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Spinor representation of framed Mannheim curves

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Abstract: In this paper, we obtain spinor with two complex components representations of Mannheim curves of framed curves. Firstly, we give the spinor formulas of the frame corresponding to framed Mannheim curve. Later, we obtain the spinor formulas of the frame corresponding to framed Mannheim partner curve. Moreover, we explain the relationships between spinors corresponding to framed Mannheim pairs and their geometric interpretations. Finally, we present some geometrical results of spinor representations of framed Mannheim curves.

Key words: Spinors, Mannheim curves, framed curves

1. Introduction

The theory of spinors, especially used in applications to electron spin and theory of relativity in quantum mechanics, was expressed by B.L. van der Waerden in 1929. In three dimensions, spinors are used to describe the spin of the nonrelativistic electron and other spin-1/2 particles. They are also used to describe the state of relativistic many-particle systems in quantum field theory. Nowadays, spinors have a wide place in physics applications, mathematics, differential geometry, topology, and complex algebraic geometry. Spinors were first studied by Élie Cartan in a geometrical sense [1]. Cartan gave spinor representations of basic geometric structures. Therefore, this became an important reference for the geometric structure of spinors. Other important studies in terms of geometrical analysis of spinors were given by Vivarelli, and del Castillo and Barrales [2, 17]. Vivarelli, introduced spinor representations of rotations in three-dimensional Euclidean space and gave the relations between quaternions and spinors. On the other hand, del Castillo and Barrales expressed the spinor formulas of the Frenet frame of curves, which is an important structure in differential geometry. After this pioneering work of del Castillo and Barrales, researchers gave spinor representations of regular curves according to Darboux, Bishop, and Sabban frames [11, 14, 16].

Spinor representations of curves with respect to different frames are given for regular curves. This is due to the inability to construct an orthonormal frame for singular curves. Therefore, the natural structure of spinors cannot be constructed. However, recently, so-called “framed curves” in space have been introduced by Honda and Takahashi to study singular curves [7]. Framed curves are smooth curves that can be defined by a moving frame, although they have singular points. Framed curves have become quite popular in the literature.

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In fact, framed curves can be thought of as the construction in space of Legendre curves or spherical Legendre curves in plane and space [6, 12, 15]. In the literature, there are many studies about existence and uniqueness conditions of framed curves [5], framed special curves [18–20], the surfaces created with framed curves [8, 10]. One of these studies is Bertrand and Mannheim curves of framed curves in space were given by Honda and Takahashi [9]. A Mannheim curve is a space curve whose principal normal line is the same as the binormal line of another curve [13]. In [9], in addition to the singular Bertrand and Mannheim curves, the regular Bertrand and Mannheim curves were evaluated by adding the condition of being nondegenerate. It is also shown that framed Bertrand curves are also framed Mannheim curves. This is an interesting characterization as there is no such result for regular curves.

In this paper, we give spinor with two complex components representations of Mannheim curves of framed curves in Section 3.1. We obtain the spinor formulas for frame of framed Mannheim curve, (Theorems 4.2 and 4.3). Later, we give the spinor formulas of the frame corresponding to framed Mannheim partner curve, (Theorems 4.4 and 4.5). Moreover, we investigate the relationships between spinors corresponding to framed Mannheim pairs (Theorem 4.6). Finally, we give some geometrical interpretations of spinor representations of framed Mannheim curves.

2. Spinors

A spinor can be represented by

$$\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

for

$$x + iy = \varepsilon^t \sigma \varepsilon, \quad z = -\widehat{\varepsilon}^t \sigma \varepsilon, \tag{2.1}$$

where the reel vectors x, y, z and the "t" denotes transposition and $\widehat{\varepsilon}$ is the conjugate and $\overline{\varepsilon}$ is the complex conjugation of ε, as follows:

$$\widehat{\varepsilon} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\varepsilon} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\varepsilon}_1 \\ \overline{\varepsilon}_2 \end{pmatrix} = \begin{pmatrix} -\overline{\varepsilon}_2 \\ \overline{\varepsilon}_1 \end{pmatrix}.$$

Moreover, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is a vector whose cartesian components are the complex symmetrics 2×2 matrices

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

For the isotropic vector $x + iy = (a_1, a_2, a_3)$, we can write:

$$a_1 = \varepsilon^t \sigma_1 \varepsilon = \varepsilon_1^2 - \varepsilon_2^2,$$

$$a_2 = \varepsilon^t \sigma_2 \varepsilon = i(\varepsilon_1^2 + \varepsilon_2^2),$$

$$a_3 = \varepsilon^t \sigma_3 \varepsilon = -2\varepsilon_1 \varepsilon_2.$$

For the real vector $z = (b_1, b_2, b_3)$, we can write:

$$b_1 = \hat{\varepsilon}^t \sigma_1 \varepsilon = \varepsilon_1 \bar{\varepsilon}_2 + \bar{\varepsilon}_1 \varepsilon_2,$$
$$b_2 = \hat{\varepsilon}^t \sigma_2 \varepsilon = i(\varepsilon_1 \bar{\varepsilon}_2 - \bar{\varepsilon}_1 \varepsilon_2),$$
$$b_3 = \hat{\varepsilon}^t \sigma_3 \varepsilon = |\varepsilon_1|^2 - |\varepsilon_2|^2.$$

Then, $x + iy \in \mathbb{C}^3$ and $z \in \mathbb{R}^3$ are given as follows:

$$\begin{array}{ccl} x+iy & = & (\varepsilon_1^2 - \varepsilon_2^2, i(\varepsilon_1^2 + \varepsilon_2^2), -2\varepsilon_1\varepsilon_2), \\ z & = & (\varepsilon_1 \bar{\varepsilon}_2 + \bar{\varepsilon}_1 \varepsilon_2, i(\varepsilon_1 \bar{\varepsilon}_2 - \bar{\varepsilon}_1 \varepsilon_2), |\varepsilon_1|^2 - |\varepsilon_2|^2). \end{array} \tag{2.2}$$

Since the vectors $x + iy \in \mathbb{C}^3$ and $z \in \mathbb{R}^3$, it can be seen easily that the vectors x, y, and z are mutually orthogonal vectors, really we get:

$$\begin{array}{l} \langle x+iy, x+iy \rangle = \langle x, x \rangle + 2i\langle x, y \rangle - \langle y, y \rangle = 0, \\ \langle x, x \rangle = \langle y, y \rangle, \langle x, y \rangle = 0, \\ \langle x+iy, z \rangle = \langle \left(\varepsilon_1^2 - \varepsilon_2^2, i\left(\varepsilon_1^2 + \varepsilon_2^2\right), -2\varepsilon_1\varepsilon_2\right), \left(\varepsilon_1 \bar{\varepsilon}_2 + \bar{\varepsilon}_1 \varepsilon_2, i\left(\varepsilon_1 \bar{\varepsilon}_2 - \bar{\varepsilon}_1 \varepsilon_2\right), |\varepsilon_1|^2 - |\varepsilon_2|^2\right) \rangle = 0, \\ \langle x+iy, z \rangle = \langle x, z \rangle + i\langle y, z \rangle = 0, \\ \langle x, z \rangle = \langle y, z \rangle = 0. \end{array}$$

Moreover, since $\|z\| = (|\varepsilon_1|^2 + |\varepsilon_2|^2)$, and $\|x + iy\|^2 = \langle x + iy, \bar{x} + \overline{iy} \rangle = 2(|\varepsilon_1|^2 + |\varepsilon_2|^2)^2$, we get $\|x + iy\|^2 = \|x\|^2 + \|y\|^2 = 2(|\varepsilon_1|^2 + |\varepsilon_2|^2)^2$ and $\|x\| = \|y\| = \|z\| = \bar{\varepsilon}^t \epsilon$ and $\langle x \wedge y, z \rangle = det(x, y, z) > 0$. Conversely, if the vectors x, y and z one another orthogonal vectors of same magnitude $(det(x, y, z) > 0)$, then there is a spinor defined up to sign such that the equation (2.2) holds. The correspondence between the spinors and the orthogonal bases (for the equation (2.2)) is two-to-one because the spinors ε and $-\varepsilon$ correspond to the same ordered orthogonal bases $\{x, y, z\}$ with $\|x\| = \|y\| = \|z\|$ and $det(x, y, z) > 0$. Moreover, the ordered triads $\{x, y, z\}, \{y, z, x\}, \{z, x, y\}$ correspond to different spinors. The set $\{\hat{\varepsilon}, \varepsilon\}$ is linearly independent for the spinor $\varepsilon \neq 0$ [1, 2]. In addition, the following proposition can be given:

Proposition 2.1 *[2] Let ε and ϱ be arbitrary spinors and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then, we get:*

i. $\overline{\varrho^t \sigma \varepsilon} = -\hat{\varrho}^t \sigma \hat{\varepsilon}$,

ii. $(\lambda_1 \varrho + \lambda_2 \varepsilon)\hat{} = \overline{\lambda_1} \hat{\varrho} + \overline{\lambda_2} \hat{\varepsilon}$,

iii. $\hat{\hat{\varepsilon}} = -\varepsilon$,

iv. $\varrho^t \sigma \varepsilon = \varepsilon^t \sigma \varrho$.

3. Framed curves

Let $\gamma : I \to \mathbb{R}^3$ be a curve with singular points. To construct framed curves, a three-dimensional smooth manifold structure is defined as follows:

$$\Delta_2 = \{\mu = (\mu_1, \mu_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle \mu_i, \mu_j \rangle = \delta_{ij}, \quad i, j = 1, 2\}.$$

The vector ν is a unit vector defined by $\nu = \mu_1 \wedge \mu_2$. Hence, the definition of a framed curve is given as follows:

Definition 3.1 *$(\gamma,\mu_1,\mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ is called a framed curve if $\langle \gamma'(s), \mu_i(s)\rangle = 0$ for all $s \in I$ and $i = 1, 2$. $\gamma : I \to \mathbb{R}^3$ is said that a framed base curve if there exists $(\mu_1,\mu_2) : I \to \Delta_2$ such that (γ,μ_1,μ_2) is a framed curve [7].*

Moreover, $\{\nu,\mu_1,\mu_2\}$ is a moving frame for $(\gamma,\mu_1,\mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ and the Frenet-type formulas are as follows:

$$\begin{pmatrix} \nu'(s) \\ \mu_1'(s) \\ \mu_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & -m(s) & -n(s) \\ m(s) & 0 & l(s) \\ n(s) & -l(s) & 0 \end{pmatrix} \begin{pmatrix} \nu(s) \\ \mu_1(s) \\ \mu_2(s) \end{pmatrix}. \tag{3.1}$$

Here it is clearly seen that $l(s) = \langle \mu_1'(s), \mu_2(s)\rangle$, $m(s) = \langle \mu_1'(s), \nu(s)\rangle$ and $n(s) = \langle \mu_2'(s), \nu(s)\rangle$. Moreover, a smooth mapping $\alpha : I \to \mathbb{R}$ characterizing the singular points of the curve is given as

$$\gamma'(s) = \alpha(s)\nu(s).$$

In addition, s_0 is a singular point of the framed curve γ if and only if $\alpha(s_0) = 0$. The mapping (l, m, n, α) are called the curvature of the framed curve (γ,μ_1,μ_2). Moreover, the existence and uniqueness theorems for these curvatures are given in [5]. Moreover, the $(\breve{\mu}_1, \breve{\mu}_2) \in \Delta_2$ is defined by

$$\begin{pmatrix} \breve{\mu}_1(s) \\ \breve{\mu}_2(s) \end{pmatrix} = \begin{pmatrix} \cos\psi(s) & -\sin\psi(s) \\ \sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \mu_1(s) \\ \mu_2(s) \end{pmatrix} \tag{3.2}$$

where $\psi(s)$ is a smooth function. By using (3.2), the Bishop frame of the framed curves with the condition $l(s) = \psi'(s)$ and the Frenet frame with the condition $m(s)\sin\psi(s) + n(s)\cos\psi(s) = 0$ are obtained. See [5, 7, 18] for a detailed review of framed curves.

3.1. Mannheim curves of framed curves

Definition 3.2 *[9] Let (γ,μ_1,μ_2) and $(\overline{\gamma},\overline{\mu}_1,\overline{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ be framed curves. If there exists a smooth function $\lambda : I \to \mathbb{R}$ such that $\overline{\gamma}(s) = \gamma(s) + \lambda(s)\mu_1(s)$ and $\mu_1(s) = \overline{\mu}_2(s)$ for every $s \in I$, the framed curves (γ,μ_1,μ_2) and $(\overline{\gamma},\overline{\mu}_1,\overline{\mu}_2)$ are called Mannheim mates. Here, the curves (γ,μ_1,μ_2) and $(\overline{\gamma},\overline{\mu}_1,\overline{\mu}_2)$ are called as framed Mannheim curve and framed Mannheim partner curve, respectively.*

Additionally, the following equation holds since $\mu_1(s) = \overline{\mu}_2(s)$:

$$\begin{pmatrix} \overline{\nu}(s) \\ \overline{\mu}_1(s) \end{pmatrix} = \begin{pmatrix} \cos\psi(s) & -\sin\psi(s) \\ \sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \mu_2(s) \\ \nu(s) \end{pmatrix}. \tag{3.3}$$

On other respect, the following relations are satisfied between the framed Mannheim mates:

$$\begin{aligned} \bar{l}(s) &= -l(s)\sin\psi(s) - m(s)\cos\psi(s), \\ \overline{m}(s) &= \psi'(s) - n(s), \\ \overline{n}(s) &= l(s)\cos\psi(s) - m(s)\sin\psi(s), \\ \overline{\alpha}(s) &= \lambda l(s)\cos\psi(s) - (\alpha(s) + \lambda m(s))\sin\psi. \end{aligned} \tag{3.4}$$

Theorem 3.3 *Let* $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ *be a framed curve with the curvature* (l, m, n, α). *Then* (γ, μ_1, μ_2) *is a Mannheim curve if and only if there exist a nonzero constant* λ *and a smooth function* $\psi : I \to \mathbb{R}$ *such that*

$$\lambda l(s) \sin \psi(s) + (\alpha(s) + \lambda m(s)) \cos \psi(s) = 0 \tag{3.5}$$

for all $s \in I$ *[14].*

4. Spinor formulations of framed Mannheim curves

In this section, firstly, we investigate the spinor formulations of Mannheim partner curves of framed curves in the three dimensional Euclidean space.

Definition 4.1 *Let* $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ *be a framed Mannheim partner curve, and the spinor* ξ *corresponds to the triad* $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$. *Then, the following equations*

$$\overline{\nu} + i\,\overline{\mu}_1 = \xi^t \sigma \xi \tag{4.1}$$

and

$$\overline{\mu}_2 = -\widehat{\xi}^t \sigma \xi \tag{4.2}$$

can be written. Here $\overline{\xi}^t \xi = 1$.

Now, we present the following theorem:

Theorem 4.2 *Let* $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ *be a framed Mannheim partner curve and the spinor* ξ *corresponds to the triad* $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$. *Then, the following single spinor equation which contains the curvatures of Mannheim curves of framed curves is satisfied:*

$$\frac{d\xi}{ds} = \frac{1}{2}\left(i\,\overline{m}\xi + \left(\overline{n} - i\overline{l}\right)\widehat{\xi}\right). \tag{4.3}$$

Proof Via differentiation of the equation (4.1) with respect to s, then we get,

$$\frac{d\overline{\nu}}{ds} + i\frac{d\overline{\mu}_1}{ds} = \left(\frac{d\xi}{ds}\right)^t \sigma\xi + \xi^t \sigma \left(\frac{d\xi}{ds}\right). \tag{4.4}$$

In other respect, the following equation can be written because of the fact that $\{\xi, \widehat{\xi}\}$ is a basis for spinors.

$$\frac{d\xi}{ds} = f\xi + g\widehat{\xi} \tag{4.5}$$

where f and g are any complex valued functions. According to equation (4.4), by using the equations (3.1), (4.1), (4.2), and (4.5), we get:

$$(-\overline{m}\,\overline{\mu}_1 - \overline{n}\,\overline{\mu}_2) + i\left(\overline{l}\,\overline{\mu}_2 + \overline{m}\,\overline{\nu}\right) = \left(f\xi + g\widehat{\xi}\right)^t \sigma\xi + \xi^t\sigma(f\xi + g\widehat{\xi}),$$

$$i\,\overline{m}\left(\overline{\nu} + i\,\overline{\mu}_1\right) + \left(-\overline{n} + i\,\overline{l}\right)\overline{\mu}_2 = 2f\left(\overline{\nu} + i\,\overline{\mu}_1\right) - 2g\overline{\mu}_2.$$

From the last equation, the following can be written:

$$\begin{cases} f = \dfrac{i\,\overline{m}}{2}, & (4.6a)\\ g = \dfrac{\overline{n} - i\,\overline{l}}{2}. & (4.6b)\end{cases}$$

That is,

$$\frac{d\xi}{ds} = \left(\frac{i\,\overline{m}}{2}\right)\xi + \left(\frac{\overline{n} - i\,\overline{l}}{2}\right)\widehat{\xi}.$$

□

Theorem 4.3 *Let* $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2) : I \to \mathbb{R}^3 \times \Delta_2$ *be a framed Mannheim partner curve and the spinor* ξ *corresponds to the triad* $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$. *Then, the spinor equations of the framed vectors are given as follows:*

$$\overline{\nu} = \frac{1}{2}\left(\xi^t\sigma\xi - \widehat{\xi}^t\sigma\widehat{\xi}\right),$$

$$\overline{\mu}_1 = -\frac{i}{2}\left(\xi^t\sigma\xi + \widehat{\xi}^t\sigma\widehat{\xi}\right),$$

$$\overline{\mu}_2 = -\widehat{\xi}^t\sigma\widehat{\xi}.$$

Proof Suppose that the spinor ξ corresponds to the triad $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$ of the framed Mannheim partner curve $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$. According to the equation (4.1), we get $\overline{\nu} = Re(\xi^t\sigma\xi)$ and $\overline{\mu}_1 = Im(\xi^t\sigma\xi)$. Then,

$$\overline{\mu}_1 = -\frac{i}{2}\left(\xi^t\sigma\xi - \overline{\xi^t\sigma\xi}\right)$$

and

$$\overline{\nu} = \frac{1}{2}\left(\xi^t\sigma\xi + \overline{\xi^t\sigma\xi}\right)$$

are written. By using the equation $\overline{\varphi^t\sigma\xi} = -\widehat{\varphi}^t\sigma\widehat{\xi}$, we obtain:

$$\overline{\mu}_1 = -\frac{i}{2}\left(\xi^t\sigma\xi + \widehat{\xi}^t\sigma\widehat{\xi}\right)$$

and

$$\overline{\nu} = \frac{1}{2}\left(\xi^t\sigma\xi - \widehat{\xi}^t\sigma\widehat{\xi}\right).$$

In addition to these, we know already $\overline{\mu}_2 = -\widehat{\xi}^t\sigma\xi$ from the equation (4.2). □

Now, let us think the framed Mannheim curve (γ, μ_1, μ_2). The spinor φ corresponds to the triad $\{\mu_2, \nu, \mu_1\}$ of the curve. Therefore,

$$\mu_2 + i\nu = \varphi^t\sigma\varphi, \tag{4.7}$$

$$\mu_1 = -\widehat{\varphi}^t\sigma\varphi, \tag{4.8}$$

where $\overline{\varphi}^t\varphi = 1$ Then, the following theorem can be given.

Theorem 4.4 *Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed Mannheim curve and the spinor φ corresponds to the triad $\{\mu_2, \nu, \mu_1\}$. Hence, the following single spinor equation that includes the curvatures of framed curves is written as:*

$$\frac{d\varphi}{ds} = \frac{1}{2}\left[(-i\,n)\,\varphi + (l + i\,m)\,\widehat{\varphi}\right].$$

Proof By differentiation of the equation (4.7), we get:

$$\frac{d\mu_2}{ds} + i\frac{d\nu}{ds} = \left(\frac{d\varphi}{ds}\right)^t \sigma\varphi + \varphi^t\sigma\frac{d\varphi}{ds}. \tag{4.9}$$

Moreover, since $\{\varphi, \widehat{\varphi}\}$ is a basis of the spinors, the following equation can be obtained:

$$\frac{d\varphi}{ds} = f\varphi + g\widehat{\varphi}. \tag{4.10}$$

where f and g are any complex valued functions. According to equation (4.9), by using the equations (3.1), (4.7), (4.8), and (4.10), we get:

$$(-l\,\mu_1 + n\,\nu) + i\,(-m\,\mu_1 - n\,\mu_2) = (f\varphi + g\widehat{\varphi})^t\,\sigma\varphi + \varphi^t\sigma(f\varphi + g\widehat{\varphi}),$$
$$-i\,n\,(\mu_2 + i\,\nu) + (-l - i\,m)\,\mu_1 = 2f\,(\mu_2 + i\,\nu) - 2g\mu_1.$$

Hence, we have:

$$\begin{cases} f = \dfrac{-\,i\,n}{2}, \\[2ex] g = \dfrac{l + i\,m}{2}. \end{cases}$$

Consequently, we get:

$$\frac{d\varphi}{ds} = \frac{1}{2}\left[(-i\,n)\,\varphi + (l + i\,m)\,\widehat{\varphi}\right].$$

□

Theorem 4.5 *Let $(\gamma, \mu_1, \mu_2) : I \to \mathbb{R}^3 \times \Delta_2$ be a framed Mannheim curve and the spinor φ corresponds to the triad $\{\mu_2, \nu, \mu_1\}$. Then, the spinor equations for the framed vectors are as follows:*

$$\nu = -\frac{i}{2}\left(\varphi^t\sigma\varphi + \widehat{\varphi}^t\sigma\widehat{\varphi}\right),$$
$$\mu_1 = -\widehat{\varphi}^t\sigma\varphi,$$
$$\mu_2 = \frac{1}{2}\left(\varphi^t\sigma\varphi - \widehat{\varphi}^t\sigma\widehat{\varphi}\right).$$

Proof By using the equation (4.7), we have $\mu_2 = Re(\varphi^t\sigma\varphi)$ and $\nu = Im(\varphi^t\sigma\varphi)$. Then, we can write:

$$\nu = -\frac{i}{2}\left(\varphi^t\sigma\varphi - \overline{\varphi^t\sigma\varphi}\right),$$
$$\mu_2 = \frac{1}{2}\left(\varphi^t\sigma\varphi + \overline{\varphi^t\sigma\varphi}\right).$$

Therefore, similar to Theorem 4.3, by using the equation $\overline{\varphi^t\sigma\varphi} = -\widehat{\varphi}^t\sigma\widehat{\varphi}$, desired is achieved, and the proof is completed since $\mu_1 = -\widehat{\varphi}^t\sigma\varphi$ from the equation (4.8) we know already. □

Theorem 4.6 *Let $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ and (γ, μ_1, μ_2) be framed Mannheim mates, and the spinors ξ and φ correspond to triad $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$ and $\{\mu_2, \nu, \mu_1\}$, respectively. Then, the following relation is satisfied:*

$$\xi = \pm e^{i\frac{\psi}{2}}\varphi \tag{4.11}$$

where ψ is an angle between the $\overline{\nu}$ and μ_2.

Proof According to equation (3.3), we know the following equations:

$$\overline{\nu} = \cos\psi\mu_2 - sin\psi\nu,$$
$$\overline{\mu}_1 = \sin\psi\mu_2 + \cos\psi\nu.$$

In that case, for the vector $\overline{\nu} + i\,\overline{\mu}_1 \in \mathbb{C}^3$, we get:

$$\begin{aligned}
\overline{\nu} + i\,\overline{\mu}_1 &= (\cos\psi\mu_2 - \sin\psi)\,\nu + i\,(\sin\psi\mu_2 + \cos\psi\nu)\,,\\
&= (-\sin\psi + i\cos\psi)\,\nu + (\cos\psi + i\sin\psi)\,\mu_2,\\
&= i\,(\cos\psi + i\sin\psi)\,\nu + (\cos\psi + i\sin\psi)\,\mu_2,\\
&= (\cos\psi + i\sin\psi)\,(\mu_2 + i\nu)\\
&= e^{i\psi}\,(\mu_2 + i\nu)\,.
\end{aligned}$$

Then, we have:

$$\overline{\nu} + i\,\overline{\mu}_1 = e^{i\psi}\,(\mu_2 + i\nu)$$

where ψ is an angle between the $\overline{\nu}$ and μ_2. Hence,

$$\xi^t\sigma\xi = e^{i\psi}\left(\varphi^t\sigma\varphi\right)$$

can be written. Considering that

$$\xi^t\sigma\xi = \left(\xi_1^2 - \xi_2^2, i\left(\xi_1^2 + \xi_2^2\right), -2\xi_1\xi_2\right),$$
$$\varphi^t\sigma\varphi = \left(\varphi_1^2 - \varphi_2^2, i\left(\varphi_1^2 + \varphi_2^2\right), -2\varphi_1\varphi_2\right),$$

then we get $\xi_1^2 = e^{i\psi}\varphi_1^2$ and $\xi_2^2 = e^{i\psi}\varphi_2^2$. Therefore,

$$\xi = \pm e^{i\frac{\psi}{2}}\varphi$$

can be seen. □

Corollary 4.7 *Let $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ and (γ, μ_1, μ_2) be framed Mannheim mates, and the spinors ξ and φ correspond to triad $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$ and $\{\mu_2, \nu, \mu_1\}$, respectively. Then, the angle between ξ and φ is $\dfrac{\psi}{2}$.*

Corollary 4.8 *Let* $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ *and* (γ, μ_1, μ_2) *be framed Mannheim mates and the spinors* ξ *and* φ *correspond to triad* $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$ *and* $\{\mu_2, \nu, \mu_1\}$*, respectively. The relation between conjugations of the spinors is*

$$\widehat{\xi} = \pm e^{-i\frac{\psi}{2}}\widehat{\varphi}. \tag{4.12}$$

Proof By taking the conjugation of the equation of the equation (4.11), then we get:

$$\widehat{\xi} = (\pm e^{i\frac{\psi}{2}}\varphi)\widehat{}.$$

Additionally, by using the equation $(\lambda\varphi + \mu\xi)\widehat{} = \overline{\lambda}\widehat{\varphi} + \overline{\mu}\widehat{\xi}$, we get the equation (4.12). □

Corollary 4.9 *Let* $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ *and* (γ, μ_1, μ_2) *be framed Mannheim mates and the spinors* ξ *and* φ *correspond to triad* $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$ *and* $\{\mu_2, \nu, \mu_1\}$*, respectively. While there exists a rotation with the angle* $\frac{\psi}{2}$ *between spinors* ξ *and* φ*, there exists a rotation with the reverse angle* $\frac{\psi}{2}$ *between the spinors* $\widehat{\xi}$ *and* $\widehat{\varphi}$*.*

Corollary 4.10 *Let* $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ *and* (γ, μ_1, μ_2) *be framed Mannheim mates and the spinors* ξ *and* φ *correspond to triad* $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$ *and* $\{\mu_2, \nu, \mu_1\}$*, respectively. Then, the following equation holds:*

$$\frac{d\xi}{ds} = \frac{1}{2}\left[i\left(\psi' - n\right)\xi + e^{i\psi}\left(l + im\right)\widehat{\xi}\right]. \tag{4.13}$$

Proof Let the spinors ξ and φ correspond to triad $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$ and $\{\mu_2, \nu, \mu_1\}$, respectively. With the help of the equation (4.3) and (3.4), we get:

$$\begin{aligned}
\frac{d\xi}{ds} &= \frac{1}{2}\left[i(\psi' - n)\xi + (l\cos\psi - m\sin\psi - i(-l\sin\psi - m\cos\psi))\widehat{\xi}\right] \\
&= \frac{1}{2}\left[i(\psi' - n)\xi + (l(\cos\psi + i\sin\psi) - m(\sin\psi - i\cos\psi))\widehat{\xi}\right] \\
&= \frac{1}{2}\left[i(\psi' - n)\xi + l(\cos\psi + i\sin\psi) + im(\cos\psi + i\sin\psi)\widehat{\xi}\right] \\
&= \frac{1}{2}\left[i\left(\psi' - n\right)\xi + e^{i\psi}\left(l + im\right)\widehat{\xi}\right].
\end{aligned}$$

□

Corollary 4.11 *Let* $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ *and* (γ, μ_1, μ_2) *be framed Mannheim mates and the spinors* ξ *and* φ *correspond to triad* $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$ *and* $\{\mu_2, \nu, \mu_1\}$*, respectively. If the angle between the* $\overline{\nu}$ *and* μ_2 *is* ψ*, then the angle between* $\frac{d\xi}{ds}$ *and* $\widehat{\xi}$ *is also* ψ *provided that* $\psi' = n$*.*

Corollary 4.12 *Let* $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ *and* (γ, μ_1, μ_2) *be framed Mannheim mates and the spinors* ξ *and* φ *correspond to triad* $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$ *and* $\{\mu_2, \nu, \mu_1\}$ *respectively. In that case, the angle between the spinors* $\frac{d\xi}{ds}$ *and* $\widehat{\varphi}$ *is* $\frac{\psi}{2}$ *on condition that* $\psi' = n$*.*

Proof According to the equation (4.13), we have:

$$\frac{d\xi}{ds} = \frac{1}{2}\left[i(\psi^{'} - n)\xi + e^{i\psi}(l + im)\widehat{\xi}\right].$$

Then, by the equation (4.11) and (4.12), we obtain:

$$\begin{aligned}\frac{d\xi}{ds} = & \pm\frac{1}{2}\left[i(\psi^{'} - n)e^{\frac{i\psi}{2}}\varphi \pm e^{i\psi}(l + im)e^{\frac{-i\psi}{2}}\widehat{\varphi}\right] \\ = & \pm\frac{1}{2}\left[i(\psi^{'} - n)e^{\frac{i\psi}{2}}\varphi \pm (l + im)e^{\frac{i\psi}{2}}\widehat{\varphi}\right] \\ = & \pm\frac{1}{2}e^{\frac{i\psi}{2}}\left[i(\psi^{'} - n)\varphi + (l + im)\widehat{\varphi}\right].\end{aligned}$$

□

Remark 4.13 *Let $(\overline{\gamma}, \overline{\mu}_1, \overline{\mu}_2)$ and (γ, μ_1, μ_2) be framed Mannheim mates and the spinors ξ and φ correspond to triad $\{\overline{\nu}, \overline{\mu}_1, \overline{\mu}_2\}$ and $\{\mu_2, \nu, \mu_1\}$, respectively. It can be written in terms of spinor equations (l, m, n, α) using the equation (3.4) and the Theorem 3.3. Therefore, spinor equations can be expressed according to the function α that characterizes the singular points.*

5. Conclusion

In this study, spinor representations of Mannheim curves of singular curves are explained with the help of framed curve theory. In addition, some geometric interpretations of these representations are given. Similarly, spinor formulas for Bertrand curves of framed curves can be given based on the study [9] for framed curves and the study [4] for regular curves, and spinor formulas for involute-evolute curves of framed curves can be given using [3] study. Moreover, some physical results can be obtained according to the physical structure of spinors.

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References

[1] Cartan É. The Theory of Spinors. Paris: Hermann, 1966 [New York: Dover, (reprinted 1981)].

[2] del Castillo GFT, Barrales GS. Spinor formulation of the differential geometry of curve. Revista Colombiana de Matemáticas 2004; 38 (1): 27-34.

[3] Erişir T, Kardağ NC. Spinor Representations of involute evolute curves in $\mathbb{E}^3$. Fundamental Journal of Mathematics and Applications 2019; 2 (2): 148-155.

[4] Erişir T. On spinor construction of Bertrand curves, AIMS Mathematics 2021; 6 (4): 3583-3591.

[5] Fukunaga T, Takahashi M. Existence conditions of framed curves for smooth curves. Journal of Geometry 2017; (108): 763-774.

[6] Fukunaga T, Takahashi M. Existence and uniqueness for Legendre curves, Journal of Geometry 2013; 104 (2): 297-307.

[7] Honda S, Takahashi M. Framed curves in the Euclidean space. Advances in Geometry 2016; 16 (3): 265-276.

[8] Honda S, Takahashi M. Evolutes and focal surfaces of framed immersions in the Euclidean space. Proceedings of the Royal Society of Edinburgh: Section A Mathematics 2020; 150 (1): 497-516.

[9] Honda S, Takahashi M. Bertrand and Mannheim curves of framed curves in the 3-dimensional Euclidean space. Turkish Journal of Mathematics 2020; 44 (3): 883-899.

[10] Honda S. Rectifying developable surfaces of framed base curves and framed helices. Singularities in Generic Geometry, Mathematical Society of Japan, 2018.

[11] Kişi İ, Tosun M. Spinor Darboux equations of curves in Euclidean 3-space. Mathematica Moravica 2015; 19 (1): 87-93.

[12] Li Y, Pei D, Takahashi M, Yu H. Envelopes of Legendre curves in the unit spherical bundle over the unit sphere. The Quarterly Journal of Mathematics 2018; 69 (2): 631-653.

[13] Liu H, Wang F. Mannheim partner curves in 3-space. Journal of Geometry 2008; 88 (1): 120-126.

[14] Şenyurt S, Çalışkan A. Spinor formulation of Sabban frame of Curve on S^2, Pure Mathematical Sciences 2015; 4 (1): 37-42.

[15] Takahashi M. Legendre curves in the unit spherical bundle over the unit sphere and evolutes. Real and complex singularities, Contemporary Mathematics, 2016; 675: 337-355.

[16] Ünal D, Kişi İ, Tosun M. Spinor Bishop equations of curves in Euclidean 3-space. Advances in Applied Clifford Algebras 2013; 23 (3): 757-765.

[17] Vivarelli MD. Development of spinors descriptions of rotational mechanics from Euler's rigid body displacement theorem. Celestial Mechanics 1984; 32: 193-207.

[18] Wang Y, Pei D, Gao R. Generic properties of framed rectifying curves. Mathematics 2019; 7 (1): 37.

[19] Yazıcı BD, Karakuş SÖ, Tosun M. On the classification of framed rectifying curves in Euclidean space. Mathematical Methods Application Sciences 2021; 1-10.

[20] Yazıcı BD, Karakuş SÖ, Tosun M. Framed normal curves in Euclidean space. Tbilisi Mathematical Journal 2021; 27-37.