

# Generalized osculating-type ruled surfaces of singular curves

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In this study, we introduce generalized osculating-type ruled surfaces of special singular curves. We give some theories and results about the geometric structure of the surface. In addition, the singular point classes of the surface are examined, and the conditions for being a cross-cap surface are expressed. Generalized osculating-type ruled surface is considered as a framed surface and its basic invariants are found and some results are given. Finally, we give some examples and figures to support the theories.

## KEYWORDS

cross-cap, framed curve, framed surface, osculating-type ruled surface

## MSC CLASSIFICATION

58K05, 53A05, 57R45

## 1 | INTRODUCTION

The theory of ruled surfaces or developable surfaces is a fundamental topic in differential geometry, computer-aided design, and architecture.<sup>1-3</sup> A ruled surface is determined as a surface such that through each of its points passes a straight line contained in this surface. When the literature is examined, there are interesting studies on both developable and non-developable ruled surfaces. For example, developable surfaces can be divided into three classes: cylindrical surfaces, conical surfaces, and tangent developable surfaces. Basically, ruled surfaces whose anchor curve is a regular curve and whose direction vectors are tangent vector, principal normal vector, and binormal vector are called tangent developable surface, principal-normal ruled surface, and binormal ruled surface. Many geometrical and topological properties of such surfaces have taken place in the literature.<sup>4-7</sup> In addition to these, osculating-type ruled surfaces, which the direction vector is a combination of  $T$  and  $N$  vectors, were introduced by Kaya, Kahraman, and Önder.<sup>8</sup> Then, generalized normal ruled surfaces, whose direction vector is a combination of  $N$  and  $B$  vectors, are introduced by Kaya and Önder.<sup>9</sup> Also, generalized rectifying ruled surfaces, whose direction vector is a combination of  $T$  and  $B$  vectors, were introduced by Önder.<sup>10</sup> Obviously, the condition that the anchor curve is regular is used in the examination of generalized ruled surfaces. In this study, we will consider the anchor curves as singular curves, thanks to the framed curve and surface theory.

In Euclidean space, framed curves or framed surfaces are smooth curves and surfaces with singular points but defined by moving frames. Framed curves defined by Honda and Takahashi are a generalization of regular curves under linearly independent conditions.<sup>11</sup> There are many studies that have made important contributions to the literature for these special curves and surfaces, which eliminates the problem of not being able to define a moving frame on a curve or surface with singular points. Similar to the fundamental theorems of curvature given in regular curves, the existence and uniqueness conditions of framed curves are given in Fukunaga and Takahashi.<sup>12</sup> Moreover, there are studies such as framed rectifying curves,<sup>13,14</sup> framed normal curves,<sup>15</sup> framed helices,<sup>16</sup> framed slant helices,<sup>17</sup> and Bertrand and Mannheim

curves of framed curves,<sup>18</sup> which are special curves with singular points. Other studies on framed curves and framed surfaces can be viewed in previous studies.<sup>19–24</sup>

In this study, generalized osculating-type ruled surfaces of Frenet-type framed curves are introduced. These surfaces are a generalization of tangent developable surfaces and principally normal ruled surfaces for both singular and regular cases. First of all, basic definitions and theorems for framed curves and framed surfaces are given in Section 2. In Section 3, generalized osculating-type ruled surfaces of Frenet-type framed curves are introduced. These surfaces are examined for singularity, and cross-cap conditions are given. In addition, geometrical examinations of surfaces, that is, being cylindrical, developable, and stricture curves, are made. In Section 3.1, generalized osculating-type ruled surfaces of Frenet-type framed curves are treated as a framed surface. The fundamental invariants and curvatures of the surface are calculated. In particular, if the surface is regular, the relations with Gaussian and mean curvatures are minimal, and flat conditions are given. In Section 4, some examples and visuals that support the theory and are examples of examining singularity classes are given.

## 2 | FRAMED CURVES AND FRAMED SURFACES IN EUCLIDEAN SPACE

Framed curves and framed surfaces in Euclidean space are special curves and surfaces with singular points that can be defined by a moving frame. Let us give the definition of framed curve defined by Honda and Takahashi<sup>11</sup>:

**Definition 1.**  $(\gamma, \delta_1, \delta_2) : I \rightarrow \mathbb{R}^3 \times \Delta_2$  is called a framed curve if  $\langle \gamma'(s), \delta_i(s) \rangle = 0$  for every  $s \in I$  and  $i = 1, 2$  where the three-dimensional manifold set

$$\Delta_2 = \{ \delta = (\delta_1, \delta_2) \in \mathbb{R}^3 \times \mathbb{R}^3, \langle \delta_i, \delta_i \rangle = \langle \delta_j, \delta_j \rangle = 1, \langle \delta_i, \delta_j \rangle = 0 \}$$

The Frenet-Serret type formulas are given by

$$\begin{pmatrix} v'(s) \\ \delta_1'(s) \\ \delta_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & -m(s) & -n(s) \\ m(s) & 0 & l(s) \\ n(s) & -l(s) & 0 \end{pmatrix} \begin{pmatrix} v(s) \\ \delta_1(s) \\ \delta_2(s) \end{pmatrix}.$$

where a unit vector  $v = \delta_1 \times \delta_2$ .  $s_0$  is a singular point of  $\gamma$  if and only if  $\alpha(s_0) = 0$ . So  $(l, m, n, \alpha)$  is the framed curvature of the framed curve.<sup>11</sup>

If  $\gamma$  has a singular point, we cannot construct the Frenet frame along  $\gamma(s)$ . However, the Frenet-type frame is defined along  $\gamma(s)$  under the  $m^2 + n^2 \neq 0$  condition. These curves are called Frenet-type framed curves and are defined by Honda as follows<sup>16</sup>:

**Definition 2.**  $\gamma : I \rightarrow \mathbb{R}^3$  is a Frenet-type framed base curve if  $\mathcal{T} : I \rightarrow S^2$  is a regular spherical curve and  $\alpha : I \rightarrow \mathbb{R}$  is a smooth function where  $\gamma'(s) = \alpha(s)\mathcal{T}(s)$ . Then,  $\mathcal{T}(s)$  is called a unit tangent vector and  $\alpha(s)$  is called a speed function of  $\gamma(s)$ . The unit principal normal vector  $\mathcal{N}(s) = \mathcal{T}'(s) / \|\mathcal{T}'(s)\|$ , and a unit binormal vector  $\mathcal{B}(s) = \mathcal{T}(s) \times \mathcal{N}(s)$  of  $\gamma(s)$ . Then, an orthonormal frame  $\{\mathcal{T}(s), \mathcal{N}(s), \mathcal{B}(s)\}$  is constructed and this frame is called Frenet-type framed base curve along  $\gamma(s)$ . The following Frenet-Serret type formula can be given:

$$\begin{pmatrix} \mathcal{T}'(s) \\ \mathcal{N}'(s) \\ \mathcal{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathcal{T}(s) \\ \mathcal{N}(s) \\ \mathcal{B}(s) \end{pmatrix} \quad (1)$$

where  $\kappa(s) = \|\mathcal{T}'(s)\|$  is curvature and  $\tau(s) = \frac{\det(\mathcal{T}(s), \mathcal{T}'(s), \mathcal{T}''(s))}{\|\mathcal{T}'(s)\|^2}$  is torsion of  $\gamma$ .<sup>16</sup>

Since the normal vector of a surface with singular points cannot be found, framed surfaces can be treated as smooth surfaces defined by a moving frame with singular points:

**Definition 3.**  $(\Psi, \Psi_1, \Psi_2) : U \rightarrow \mathbb{R}^3 \times \Delta$  is called a framed surface if  $\langle \Psi_s(s, u), \Psi_1(s, u) \rangle = 0$  and  $\langle \Psi_u(s, u), \Psi_1(s, u) \rangle = 0$  for every  $(s, u) \in U$  where  $\Psi_s(s, u) = \frac{\partial \Psi}{\partial s}(s, u)$  and  $\Psi_u(s, u) = \frac{\partial \Psi}{\partial u}(s, u)$ .  $\Psi : U \rightarrow \mathbb{R}^3$  is called a framed base surface if there exists  $(\Psi_1, \Psi_2) : U \rightarrow \Delta$  such that  $(\Psi, \Psi_1, \Psi_2)$  framed surface.<sup>22</sup>

Also, the  $\{\Psi_1(s, u), \Psi_2(s, u), \Psi_3(s, u)\}$  is a moving frame along  $\Psi(s, u)$  where  $\Psi_3(s, u) = \Psi_1(s, u) \wedge \Psi_2(s, u)$ . The basic invariants functions of framed surface  $(\Psi, \Psi_1, \Psi_2)$  are given by

$$\begin{pmatrix} \Psi_s \\ \Psi_u \end{pmatrix} = \begin{pmatrix} \varepsilon_{01} & \varepsilon_{02} \\ \varepsilon_{03} & \varepsilon_{04} \end{pmatrix} \begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix}, \quad (2)$$

$$\begin{pmatrix} \Psi_{1s} \\ \Psi_{2s} \\ \Psi_{3s} \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_{11} & \varepsilon_{12} \\ -\varepsilon_{11} & 0 & \varepsilon_{13} \\ -\varepsilon_{12} & -\varepsilon_{13} & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \quad (3)$$

$$\begin{pmatrix} \Psi_{1u} \\ \Psi_{2u} \\ \Psi_{3u} \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_{21} & \varepsilon_{22} \\ -\varepsilon_{21} & 0 & \varepsilon_{23} \\ -\varepsilon_{22} & -\varepsilon_{23} & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \quad (4)$$

where  $\varepsilon_{ij} : U \rightarrow \mathbb{R}, i = 0, 1, 2, j = 1, 2, 3, 4$  are smooth functions. The above matrices are denoted by  $F_0, F_1, F_2$ , respectively. The matrices  $(F_0, F_1, F_2)$  are called basic invariants of the framed surface. Also,  $(s, u)$  is a singular point of  $\Psi$  if and only if  $\det F_0(s, u) = 0$ . By the integrability conditions of the framed surface, then  $\varepsilon_{01}\varepsilon_{21} + \varepsilon_{02}\varepsilon_{22} = \varepsilon_{03}\varepsilon_{11} + \varepsilon_{04}\varepsilon_{12}$ . The  $C_f = (J_f, K_f, H_f) : U \rightarrow \mathbb{R}^3$  is called a curvature of the framed surface where

$$\begin{aligned} J_\Psi &= \det \begin{pmatrix} \varepsilon_{01} & \varepsilon_{02} \\ \varepsilon_{03} & \varepsilon_{04} \end{pmatrix}, K_\Psi = \det \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}, \\ H_\Psi &= -\frac{1}{2} \left\{ \det \begin{pmatrix} \varepsilon_{01} & \varepsilon_{12} \\ \varepsilon_{03} & \varepsilon_{22} \end{pmatrix} - \det \begin{pmatrix} \varepsilon_{02} & \varepsilon_{11} \\ \varepsilon_{04} & \varepsilon_{21} \end{pmatrix} \right\} \end{aligned} \quad (5)$$

If  $\Psi : U \rightarrow \mathbb{R}^3$  is a regular surface, the first and second fundamental forms are given by

$$\begin{aligned} E &= \varepsilon_{01}^2 + \varepsilon_{02}^2, F = \varepsilon_{01}\varepsilon_{02} + \varepsilon_{03}\varepsilon_{04}, G = \varepsilon_{03}^2 + \varepsilon_{04}^2 \\ L &= -\varepsilon_{01}\varepsilon_{11} - \varepsilon_{02}\varepsilon_{12}, M = -\varepsilon_{01}\varepsilon_{21} - \varepsilon_{02}\varepsilon_{22}, N = -\varepsilon_{03}\varepsilon_{21} - \varepsilon_{04}\varepsilon_{22} \end{aligned} \quad (6)$$

**Proposition 1** (Fukunaga and Takahashi<sup>22</sup>). *Let  $\Psi : U \rightarrow \mathbb{R}^3$  is a regular surface. The Gauss curvature and the mean curvature of the framed surface  $(\Psi, \Psi_1, \Psi_2)$  are given by*

$$K = \frac{K_\Psi}{J_\Psi}, H = \frac{H_\Psi}{J_\Psi}.$$

### 3 | GENERALIZED OSCULATING-TYPE RULED SURFACES OF FRENET-TYPE FRAMED BASE CURVES

The aim of this section is to determine the generalized osculating-type ruled surface by using singular curves. Additionally, we investigate geometric properties and the singular point classes of generalized osculating-type ruled surface.

**Definition 4.** Let  $\gamma(s) : I \rightarrow \mathbb{R}^3$  a Frenet-type framed base curve. The ruled surface  $\Psi_{(\gamma, \varrho_n)}(s, u) = I \times \mathbb{R} \rightarrow \mathbb{R}^3$  defined by

$$\Psi_{(\gamma, \varrho_n)}(s, u) = \gamma(s) + u\varrho_n(s), \quad \varrho_n(s) = \varrho_1(s)\mathcal{T}(s) + \varrho_2(s)\mathcal{N}(s) \quad (7)$$

is called the generalized osculating-type ruled surface of framed base curve where  $\varrho_1(s)$  and  $\varrho_2(s)$  are smooth functions where  $\varrho_1^2(s) + \varrho_2^2(s) = 1$ . Obviously, if  $\varrho_1(s) = 0$  and  $\varrho_2(s) = \pm 1$ , then the surface  $\Psi_{(\gamma, \varrho_n)}(s)$  is principal normal surface along Frenet-type framed base curves. If  $\varrho_2(s) = 0$  and  $\varrho_1(s) = \pm 1$ , then the surface  $\Psi_{(\gamma, \varrho_n)}(s)$  is tangent developable surface along Frenet-type framed base curves.

**Theorem 1.** *The set of singular points of generalized osculating-type ruled surface  $\Psi_{(\gamma, \varrho_n)}$  is given by*

$$S = \{(s, u) \in I \times U \mid u\varrho_2(s)\tau(s) = 0, (\alpha(s)\varrho_2(s) + u(\varrho_1'(s)\varrho_2(s) - \varrho_1(s)\varrho_2'(s) - \kappa(s))) = 0\}. \quad (8)$$

*Proof.* According to the partial derivatives of (7), we get

$$\begin{aligned}\frac{\partial \Psi_{(\gamma, \rho_n)}(s, u)}{\partial s} &= (\alpha(s) + u\rho_1'(s) - u\rho_2(s)\kappa(s))\mathcal{T}(s) + u(\rho_1(s)\kappa(s) + \rho_2'(s))\mathcal{N}(s) + u\rho_2(s)\tau(s)\mathcal{B}(s), \\ \frac{\partial \Psi_{(\gamma, \rho_n)}(s, u)}{\partial u} &= \rho_1(s)\mathcal{T}(s) + \rho_2(s)\mathcal{N}(s).\end{aligned}\quad (9)$$

From Equation (9), we get

$$\begin{aligned}\frac{\partial \Psi_{(\gamma, \rho_n)}(s, u)}{\partial s} \wedge \frac{\partial \Psi_{(\gamma, \rho_n)}(s, u)}{\partial u} &= -u\rho_2^2(s)\tau(s)\mathcal{T}(s) + u\rho_1(s)\rho_2(s)\tau(s)\mathcal{N}(s) \\ &\quad + (\alpha(s)\rho_2(s) + u(\rho_1'(s)\rho_2(s) - \rho_1(s)\rho_2'(s) - \kappa(s)))\mathcal{B}(s)\end{aligned}$$

The singular points of surface construct with the set (8).  $\square$

Now, let us examine the singularity types of the generalized osculating-type ruled surface of Frenet-type framed curves. In order to examine the types of singularity of the surface, we can divide them into two classes  $S_1$  and  $S_2$ . For singular points, from Equation (8),

$$u\rho_2(s)\tau(s) = 0 \quad (10)$$

$$\alpha(s)\rho_2(s) + u(\rho_1'(s)\rho_2(s) - \rho_1(s)\rho_2'(s) - \kappa(s)) = 0 \quad (11)$$

First, let us assume that  $u = 0$ . Then, the class  $S_1$  is denoted by

$$S_1 = \{(s, 0) \in I \times U \mid \alpha(s)\rho_2(s) = 0\}.$$

Next, assume that  $u \neq 0$ . According to Equation (11), we have

$$u = -\frac{\alpha(s)\rho_2(s)}{\rho_1'(s)\rho_2(s) - \rho_1(s)\rho_2'(s) - \kappa(s)}. \quad (12)$$

If  $u \neq 0$ , then we have  $\rho_2(s) = 0$  or  $\tau(s) = 0$ . However, if  $\rho_2(s) = 0$ , from Equation (12), we get  $u = 0$ . This contradicts the fact that  $u \neq 0$ . Therefore,  $\tau(s) = 0$ . After all,  $\tau(s) = 0$  for every  $s \in I$  is not. That is, this does not require the curve to be planar. Finally, the class  $S_2$  is denoted by

$$S_2 = \{(s, u) \in I \times U \mid \tau(s) = 0, \alpha(s)\rho_2(s) + u(\rho_1'(s)\rho_2(s) - \rho_1(s)\rho_2'(s) - \kappa(s)) = 0, u \neq 0\}$$

### Corollary 1.

1. If  $\rho_1(s) = 0$  and  $\rho_2(s) = 1$ , generalized osculating-type ruled surface  $\Psi_{(\gamma, \rho_n)}$  is a principal normal surface of Frenet-type framed curves. Consequently, from definition sets  $S_1$  and  $S_2$ , we have

$$S_1 = \{(s, 0) \in I \times U \mid \alpha(s) = 0\}.$$

$$S_2 = \{(s, u) \in I \times U \mid \tau(s) = 0, \alpha(s) - u\kappa(s) = 0, u \neq 0\}$$

Obviously, this particular case presents the results of study.<sup>19</sup>

2. If  $\rho_2(s) = 0$  and  $\rho_1(s) = 1$ , generalized osculating-type ruled surface  $\Psi_{(\gamma, \rho_n)}$  is a tangent surface of Frenet-type framed curves. Since  $\rho_2(s) = 0$ , from definition set  $S_2$  for  $u \neq 0$ , we have  $\kappa(s) = 0$ . This is a contradiction since  $\kappa(s) \neq 0$  for Frenet-type framed curves. Therefore, there is no  $S_2$  singularity for tangent surfaces.

After this part, for convenience, according to Equation (8), it will be accepted as

$$\begin{aligned}x(s) &= \rho_2(s)\tau(s) \\ y(s, u) &= \alpha(s)\rho_2(s) + u(\rho_1'(s)\rho_2(s) - \rho_1(s)\rho_2'(s) - \kappa(s))\end{aligned}$$

Now, let us examine the conditions for having cross-cap singularity of generalized osculating-type ruled surfaces of Frenet-type framed base curves.

**Theorem 2.** Let  $\Psi_{(\gamma, \rho_n)}(s, u)$  is generalized osculating-type ruled surface of Frenet-type framed base curve  $\gamma(s)$ . Then, we have

1. If  $(s_1, 0) \in S_1$  and  $\alpha'(s_1)\rho_2(s_1)x(s_1) \neq 0$ , then  $\Psi_{(\gamma, \rho_n)}(s, u)$  is a cross-cap at the point  $(s_1, 0)$ .
2. If  $(s_2, u_2) \in S_2$  and  $\alpha(s_2)\rho_2(s_2)x'(s_2) \neq 0$ , then  $\Psi_{(\gamma, \rho_n)}(s, u)$  is a cross-cap at the point  $(s_2, u_2)$ .

*Proof.* From Equation (9), we get

$$\begin{aligned} \frac{\partial^2 \Psi_{(\gamma, \rho_n)}}{\partial u \partial s}(s, u) &= (\rho_1'(s) - \rho_2(s)\kappa(s))\mathcal{T}(s) + (\rho_2'(s) + \rho_1(s)\kappa(s))\mathcal{N}(s) + \rho_2(s)\tau(s)\mathcal{B}(s) \\ \frac{\partial^2 \Psi_{(\gamma, \rho_n)}}{\partial s^2}(s, u) &= [(\alpha(s) - u\rho_1(s))' - u\rho_2(s)\kappa(s)]' - u\rho_1(s)\kappa^2(s) - u\rho_2'(s)\kappa(s)]\mathcal{T}(s) \\ &\quad + [\alpha(s)\kappa(s) - u(\rho_2(s)(\kappa^2(s) + \tau^2(s)) - \rho_1(s)\kappa'(s) - \rho_2''(s))]\mathcal{N}(s) \\ &\quad + [u(2\rho_2'(s)\tau(s) + \rho_2(s)\tau'(s) + \rho_1(s)\kappa(s)\tau(s))]\mathcal{B}(s) \end{aligned}$$

Consequently, we have

$$\begin{aligned} \det \left( \frac{\partial \Psi}{\partial u}(s, u), \frac{\partial^2 \Psi}{\partial u \partial s}(s, u), \frac{\partial^2 \Psi}{\partial s^2}(s, u) \right) &= u(\rho_1(s)\rho_2'(s) - \rho_1'(s)\rho_2(s)) \\ &\quad + \kappa(s)(2\rho_2'(s)\tau(s) + \rho_2(s)\tau'(s) + \rho_1(s)\kappa(s)\tau(s)) \\ &\quad + \tau(s)\rho_2(s)(\alpha'(s)\rho_2(s) - \alpha(s)\rho_1(s)\kappa(s)) \\ &\quad - u\tau(s)\rho_2(s)(\rho_2(s)\rho_1''(s) + \rho_2''(s)\rho_1(s)) \\ &\quad - u\tau(s)\rho_2(s)(\kappa'(s) - \rho_1(s)\rho_2(s)\tau^2(s) + 2\rho_2(s)\rho_2'(s)\kappa(s)) \end{aligned}$$

If the singular points of the surface belong to the set  $S_1$  (i.e.,  $(s_1, 0) \in S_1$ ), then

$$\det \left( \frac{\partial \Psi}{\partial u}(s_1, 0), \frac{\partial^2 \Psi}{\partial u \partial s}(s_1, 0), \frac{\partial^2 \Psi}{\partial s^2}(s_1, 0) \right) = \alpha'(s_1)\rho_2^2(s_1)\tau(s_1) = \alpha'(s_1)\rho_2(s_1)x(s_1)$$

If the singular points of the surface belong to the set  $S_2$  (i.e.,  $(s_2, u_2) \in S_2$ ), then

$$\det \left( \frac{\partial \Psi}{\partial u}(s_2, u_2), \frac{\partial^2 \Psi}{\partial u \partial s}(s_2, u_2), \frac{\partial^2 \Psi}{\partial s^2}(s_2, u_2) \right) = \alpha(s_2)\rho_2(s_2)x'(s_2).$$

The desired from Izumiya and Takeuchi<sup>6</sup> is obtained. □

**Corollary 2.** Let Frenet-type framed curve  $\gamma(s)$  be a Bertrand curve and  $\Psi_{(\gamma, \rho_n)}(s, u)$  be the generalized osculating-type ruled surface of  $\gamma(s)$ . Then, we have

- i. If  $s_1$  is the ordinary cusp singularity of  $\gamma$  and  $\rho_2(s_1) \neq 0$ , then  $\Psi_{(\gamma, \rho_n)}(s, u)$  is a cross-cap at the point  $(s_1, 0)$ .
- ii. If  $s_2$  is the ordinary cusp singularity of the Bertrand mate  $\gamma$  and  $\rho_1(s) = 0$  and  $\rho_2(s) = 1$ , then  $\Psi_{(\gamma, \rho_n)}(s, u)$  is a cross-cap at the point  $(s_2, u_2)$ .

*Proof.* i. From the definition of ordinary cusp singularity,<sup>25</sup> since  $s_1$  is the ordinary cusp singularity of  $\gamma$ ,  $\alpha(s_1) = 0$ , and  $\alpha'(s_1) \neq 0$ . On the other hand, the Frenet-type framed curve  $\gamma(s)$  is a Bertrand curve if and only if there exist two constants  $\lambda_1 \neq 0$  and  $\lambda_2$  such that  $\lambda_1\kappa(s) + \lambda_2\tau(s) = \alpha(s)$  and  $\lambda_2\kappa(s) - \lambda_1\tau(s) \neq 0$ .<sup>19</sup> For Frenet-type framed curves, since  $\kappa(s) \neq 0$  for every  $s \in I$ , we have  $\tau(s_1) \neq 0$ . Therefore, since  $\tau(s_1) \neq 0$ ,  $\alpha'(s_1) \neq 0$ , if  $\rho_2(s_1) \neq 0$ , then  $\Psi_{(\gamma, \rho_n)}(s, u)$  is a cross-cap at the point  $(s_1, 0)$ .

ii. Let us assume that  $\bar{\gamma}(s)$  is the Bertrand mate  $\gamma(s)$ . If  $s_2$  is the ordinary cusp singularity of the Bertrand mate  $\gamma$  and  $\rho_1(s) = 0$  and  $\rho_2(s) = 1$ , since  $\Psi_{(\gamma, \mathcal{N})}(s, u)$  is a principal normal surface, from Corollary 1 in Huang and Pei,<sup>19</sup> the desired result is obtained.  $\square$

**Theorem 3.** Let  $\Psi_{(\gamma, \rho_n)}(s, u)$  is generalized osculating-type ruled surface of Frenet-type framed base curve  $\gamma(s)$ . For  $\rho_1(s), \rho_2(s) \neq 0$  are non-constant, we have the following:

1. The generalized osculating-type ruled surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  of Frenet-type framed base curve  $\gamma(s)$  is cylindrical if and only if  $\tau(s) = 0$  and  $\kappa(s) = \rho'_1(s)\rho_2(s) - \rho'_2(s)\rho_1(s)$ .
2. The Frenet-type framed base curve  $\gamma(s)$  of surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  is its striction curve if and only if  $\alpha(s) = 0$  or  $\rho'_1(s) - \rho_2(s)\kappa(s) = 0$ .

*Proof.*

1. By differentiating of  $\rho_n(s) = \rho_1(s)\mathcal{T}(s) + \rho_2(s)\mathcal{N}(s)$ , we get

$$\rho'_n(s) = (\rho'_1(s) - \rho_2(s)\kappa(s))\mathcal{T}(s) + (\rho'_2(s) + \rho_1(s)\kappa(s))\mathcal{N}(s) + \rho_2(s)\tau(s)\mathcal{B}(s).$$

If surface is cylindrical  $\rho'_n(s) = 0$ . Then, we have

$$\begin{aligned} \rho'_1(s) - \rho_2(s)\kappa(s) &= 0, \\ \rho'_2(s) + \rho_1(s)\kappa(s) &= 0, \\ \rho_2(s)\tau(s) &= 0. \end{aligned} \tag{13}$$

If  $\rho_2(s) = 0$ , we have  $\kappa(s) = 0$ . However, this contradicts the existence of Frenet-type framed curves. Therefore,  $\rho_2(s) \neq 0$  and  $\tau(s) = 0$ . If both sides of the first equation of (13) are multiplied by  $\rho_2(s)$  and both sides of the second equation of (13) by  $-\rho_1(s)$ , we get  $\kappa(s) = \rho'_1(s)\rho_2(s) - \rho'_2(s)\rho_1(s)$ . Then, if the surface is cylindrical,  $\tau(s) = 0$  and  $\kappa(s) = \rho'_1(s)\rho_2(s) - \rho'_2(s)\rho_1(s)$ .

2. The striction parameter of the  $\Psi_{(\gamma, \rho_n)}(s, u)$  can given by

$$\begin{aligned} \mu(s) &= \gamma(s) - \frac{\langle \gamma'(s), \rho'_n(s) \rangle}{\langle \rho'_n(s), \rho'_n(s) \rangle} \rho_n(s) \\ &= \gamma(s) + \frac{\alpha(s)(\rho'_1(s) - \rho_2(s)\kappa(s))}{(\rho'_1(s) - \rho_2(s)\kappa(s))^2 + (\rho'_2(s) + \rho_1(s)\kappa(s))^2 + (\rho_2(s)\tau(s))^2} \rho_n(s). \end{aligned}$$

The base curve  $\gamma(s)$  of surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  is its striction curve if and only if  $\rho'_1(s) - \rho_2(s)\kappa(s) = 0$  or  $\alpha(s) = 0$ .  $\square$

**Corollary 3.** If the generalized osculating-type ruled surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  of Frenet-type framed base curve  $\gamma(s)$  is cylindrical, it is a plane with the following parametric representation:

$$\Psi_{(\gamma, \rho_n)}(s, u) = \gamma(s) + u \left( \cos \left( \int \kappa(s) ds \right) \mathcal{T}(s) - \sin \left( \int \kappa(s) ds \right) \mathcal{N}(s) \right)$$

*Proof.* Let the generalized osculating-type ruled surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  of Frenet-type framed base curve  $\gamma(s)$  is cylindrical. Then, according to Theorem 3,  $\tau(s) = 0$  and  $\kappa(s) = \rho'_1(s)\rho_2(s) - \rho'_2(s)\rho_1(s)$ . On the other hand, since  $\rho_1^2(s) + \rho_2^2(s) = 1$ , we get  $\rho_1(s)\rho'_1(s) + \rho_2(s)\rho'_2(s) = 0$ . From the solution of this differential equation, we can write  $\rho_1(s) = \sqrt{c - \rho_2^2}$ . By using equation  $\kappa(s) = \rho'_1(s)\rho_2(s) - \rho'_2(s)\rho_1(s)$ , we have

$$\kappa(s) = -\frac{c}{\sqrt{c}} \left( \frac{\rho'_2(s)}{\sqrt{1 - \left( \frac{\rho_2(s)}{\sqrt{c}} \right)^2}} \right)$$

Let us assume that  $c = 1$ . If both sides are integrated, we get

$$\int \kappa(s) = -\arcsin(\rho_2(s)) \quad (14)$$

From Equation (14), we have

$$\rho_2(s) = -\sin \int \kappa(s) ds, \quad \rho_1(s) = \cos \int \kappa(s) ds$$

□

**Corollary 4.** *If the Frenet-type framed base curve  $\gamma$  of the generalized osculating-type ruled surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  is also its striction for  $\alpha(s) = 0$ , the striction curve is a point.*

**Proposition 2.** *The generalized osculating-type ruled surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  of Frenet-type framed base curve  $\gamma(s)$  is developable if and only if  $\alpha(s)\rho_2^2(s)\tau(s) = 0$ .*

*Proof.* If  $\Psi_{(\gamma, \rho_n)}(s, u)$  is developable,  $\det(\gamma'(s), \rho_n(s), \rho_n'(s)) = 0$ . Then, we have

$$\begin{aligned} \det(\gamma'(s), \rho_n(s), \rho_n'(s)) &= \det(\alpha(s)\mathcal{T}(s), \rho_1(s)\mathcal{T}(s) + \rho_2(s)\mathcal{N}(s), (\rho_1'(s) - \rho_2(s)\kappa(s))\mathcal{T}(s) \\ &\quad + (\rho_2'(s) + \rho_1(s)\kappa(s))\mathcal{N}(s) + \rho_2(s)\tau(s)\mathcal{B}(s)) \\ &= \alpha(s)\rho_2^2(s)\tau(s). \end{aligned}$$

Therefore, if generalized osculating-type ruled surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  is developable  $\alpha(s)\rho_2^2(s)\tau(s) = 0$ . □

**Corollary 5.** *If generalized osculating-type ruled surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  is a tangent developable surface (i.e.,  $\rho_2(s) = 0$ ),  $\Psi_{(\gamma, \rho_n)}(s, u)$  is developable. If Frenet-type framed curve  $\gamma(s)$  is planar curve (i.e.,  $\tau(s) = 0$ ),  $\Psi_{(\gamma, \rho_n)}(s, u)$  is developable.*

**Proposition 3.** *The locus of the singular points of the generalized osculating-type surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  of Frenet-type framed curves is given by*

1. *If the singular points of the surface belong to the set  $S_1$ , the locus of the singular points is the framed base curve  $\gamma$ .*
2. *If the singular points of the surface belong to the set  $S_2$ , the locus of the singular points  $\Psi_{(\gamma, \rho_n)}(s, u)$  is the curve*

$$\beta(s) = \gamma(s) + \frac{\alpha(s)\rho_n(s)}{\rho_1(s)\rho_2'(s) - \rho_1'(s)\rho_2(s) + \kappa(s)}.$$

*Proof.* The definition of singular point set  $S_2$ ,  $\tau(s) = 0$  and  $y(s, u) = 0$ . From equation  $y(s, u) = 0$ , we get  $u = \frac{\alpha(s)}{\rho_1(s)\rho_2'(s) - \rho_1'(s)\rho_2(s) + \kappa(s)}$  where  $\rho_1(s)\rho_2'(s) - \rho_1'(s)\rho_2(s) + \kappa(s) \neq 0$ . □

### 3.1 | Generalized osculating-type surfaces as framed surfaces

In this part of this study, we intend to examine the generalized osculating-type ruled surfaces of Frenet-type framed base curves, which may have singular points with the help of the theory of the framed surface which was studied in Fukunaga and Takahashi.<sup>22</sup>

**Definition 5.** Let  $\gamma(s)$  is a Frenet-type framed base curve and  $\Psi_{(\gamma, \rho_n)}(s, u)$  is the generalized osculating-type ruled surface of  $\gamma(s)$ . If there exist two smooth functions  $\Theta, \Phi : I \times U \rightarrow \mathbb{R}^3$  such that  $\langle \Psi_s(s, u), \Psi_1(s, u) \rangle = 0$  and  $\langle \Psi_u(s, u), \Psi_1(s, u) \rangle = 0$  where

$$\Psi_1(s, u) = \cos \Theta(s, u)(\rho_2(s)\mathcal{T}(s) - \rho_1(s)\mathcal{N}(s)) + \sin \Theta(s, u)\mathcal{B}(s)$$

and

$$\Psi_2(s, u) = \cos \Phi(s, u)(\rho_1(s)\mathcal{T}(s) + \rho_2(s)\mathcal{N}(s)) + \sin \Phi(s, u)(\sin \Theta(s, u)(\rho_2(s)\mathcal{T}(s) - \rho_1(s)\mathcal{N}(s)) - \cos \Theta(s, u)\mathcal{B}(s)).$$

Hence, we have the framed surface  $(\Psi, \Psi_1, \Psi_2) : I \times U \rightarrow \mathbb{R}^3 \times \Delta$  where  $\Delta = \{(\Psi_1, \Psi_2) \in \mathbb{S}^2 \times \mathbb{S}^2, \langle \Psi_1, \Psi_2 \rangle = 0\}$ . We can calculate  $\Psi_3(s, u) = \Psi_1(s, u) \wedge \Psi_2(s, u)$ . Therefore, we have

$$\begin{aligned} \Psi_3(s, u) &= (\rho_1(s) \sin \Phi(s, u) - \rho_2(s) \sin \Theta(s, u) \cos \Phi(s, u)) \mathcal{T}(s) \\ &\quad + (\rho_2(s) \sin \Phi(s, u) + \rho_1(s) \cos \Phi(s, u) \sin \Theta(s, u)) \mathcal{N}(s) \\ &\quad + \cos \Theta(s, u) \cos \Phi(s, u) \mathcal{B}(s) \end{aligned}$$

From Equations (2), (3), and (4), the basic invariants of  $(\Psi, \Psi_1, \Psi_2)$  are given by

$$\begin{aligned} \varepsilon_{01} &= \sin \Phi(s, u) [\sin \Theta(s, u)y(s, u) - u \cos \Theta(s, u)x(s)], \\ \varepsilon_{02} &= -\cos \Phi(s, u)[\sin \Theta(s, u)y(s, u) - u \cos \Theta(s, u)x(s)], \\ \varepsilon_{03} &= \cos \Phi(s, u), \\ \varepsilon_{04} &= \sin \Phi(s, u). \end{aligned}$$

$$\begin{aligned} \varepsilon_{11} &= -\cos \Phi(s, u)[\cos \Theta(s, u)(\rho_2(s)\rho_1'(s) - \rho_1(s)\rho_2'(s) - \kappa(s)) + \sin \Theta(s, u)x(s)] - \sin \Phi(s, u)(\Theta_s(s, u) - \rho_1(s)\tau(s)), \\ \varepsilon_{12} &= -\sin \Phi(s, u)[\cos \Theta(s, u)(\rho_2(s)\rho_1'(s) - \rho_1(s)\rho_2'(s) - \kappa(s)) + \sin \Theta(s, u)x(s)] + \cos \Phi(s, u)(\Theta_s(s, u) - \rho_1(s)\tau(s)), \\ \varepsilon_{13} &= -\Phi_s - \sin \Theta(s, u)(\rho_2(s)\rho_1'(s) - \rho_1(s)\rho_2'(s) - \kappa(s)) + \cos \Theta(s, u)x(s) \end{aligned}$$

and

$$\begin{aligned} \varepsilon_{21} &= -\Theta_u \sin \Phi(s, u), \\ \varepsilon_{22} &= \Theta_u \cos \Phi(s, u), \\ \varepsilon_{23} &= -\Phi_u \end{aligned}$$

where  $x(s) = \tau(s)\rho_2(s)$  and  $y(s, u) = \alpha(s)\rho_2(s) + u(\rho_1'(s)\rho_2(s) - \rho_1(s)\rho_2'(s) - \kappa(s))$ . On the other hand, from integrability conditions of the framed surface, we have

$$\Theta_u(\sin \Theta(s, u)y(s, u) - u \cos \Theta(s, u)x(s)) = [\cos \Theta(s, u)(\rho_2(s)\rho_1'(s) - \rho_1(s)\rho_2'(s) - \kappa(s)) + \sin \Theta(s, u)x(s)].$$

The curvature  $C_\Psi = (J_\Psi, K_\Psi, H_\Psi)$  of the framed surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  is given by

$$\begin{aligned} J_\Psi &= \sin \Theta(s, u)y(s, u) - u \cos \Theta(s, u)x(s), \\ K_\Psi &= -\Theta_u(\cos \Theta(s, u)(\rho_2(s)\rho_1'(s) - \rho_1(s)\rho_2'(s) - \kappa(s)) + \sin \Theta(s, u)x(s)), \\ H_\Psi &= \frac{\Theta_s(s, u) - \rho_1(s)\tau(s)}{2}. \end{aligned}$$

By according to Corollary 1 in Fukunaga and Takahashi,<sup>22</sup> we get

**Corollary 6.** *If  $p \in S_1$  or  $p \in S_2$ ,*

- i. *Since,  $J_\Psi(p) = 0$ ,  $\Psi$  is not an immersion (a regular surface) around  $p$ .*
- ii.  *$(\Psi, \Psi_1)$  is a Legendre immersion around  $p$  if and only if  $K_\Psi(p) \neq 0$  or  $H_\Psi(p) \neq 0$ . If  $p \notin S_1$  and  $p \notin S_2$ ,*

- iii.  $\Psi$  is an immersion (a regular surface) around  $p$ .  
 iv.  $(\Psi, \Psi_1)$  is a Legendre immersion around  $p$ .

**Corollary 7.** Let  $\Psi_{(\gamma, \rho_n)} : U \rightarrow \mathbb{R}^3$  is a regular surface. From Equation (6), the first and second fundamental forms are given by

$$\begin{aligned} E &= (\sin \Theta(s, u)y(s, u) - u \cos \Theta(s, u)x(s))^2, \\ F &= \cos \Theta(s, u) \sin \Theta(s, u)[1 - (\sin \Theta(s, u)y(s, u) - u \cos \Theta(s, u)x(s))^2], \\ G &= 1, \\ L &= -(\Theta_s(s, u) - \rho_1(s)\tau(s))(\sin \Theta(s, u)y(s, u) - u \cos \Theta(s, u)x(s)), \\ M &= \Theta_u(\sin \Theta(s, u)y(s, u) - u \cos \Theta(s, u)x(s)), \\ N &= 0. \end{aligned}$$

**Corollary 8.** The Gauss curvature  $K$  and the mean curvature  $H$  of the regular surface  $\Psi_{(\gamma, \rho_n)}$  are given by, respectively,

$$\begin{aligned} K &= \frac{-\Theta_u(\cos \Theta(s, u)(\rho_2(s)\rho_1'(s) - \rho_1(s)\rho_2'(s) - \kappa(s)) + \sin \Theta(s, u)x(s))}{\sin \Theta(s, u)y(s, u) - u \cos \Theta(s, u)x(s)}, \\ H &= \frac{\Theta_s(s, u) - \rho_1(s)\tau(s)}{2(\sin \Theta(s, u)y(s, u) - u \cos \Theta(s, u)x(s))}. \end{aligned}$$

**Corollary 9.** The regular surface  $\Psi_{(\rho_n)} : U \rightarrow \mathbb{R}^3$  is a flat surface if and only if  $\Theta_u(s, u) = 0$ .

*Proof.* From the integrability conditions of the framed surface, we have  $K = \Theta_u^2(s, u)$ . If  $\Psi_{(\gamma, \rho_n)} : U \rightarrow \mathbb{R}^3$  is a flat surface if and only if  $\Theta_u(s, u) = 0$ . On the other hand, since  $\Psi_{(\gamma, \rho_n)} : U \rightarrow \mathbb{R}^3$  is regular, we can write  $\cos \Theta(s, u) = \frac{y(s, u)}{\sqrt{u^2x^2(s) + y^2(s, u)}}$  and  $\sin \Theta(s, u) = \frac{ux(s)}{\sqrt{u^2x^2(s) + y^2(s, u)}}$ . Then we have  $\Theta_u(s, u) = \frac{\alpha(s)\rho_2^2(s)\tau(s)}{u^2x^2(s) + y^2(s, u)}$ . Therefore, the surface is flat if and only if  $\Theta_u(s, u) = \alpha(s)\rho_2^2(s)\tau(s) = 0$ . This situation coincides with Proposition 4.  $\square$

**Corollary 10.** The regular surface  $\Psi_{(\gamma, \rho_n)} : U \rightarrow \mathbb{R}^3$  is a minimal surface if and only if  $\Theta_s(s, u) = \rho_1(s)\tau(s)$ .

## 4 | EXAMPLES

In this section, some examples are given that explain the theorems and results from the previous sections.

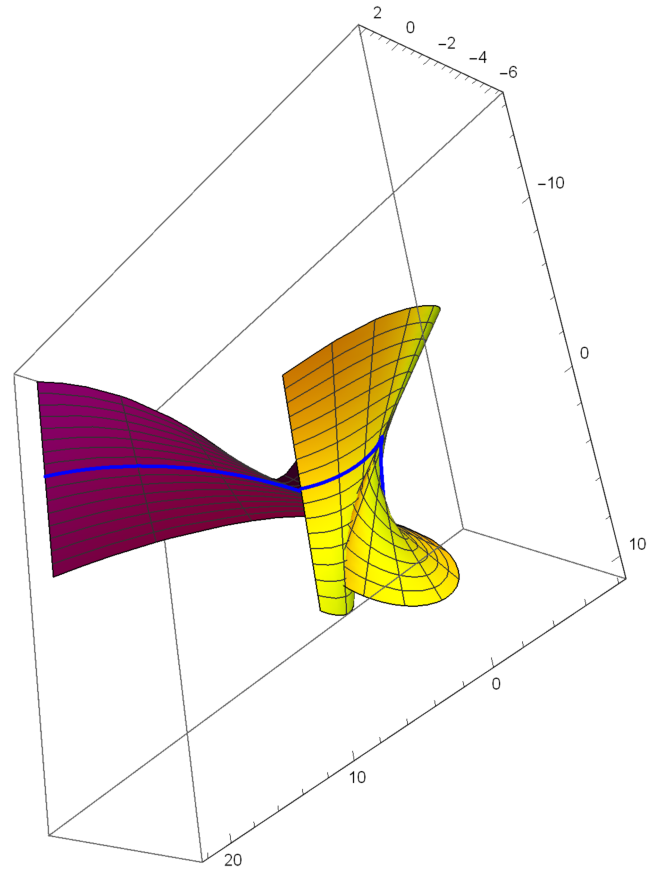
**Example 1.** Let the anchor curve framed helix  $\gamma(s)$  of generalized osculating-type ruled surface  $\Psi_{(\gamma, \rho_n)}(s, u)$  be given by the parametrization

$$\gamma(s) = \left( 2(s + \sqrt{2}) \cos s - 2 \sin s, -2(s + \sqrt{2}) \sin s - 2 \cos s, \frac{\sqrt{5}s^2}{2} + \sqrt{10}s \right)$$

with a singular point  $s_0 = -\sqrt{2}$ . The moving frame and framed curvatures  $\{\mathcal{T}, \mathcal{N}, \mathcal{B}, \kappa, \tau, \alpha\}$  are given by

$$\begin{aligned} \mathcal{T}(s) &= \left( -\frac{2}{3} \sin s, -\frac{2}{3} \cos s, \frac{\sqrt{5}}{3} \right) \\ \mathcal{N}(s) &= (-\cos s, \sin s, 0) \\ \mathcal{B}(s) &= \left( -\frac{\sqrt{5}}{3} \sin s, -\frac{\sqrt{5}}{3} \cos s, -\frac{2}{3} \right) \\ \kappa(s) &= \frac{2}{3}, \tau(s) = -\frac{\sqrt{5}}{3}, \alpha(s) = 3(s + \sqrt{2}). \end{aligned}$$

**FIGURE 1** Generalized osculating-type ruled surface of framed helice [Colour figure can be viewed at wileyonlinelibrary.com]



Let us assume that  $\varrho_1(s) = \sin s$  and  $\varrho_2(s) = \cos s$ . Therefore, we can write  $\Psi_{(\gamma, \varrho_n)}(s, u) = \gamma(s) + u\varrho_n(s)$  where

$$\varrho_n(s) = \left( -\frac{2}{3} - \frac{1}{3}\cos^2(s), \frac{1}{3}\sin s \cos s, \frac{\sqrt{5}}{3}\sin s \right)$$

Since  $\tau(s) \neq 0$  and  $\varrho_2(s) \neq 0$ ,  $\Psi_{(\gamma, \varrho_n)}$  is non-developable and non-cylindrical. Also, we can write the following expressions for singularity types:

- i. If  $u = 0$  and  $\alpha(s_1)\varrho_2(s_1) = 3(s_1 + \sqrt{2})\cos s_1 = 0$ , the  $(s_1, 0) \in S_1$  is a singular point of surface. Then, the  $(-\sqrt{2}, 0) \in S_1$ ,  $(-\frac{\pi}{2}, 0) \in S_1$  and  $(\frac{\pi}{2}, 0) \in S_1$  are singular points of surface.
- ii. Since  $\tau(s) \neq 0$  for every  $s \in I$ ,  $(s_2, u_2) \in S_2$  is not singular point.

The graph of surface  $\Psi_{(\gamma, \varrho_n)}$  for  $u \in (-2\pi, 2\pi)$  and  $s \in (-\pi, \pi)$  is given by Figure 1.

**Example 2.** Let the surface  $\Psi_{(\gamma, \varrho_n)}$  be given by the parametrization

$$\Psi_{(\gamma, \varrho_n)}(s, u) = \gamma(s) + u\varrho_n(s)$$

where  $\gamma(s) : (-2\pi, 2\pi) \rightarrow \mathbb{R}^3$  is framed slant helice is given by<sup>17</sup>

$$\gamma(s) = \left( \frac{\sqrt{6}}{5}\sin\left(\frac{3s}{5}\right) - \frac{2\sqrt{6}}{35}\sin\left(\frac{7s}{5}\right) - \frac{\sqrt{6}}{25}\sin s, -\frac{\sqrt{6}}{5}\cos\left(\frac{3s}{5}\right) + \frac{2\sqrt{6}}{35}\cos\left(\frac{7s}{5}\right) + \frac{\sqrt{6}}{25}\cos s, \frac{12s}{25} - \frac{6}{5}\sin\left(\frac{2s}{5}\right) \right)$$

and the  $\{\mathcal{T}, \mathcal{N}, \mathcal{B}, \kappa, \tau, \alpha\}$  are given by

$$\mathcal{T}(s) = \left( \frac{3}{5} \sin\left(\frac{4s}{5}\right) + \frac{2}{5} \sin\left(\frac{6s}{5}\right), -\frac{3}{5} \cos\left(\frac{4s}{5}\right) - \frac{2}{5} \cos\left(\frac{6s}{5}\right), \frac{2\sqrt{6}}{25} \sin\left(\frac{s}{5}\right) \right),$$

$$\mathcal{N}(s) = \left( \frac{2\sqrt{6}}{25} \cos s, \frac{2\sqrt{6}}{25} \sin s, \frac{1}{5} \right),$$

$$\mathcal{B}(s) = \left( \frac{2}{5} \cos\left(\frac{6s}{5}\right) - \frac{3}{5} \cos\left(\frac{4s}{5}\right), \frac{2}{5} \sin\left(\frac{6s}{5}\right) - \frac{3}{5} \sin\left(\frac{4s}{5}\right), \frac{2\sqrt{6}}{25} \cos\left(\frac{s}{5}\right) \right),$$

$$\kappa(s) = \frac{2\sqrt{6}}{5} \cos\left(\frac{s}{5}\right),$$

$$\tau(s) = \frac{2\sqrt{6}}{5} \sin\left(\frac{s}{5}\right),$$

$$\alpha(s) = 2\sqrt{6} \sin\left(\frac{s}{5}\right).$$

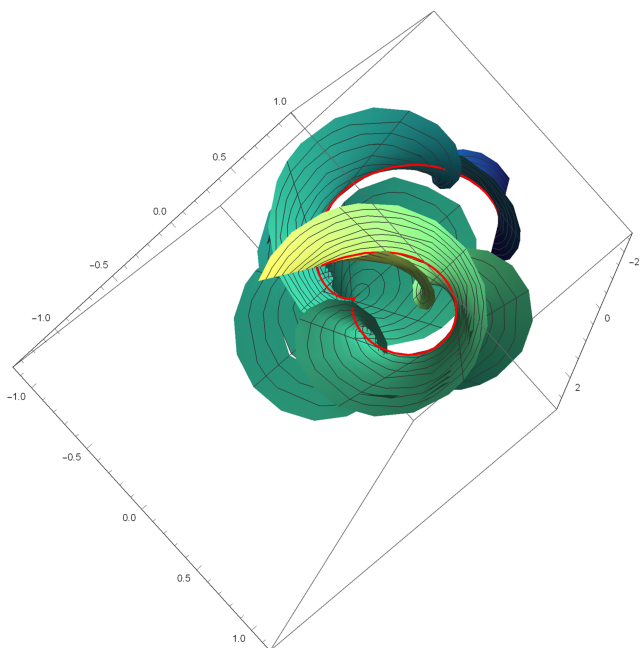
Suppose that  $\rho_1(s) = \cos s$  and  $\rho_2(s) = \sin s$ . Then, we have

$$\Psi_{(\gamma, \rho_n)}(s, u) = \gamma(s) + u(\cos s \mathcal{T}(s) + \sin s \mathcal{N}(s))$$

Then we get the following:

- Since  $\alpha(s)\rho_2(s) = 2\sqrt{6} \sin\left(\frac{s}{5}\right) \sin s$ , if  $s = k\pi$  where  $k \in \mathbb{R}$  for  $u = 0$  and  $s \in (-2\pi, 2\pi)$ , the set of singular points is  $S_1$ .
- The definition of singular point set  $S_2$ , we have  $\tau(s) = 0$  and  $y(s, u) = 0$ . Then, if  $s_2 = k\pi$ ,  $x(s_2) = 0$ . However,  $y(s_2, u_2) = -u(1 \pm \frac{2\sqrt{6}}{5})$  for  $s_2 = k\pi$ . Then, if  $y(s_2, u_2) = 0$ ,  $u_2 = 0$ . Therefore, the set of singular points is not  $S_2$ .

The graph of surface  $\Psi_{(\gamma, \rho_n)}$  for  $u \in (-1, 1)$  and  $s \in (-2\pi, 2\pi)$  is given by Figure 2.



**FIGURE 2** Generalized osculating-type ruled surface of framed slant helix [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com/terms-and-conditions)]

**Example 3.** Let us consider the (2, 3, 5)-type Frenet-type framed base curve by  $\gamma(s) = \left(\frac{s^2}{2}, \frac{s^3}{3}, \frac{s^5}{5}\right)$ . Since  $\gamma'(0) = (0, 0, 0)$ , the curve is not Frenet curve. However, we can write

$$\gamma'(s) = \alpha(s)\mathcal{T}(s)$$

where

$$\alpha(s) = s \left( \sqrt{1 + s^2 + s^6} \right)$$

$$\mathcal{T}(s) = \frac{1}{\sqrt{1 + s^2 + s^6}}(1, s, s^3)$$

It is clear that  $\alpha(0) = 0$  and  $\mathcal{T}$  is the regular spherical curve. Then the other orthonormal frame elements of the Frenet-type framed curve  $\gamma(s)$  are given by

$$\mathcal{N}(s) = \frac{1}{\sqrt{1 + 9s^4 + 4s^6}\sqrt{1 + s^2 + s^6}}(-s - 3s^5, 1 - 2s^6, 3s^2 + 2s^4),$$

$$B(s) = \frac{1}{\sqrt{1 + 9s^4 + 4s^6}}(2s^3, -3s^2, 1),$$

$$\kappa(s) = \frac{\sqrt{1 + 9s^4 + 4s^6}}{1 + s^2 + s^6},$$

$$\tau(s) = \frac{6s\sqrt{1 + s^2 + s^6}}{1 + 9s^4 + 4s^6}.$$

Let us consider the following cases:

Case 1: The generalized osculating-type ruled surface  $\Psi_{(\gamma, \varrho_n)}$  is given by

$$M_1 = \left( \frac{s^2}{2} + u \left( \frac{s}{\sqrt{1 + s^2 + s^6}} + \frac{\sqrt{1 - s^2}(-s - 3s^5)}{\sqrt{1 + 9s^4 + 4s^6}\sqrt{1 + s^2 + s^6}} \right), \right.$$

$$\left. \frac{s^3}{3} + u \left( \frac{s^2}{\sqrt{1 + s^2 + s^6}} + \frac{\sqrt{1 - s^2}(1 - 2s^5)}{\sqrt{1 + 9s^4 + 4s^6}\sqrt{1 + s^2 + s^6}} \right), \right.$$

$$\left. \frac{s^5}{5} + u \left( \frac{s^4}{\sqrt{1 + s^2 + s^6}} + \frac{\sqrt{1 - s^2}(3s^2 + 2s^4)}{\sqrt{1 + 9s^4 + 4s^6}\sqrt{1 + s^2 + s^6}} \right) \right).$$

for  $\varrho_1(s) = s$  and  $\varrho_2(s) = \sqrt{1 - s^2}$ .

Case 2: The generalized osculating-type ruled surface  $\Psi_{(\gamma, \varrho_n)}$  is given by

$$M_2 = \left( \frac{s^2}{2} + u \left( \frac{3}{5\sqrt{1 + s^2 + s^6}} + \frac{-4s - 12s^5}{5\sqrt{1 + 9s^4 + 4s^6}\sqrt{1 + s^2 + s^6}} \right), \right.$$

$$\left. \frac{s^3}{3} + u \left( \frac{3s}{5\sqrt{1 + s^2 + s^6}} + \frac{(4 - 8s^5)}{5\sqrt{1 + 9s^4 + 4s^6}\sqrt{1 + s^2 + s^6}} \right), \right.$$

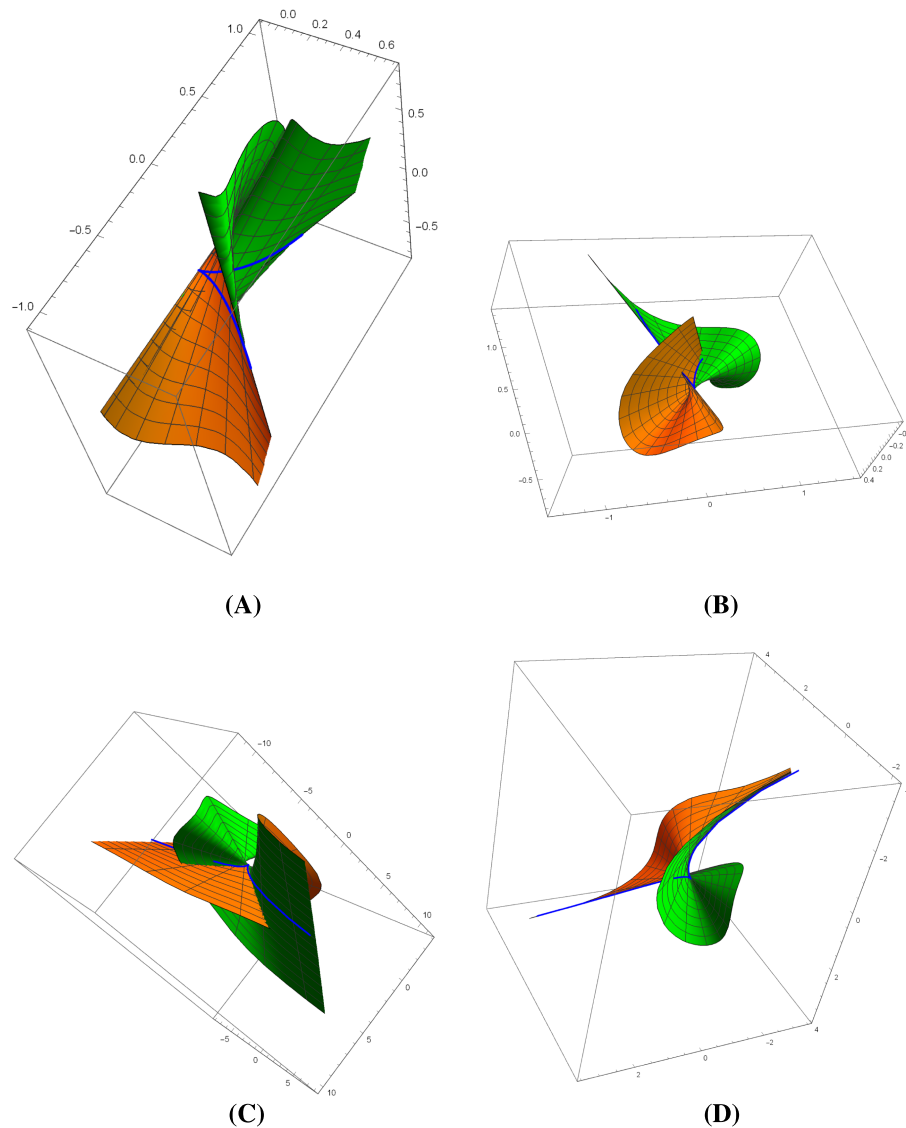
$$\left. \frac{s^5}{5} + u \left( \frac{3s^3}{5\sqrt{1 + s^2 + s^6}} + \frac{(12s^2 + 8s^4)}{5\sqrt{1 + 9s^4 + 4s^6}\sqrt{1 + s^2 + s^6}} \right) \right).$$

for  $\varrho_1(s) = \frac{3}{5}$  and  $\varrho_2(s) = \frac{4}{5}$ .

Case 3: The generalized osculating-type ruled surface  $\Psi_{(\gamma, \varrho_n)}$  is given by

$$M_3 = \left( \frac{s^2}{2} + u \frac{-s - 3s^5}{\sqrt{1 + 9s^4 + 4s^6}\sqrt{1 + s^2 + s^6}}, \frac{s^3}{3} + u \frac{1 - 2s^6}{\sqrt{1 + 9s^4 + 4s^6}\sqrt{1 + s^2 + s^6}}, \frac{s^5}{5} + u \frac{3s^2 + 2s^4}{\sqrt{1 + 9s^4 + 4s^6}\sqrt{1 + s^2 + s^6}} \right).$$

for  $\varrho_1(s) = 0$  and  $\varrho_2(s) = 1$ .



**FIGURE 3** The surface (A)  $M_1$ , (B)  $M_2$ , (C)  $M_3$ , and (D)  $M_4$  [Colour figure can be viewed at [wileyonlinelibrary.com](https://onlinelibrary.wiley.com/doi/10.1002/mm.8997)]

Case 4: The generalized osculating-type ruled surface  $\Psi_{(\gamma, \ell_n)}$  is given by

$$M_4 = \left( \frac{s^2}{2} + u \frac{1}{\sqrt{1+s^2+s^6}}, \frac{s^3}{3} + u \frac{s}{\sqrt{1+s^2+s^6}}, \frac{s^5}{5} + u \frac{s^3}{\sqrt{1+s^2+s^6}} \right).$$

for  $\rho_1(s) = 1$  and  $\rho_2(s) = 0$ .

The plots of surfaces  $M_1$ ,  $M_2$ ,  $M_3$  and  $M_4$  are given by Figure 3.

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## CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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