10th International IFS and Contemporary Mathematics and Engineering Conference

September 4-7, 2024
Mersin, TÜRKİYE

IFSCOM-E 2024

Proceeding Book

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ISBN: 978-625-97923-1-6

IFSCOM-E 2024
10TH IFS AND CONTEMPORARY MATHEMATICS AND ENGINEERING CONFERENCE
04-07 SEPTEMBER 2024, MERSİN, TÜRKİYE
ISBN:978-625-97923-1-6
pp:103-108

SIMILARITY METRIC AND ITS RELATIONS BETWEEN OTHER METRICS

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ABSTRACT. In this paper, we focus on the notion of similarity metric spaces, partial metric spaces and (weighted) quasi metric spaces with their certain properties. Furthermore, we investigate the relations of these structures on a given set and give various examples.

1. INTRODUCTION

As one of the most common ways of comparing objects is the measure of distance, interpreting and measuring the similarity of objects is also conventionally associated with distance. Similarity, as expressed by [1], could not be only based on distance, but also on the amount of common features. Although the concept of similarity is used in various fields such as physics, statistics, data science, psychology, etc., (see [1–7]) the formal axiomatic definition of similarity metric that we consider is first defined in [8]. By using this definition, the duality of similarity and distance is recently studied by [9]. Also, the partial and (weighted) quasi metric is used to examine partially defined data that is likely to occur in computer science and is defined as an extension of the distance metric, where the self-distance may not always be zero [10,11]. A considerable number of studies about quasi metric spaces and partial metric spaces exist which are relevant to the topological properties of them, the fixed point theory and theoretic computer science (See [12–20] and others in them). In this study, we focus on the relations of similarity metric with quasi metric and partial metric functions on a given set U and give elegant ways to construct a similarity metric from a quasi metric and partial metric on a non-empty set U, and state the correspondence between these structures and a similarity metric. Also, we present coherent examples about the idea.

2. THE SIMILARITY SPACES

Definition 2.1. Let $U \neq \emptyset$ and $s : U \times U \to \mathbb{R}$ be a function satisfying the axioms

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10th International IFS and Contemporary Mathematics and Engineering Conference
04-07 September 2024
Mersin TÜRKİYE

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This certificate is given to

ESRA GÜDER

in online presentation due to her research contributions to IFSCOM-E 2024 "10th International IFS and Contemporary Mathematics and Engineering Conference " in Mersin, TÜRKİYE, September 04-07, 2024

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IFSCOM-E 2024
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S1. $s(v,v) \geq 0$,
S2. $s(v,v) \geq s(v,w)$,
S3. $s(v,w) = s(w,v)$,
S4. $s(v,v) = s(v,w) = s(w,w) \Leftrightarrow v = w$,
S5. $s(v,z) \geq s(v,w) + s(w,z) - s(w,w)$

for all $v,w,z \in U$. Then, the function s is said to be a similarity metric on U and (U,s) is called a similarity space. Also, s is said to be normalized similarity metric if $|s(v,w)| \leq 1$ for every $v,w \in U$ [8].

As it is seen, with a similarity function, one can analogize the similarities of objects of a set. Contrary to distance measures, as two objects v, w get "similar" to each other, the value $s(v,w)$ gets bigger.

Example 2.2. The well known Jaccard index in statistics measures the similarity between finite sets of samples V and W from a population X and is denoted by

$$J(V,W) = \frac{|V \cap W|}{|V \cup W|}.$$

This is a simple similarity metric on X and so (X,J) is a similarity space. These and similar examples can be found in [8,9].

Also, in the studies [1,8], similarity and distance are considered interchangeably. By way of example, let (U,d) be a metric space and consider the function $s(v,w) = d(v,\bar{v}) + d(w,\bar{v}) - d(v,w)$ for a fixed element $\bar{v} \in U$ and all $v,w \in U$. It can be seen that s satisfies *S1-S5* and so (U,s) is a similarity space.

3. THE RELATIONS BETWEEN SIMILARITY METRIC AND OTHER METRIC STRUCTURES

The correspondence between a distance metric, a partial metric and a quasi metric has been studied by many authors. This section is reserved for the basic notions and definitions about the quasi metric and the partial metric. Then, we evaluate the relations of these structures with a (given) similarity metric.

Definition 3.1. Let $U \neq \emptyset$ and $q : U \times U \to \mathbb{R}$ be a function satisfying the following axioms for all $v,w,z \in U$.

Q1. $q(v,w) \geq 0$,
Q2. $q(v,w) = q(w,v) = 0 \Leftrightarrow v = w$,
Q3. $q(v,z) \leq q(v,w) + q(w,z)$.

Then, the function q is called a quasi metric on U and (U,q) is said to be a quasi metric space. It is well known that the quasi metric is an asymmetric distance function. The conjugate (quasi metric) of q is defined by $q^{*}(v,w) = q(w,v)$ for every $v,w \in U$ which is also a quasi metric. Moreover, a quasi metric space (U,q) is called "weighted" and denoted by (U,q,φ) if there exists a function $\varphi : U \to \mathbb{R}$ with

$$\varphi(v) + q(v,w) = \varphi(w) + q(w,v). \tag{3.1}$$

In particular case, if $\varphi(U) \subset [0,\infty)$, then the space is called positively weighted. For detailed information, see [21–24] and others in them.

Example 3.2. Let $q : \mathbb{R}^2 \to \mathbb{R}_0^+$ be given by

$$q(v,w) = max\{w - v, 0\},$$

for all $v, w \in \mathbb{R}$. Then, it can be seen that the function q satisfies $Q1 - Q3$. So, q is a quasi metric function. Moreover, The space can be considered as weighted with the function $\varphi : \mathbb{R} \to \mathbb{R}$, with $\varphi(v) = v$.

Example 3.3. Let $q(K, L)$ be defined as the cost of the travel from the departure city K to the arrival city L. Then, obviously $q(K, L) \geq 0$ and $q(K, L) = 0$ if and only if $K = L$. Also, one can consider that $q(K, M) \leq q(K, L) + q(L, M)$ for all cities K, L, M. On the other hand, depending on the supply-demand and taxes, the prices $q(K, L)$ and $q(L, K)$ could be slightly different. In this manner, the function q is a quasi metric function. Also, by considering $\varphi(K)$ to be the local tax for leaving the city K with

$$q(K, L) + \varphi(K) = q(L, K) + \varphi(L),$$

for any cities K, L, we get the positively (weighted) quasi metric space (U, q, φ), on the set of the cities U [24].

Definition 3.4. Let $U \neq \emptyset$ and $p : U \times U \to \mathbb{R}$ be a function with the followings for all $v, w, z \in U$.

P1. $p(v, w) \geq p(v, v) \geq 0$,
P2. $p(v, w) = p(w, v)$,
P3. $p(v, v) = p(v, w) = p(w, w) \Leftrightarrow v = w$,
P4. $p(v, z) \leq p(v, w) + p(w, z) - p(w, w)$.

Then, the function p is called a partial metric on U and (U, p) is said to be a partial metric space.

Obviously, any metric space is a partial metric space. Another partial metric example is as follows:

Example 3.5. Let U be the collection of all closed intervals, $[v, w] \subset \mathbb{R}$, and $p : U^2 \to [0, \infty)$ be defined as

$$p([v, w], [z, t]) = max\{w, t\} - min\{v, z\}.$$

Then, p is a partial metric on U.

Example 3.6. Let $U = \mathbb{R}$ and the function p be as, $p(\alpha, \beta) = e^{max\{\alpha, \beta\}}$ for all $\alpha, \beta \in U$. Then, by direct calculation, in can be seen that (U, p) is a partial metric space.

In [11], it is shown that one can obtain a (weighted) quasi metric from a partial metric and vice versa. For detailed information, see [10, 11].

Now, we give the relations between these structures and similarity metric on a given set U.

Proposition 1. Let (U, s) be a similarity space and define $q_s : U \times U \to \mathbb{R}$ as

$$q_s(v, w) = s(v, v) - s(v, w)$$

for all $v, w \in U$. Then, (U, q_s) is a quasi metric space.

Proof. The mapping q_s satisfies the axioms *Q1-Q3* as follows:

Q1. It follows from the axiom *S2*.
Q2. If $q_s(v, w) = q_s(w, v) = 0$ for all $v, w \in U$, then $q_s(v, w) = s(v, v) - s(v, w) = 0$ and $q_s(w, v) = s(w, w) - s(w, v) = 0$. So, we have $s(v, v) = s(v, w) = s(w, w)$ from the axiom *S3* and hence $v = w$ from the axiom *S4*. Also, the converse is obvious.

Q3. For all $v, w, z \in U$, the result is obtained by using the axiom *S5* as follows:

$$\begin{aligned} q_s(v,z) &= s(v,v) - s(v,z) \\ &\leq s(v,v) - [s(v,w) + s(w,z) - s(w,w)] \\ &= s(v,v) - s(v,w) + s(w,w) - s(w,z) \\ &= q_s(v,w) + q_s(w,z). \end{aligned}$$

Thus, q_s is a quasi metric. □

Remark 3.7. In particular, if there exists $M \in \mathbb{R}$ such that, $s(v,w) < M$ for any $v, w \in U$, then the function $\varphi_s(v) := M - s(v,u)$ is indeed a weight function for (U, q_s). Thus, one can obtain a (weighted) quasi metric from a given s, under the condition of the existence such M. As a corollary, a (weighted) quasi metric can be obtained from a normalized similarity metric.

On the contrary to the above proposition, one may not be able to obtain a similarity metric from a quasi metric. To do that, we also need a weight function on (U, q).

Proposition 2. Let (U, q, φ_q) be a (weighted) quasi metric space and $s_q : U \times U \to \mathbb{R}$ be a function as

$$s_q(v,w) = \varphi_q(v) - q(w,v)$$

for all $v, w \in U$. Then, (U, s_q) is a similarity metric space.

Proof. The mapping s_q satisfies the axioms *S1-S5* as follows:

S1. It follows from the axiom *Q2* and the definition of φ.

S2. For all $v, w \in U$, we get the result by using the axioms *Q1* and *Q2* as below:

$$s_q(v,v) - s_q(v,w) = \varphi_q(v) - q(v,v) - \varphi_q(v) + q(w,v) \geq 0.$$

S3. For all $v, w \in U$, we get the result by using the condition (3.1) as below:

$$\begin{aligned} s_q(v,w) - s_q(w,v) &= \varphi_q(v) - q(w,v) - \varphi_q(w) + q(v,w) \\ &= \varphi_q(v) - \varphi_q(w) - q(w,v) + q(v,w) \\ &= q(w,v) - q(v,w) - q(w,v) + q(v,w) \\ &= 0. \end{aligned}$$

S4. For all $v, w \in U$, if $s_q(v,v) = s_q(v,w)$, then we have $\varphi_q(v) = \varphi_q(v) - q(w,v)$ and hence $q(w,v) = 0$. Similarly, if $s_q(w,w) = s_q(v,w)$ and we use the axiom *S3*, then we have $\varphi_q(w) = \varphi_q(w) - q(v,w)$ and hence $q(v,w) = 0$. Thus, we obtain $v = w$ from the axiom *Q2*. Also, the converse is obvious.

S5. For all $v, w, z \in U$, we obtain the result by using the axioms *Q2* and *Q3* as below:

$$\begin{aligned} s_q(v,z) &= \varphi_q(v) - q(z,v) \\ &\geq \varphi_q(v) - [q(z,w) + q(w,v)] \\ &= \varphi_q(v) - q(w,v) + \varphi_q(w) - q(z,w) - \varphi_q(w) - q(w,w) \\ &= s_q(v,w) + s_q(w,z) - s_q(w,w). \end{aligned}$$

Thus, s_q is a similarity metric. □

Remark 3.8. Note that, with keeping the notation above in mind, by constructing the quasi metric q_{s_q} that is deduced from the similarity metric s_q, we indeed obtained q^*, the conjugate of q. In a word, $q_{s_q} = q^*$. In addition, if there exists M with $s_q(v,w) < M$ for all $v, w \in U$, $\varphi_{q_{s_q}} = M - \varphi_q$.

For a given similarity metric s on U, it is proven that the function $d(v,w) = s(v,v) + s(w,w) - 2s(v,w)$ is a distance metric [9]. So, we can state the following corollary:

Corollary 3.9. Let (U,s) be a similarity space. Then, the function $d_s = q_s + q_s^*$ obtained as

$$d_s(v,w) = s(v,v) + s(w,w) - 2s(v,w)$$

is a distance metric.

We can state the following two propositions by considering the correspondence of the partial metrics and the (weighted) quasi metrics, studied by [11].

Proposition 3. Let (U,p) be a partial metric space and $s_p : U \times U \to \mathbb{R}$ be a function such that

$$s_p(v,w) = p(v,v) + p(w,w) - p(v,w)$$

for all $v, w \in U$. Then, (U, s_p) is a similarity metric space.

Proof. From [11], (U, q, φ) is a (weighted) quasi metric space with $q(v,w) = p(v,w) - p(v,v)$ and $\varphi(v) = p(v,v)$. Then, from Proposition 2, we have a similarity metric $s(v,w) = \varphi(v) - q(w,v)$. Hence, $s(v,w) = p(v,v) - p(v,w) + p(w,w)$. ☐

Example 3.10. Let $(\mathbb{R}^+, p)$ be partial metric space with $p(v,w) = \max\{v,w\}$. Then, from Proposition 3, one can see that $s(v,w) = v + w - \max\{v,w\}$ is a similarity metric on $\mathbb{R}^+$.

Proposition 4. Let (U,s) be a similarity metric space with $M \in \mathbb{R}$ with $s(v,w) < M$, and $p_s : U \times U \to \mathbb{R}$ be a function such that

$$p_s(v,w) = M - s(v,w)$$

for every $v, w \in U$. Then, (U, p_s) is a partial metric space.

Proof. As we mentioned before, (U, q_s, φ) is a (weighted) quasi metric with $q_s(v,w) = s(v,v) - s(v,w)$ and $\varphi(v) = M - s(v,v)$. Then, the partial metric p corresponding to q_s is $p(v,w) = \varphi(v) + q_s(v,w)$. Hence, $p(v,w) = M - s(v,w)$. ☐

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