - \frac{4}{5} \sin s \sin z, \cos 2z - \frac{3 \sin s}{5} \Big),$$

where $\rho = 1$ (Figure 2).

4 | CONCLUSION

The paper is about the tubular surfaces associated with framed base curves in Euclidean 3-space. The TFB surface is considered as a singular surface. By the help of the smooth basic invariants of the TFB surface, the curvatures of this surface were determined. Later, the necessary conditions of the TFB surface to be or not to be minimal surface are obtained. Finally, necessary conditions are given for the parameter curves of the TFB surface to be asymptotic, geodesic, and line of curvature.

ACKNOWLEDGEMENTS

The authors thank the editors and referees for their valuable comments and contributions.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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How to cite this article: Eren K, Yıldız ÖG, Akyiğit M. Tubular surfaces associated with framed base curves in Euclidean 3-space. *Math Meth Appl Sci*. 2022;45(18):12110-12118. https://doi.org/10.1002/mma.7590