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Estimation of renewal function under progressively censored data and its applications

Ömer Altındağ [a,*], Halil Aydoğdu [b]

[a] Department of Statistics and Computer Sciences, Bilecik Şeyh Edebali University, Gülümbe, Bilecik 11230, Turkey
[b] Department of Statistics, Ankara University, Tandoğan, Ankara 06100, Turkey

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ABSTRACT

Renewal function is an important tool used by researchers in the fields of applied probability such as reliability theory, risk analysis, inventory theory and warranty analysis etc. Estimation problem of this function under complete and right censored samples is well studied in the literature. However, there isn't any study dealing with the estimation problem of this function under progressive censoring which is used widely in survival and failure analyzes. In this study, estimation problem of renewal function as well as variance function of a renewal process under progressively censored data is considered. Some parametric plug-in estimators are proposed, and their statistical properties are investigated. Consistency and asymptotic unbiasedness of these estimators are established. Possible applications of the estimators in maintenance, warranty and spare parts analyzes are investigated. Numerical procedures are provided to compute renewal and variance functions and their plug-in estimators. Small sample performances of the estimators are evaluated by a simulation study. Finally, two real data sets are examined to exhibit applicability of the estimators in some reliability problems.

1. Introduction

Renewal process (RP), a counting process based on a sequence of independent and identically distributed (iid) random variables, is an important model utilized in fields of applied probability. Some applications of RP but not limited to are listed below:

- Reliability theory; to model replacement or repair process, to determine optimal maintenance policy, to analyze availability of a system, see [5,7,16,19].
- Inventory theory; to model consecutive demands for a product, to control inventory, to make demand forecasting of spare parts, see [11,13,24,35,40].
- Warranty analysis; to determine optimal warranty policy, see [8,9,14,15,17,27,36].

One of the main characteristics of a RP is its mean value function. It is a well-known function and is called as renewal function. Another important characteristic of a RP is its variance function. These functions, especially the renewal function, are main tools to deal with some reliability problems such as maintenance and warranty cost analyzes, and future demand estimation for spare parts.

For statistical considerations, the former studies have focused on complete and (random) right censored data cases to estimate renewal and variance functions. These data structures occur naturally when observing consecutive failure times which correspond to iid random variables forming a RP. Pioneering and some important early works about statistical inference for renewal and variance functions under complete and right censored samples are [6,17,18,20,21,38,39].

However, the renewal and variance functions are also estimable even if there exist non-consecutive failure times corresponding to iid random variables. This feature allows to estimate renewal and variance functions based on failure times observed simultaneously which correspond to iid random variables not necessarily forming a RP. Then, some reliability issues requiring the knowledge of renewal and variance functions may be further evaluated based on different data types beside complete and right censored. For this purpose, [2] have considered the estimation problem of renewal and variance functions both parametrically and non-parametrically under progressively censored data which is often used in lifetime and reliability applications. This study considers estimation of renewal and variance functions under progressively censored data in parametric point of view and emphasizes its possible

* Corresponding author
E-mail addresses: omer.altindag@bilecik.edu.tr (Ö. Altındağ), aydogdu@ankara.edu.tr (H. Aydoğdu).

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Table 1
Censoring Schemes used in simulation.

Scheme	n	m	$(R_1, R_2, \ldots, R_m)$
1	16	8	$(3,3,2,0_{1\times5})$
2	16	8	$(1_{1\times8})$
3	16	8	$(0_{1\times5},2,3,3)$
4	16	10	$(2,2,2,0_{1\times7})$
5	16	10	$(1_{1\times6},0_{1\times4})$
6	16	10	$(0_{1\times7},2,2,2)$
7	30	15	$(5,5,5,0_{1\times12})$
8	30	15	$(1_{1\times15})$
9	30	15	$(0_{1\times12},5,5,5)$
10	30	20	$(4,4,2,0_{1\times17})$
11	30	20	$(1_{1\times10},0_{1\times10})$
12	30	20	$(0_{1\times17},2,4,4)$
13	50	25	$(9,8,8,0_{1\times22})$
14	50	25	$(1_{1\times25})$
15	50	25	$(0_{1\times22},8,8,9)$
16	50	35	$(5,5,5,0_{1\times32})$
17	50	35	$(1_{1\times15},0_{1\times20})$
18	50	35	$(0_{1\times32},5,5,5)$

applications in some reliability problems.

Some preliminaries and motivation of the study are provided in Section 2. Parametric estimators for renewal and variance functions are given in Section 3. Statistical properties and possible applications of the estimators are investigated in Section 4. Section 5 deals with computational procedures. In Section 6, a simulation study is carried out to observe performance of the estimators for different sample size and censoring settings. In Section 7, two data sets are examined to illustrate possible applications of the estimators in some reliability problems. Finally, some conclusions and discussions are given in Section 8.

2. Preliminaries and motivation

Let $\{X_1, X_2, \ldots\}$ be a sequence of iid and positive defined random variables with distribution function F. Set the partial sums as $S_0 = 0$ and $S_n = \sum_{k=1}^{n} X_k$, $n = 1, 2, \ldots$. Define $N(t) = \max\{n : S_n \leq t\}$ for each fixed $t \geq 0$. Then the stochastic process $\{N(t), t \geq 0\}$ is called as a RP which counts the number of renewals up to time t. The random variables X_1, $X_2, \ldots$ denote interoccurrence times of successive events forming a RP. In reliability and warranty studies these times especially correspond to failure times of renewed units. S_n denotes occurrence time of nth event. Mean value and variance functions of the RP $\{N(t), t \geq 0\}$ is given by

Table 2
Results for renewal and variance functions when F is Exp(λ) with $\lambda = 1$.

	t	0.5	1	2	3	5	10
Censoring Scheme	$M(t), V(t)$	0.5	1	2	3	5	10
1	$\widehat{M}(t), \widehat{V}(t)$	0.5724	1.1447	2.2895	3.4342	5.7237	11.4475
		0.0585	0.2338	0.9353	2.1044	5.8455	23.3820
2	$\widehat{M}(t), \widehat{V}(t)$	0.5738	1.1476	2.2952	3.4428	5.7380	11.4761
		0.0613	0.2454	0.9816	2.2085	6.1348	24.5393
3	$\widehat{M}(t), \widehat{V}(t)$	0.5691	1.1383	2.2766	3.4149	5.6914	11.3829
		0.0586	0.2343	0.9370	2.1083	5.8565	23.4259
4	$\widehat{M}(t), \widehat{V}(t)$	0.5560	1.1120	2.2240	3.3359	5.5599	11.1198
		0.0416	0.1662	0.6648	1.4959	4.1553	16.6212
5	$\widehat{M}(t), \widehat{V}(t)$	0.5570	1.1141	2.2282	3.3423	5.5705	11.1409
		0.0409	0.1638	0.6551	1.4740	4.0944	16.3776
6	$\widehat{M}(t), \widehat{V}(t)$	0.5580	1.1160	2.2320	3.3479	5.5799	11.1598
		0.0426	0.1702	0.6810	1.5322	4.2562	17.0248
Complete Sample $n = 16$	$\widehat{M}(t), \widehat{V}(t)$	0.5344	1.0688	2.1375	3.2063	5.3438	10.6876
		0.0218	0.0872	0.3487	0.7845	2.1791	8.7166
7	$\widehat{M}(t), \widehat{V}(t)$	0.5361	1.0722	2.1443	3.2165	5.3608	10.7216
		0.0227	0.0909	0.3638	0.8185	2.2736	9.0942
8	$\widehat{M}(t), \widehat{V}(t)$	0.5371	1.0742	2.1485	3.2227	5.3712	10.7423
		0.0231	0.0926	0.3703	0.8332	2.3143	9.2573
9	$\widehat{M}(t), \widehat{V}(t)$	0.5347	1.0695	2.1389	3.2084	5.3473	10.6947
		0.0242	0.0967	0.3869	0.8705	2.4181	9.6723
10	$\widehat{M}(t), \widehat{V}(t)$	0.5266	1.0532	2.1063	3.1595	5.2659	10.5317
		0.0165	0.0659	0.2635	0.5928	1.6467	6.5868
11	$\widehat{M}(t), \widehat{V}(t)$	0.5256	1.0513	2.1026	3.1538	5.2564	10.5128
		0.0162	0.0649	0.2594	0.5837	1.6214	6.4854
	$\widehat{M}(t), \widehat{V}(t)$	0.5269	1.0537	2.1074	3.1611	5.2685	10.5370
		0.0158	0.0631	0.2523	0.5676	1.5766	6.3065
Complete Sample $n = 30$	$\widehat{M}(t), \widehat{V}(t)$	0.5168	1.0336	2.0673	3.1009	5.1681	10.3363
		0.0093	0.0373	0.1494	0.3361	0.9336	3.7342
13	$\widehat{M}(t), \widehat{V}(t)$	0.5196	1.0392	2.0783	3.1175	5.1958	10.3915
		0.0117	0.0467	0.1867	0.4201	1.1670	4.6681
14	$\widehat{M}(t), \widehat{V}(t)$	0.5215	1.0431	2.0861	3.1292	5.2153	10.4305
		0.0124	0.0498	0.1992	0.4482	1.2449	4.9798
15	$\widehat{M}(t), \widehat{V}(t)$	0.5210	1.0421	2.0841	3.1262	5.2104	10.4207
		0.0121	0.0484	0.1934	0.4352	1.2089	4.8354

(continued on next page)

Table 2 (*continued*)

Censoring Scheme	t / $M(t), V(t)$	0.5 / 0.5	1 / 1	2 / 2	3 / 3	5 / 5	10 / 10
16	$\widehat{M}(t), \widehat{V}(t)$	0.5134	1.0269	2.0538	3.0807	5.1345	10.2690
		0.0082	0.0327	0.1307	0.2940	0.8166	3.2664
17	$\widehat{M}(t), \widehat{V}(t)$	0.5140	1.0280	2.0560	3.0840	5.1401	10.2802
		0.0083	0.0333	0.1331	0.2996	0.8322	3.3287
18	$\widehat{M}(t), \widehat{V}(t)$	0.5152	1.0305	2.0609	3.0914	5.1523	10.3046
		0.0081	0.0326	0.1303	0.2931	0.8141	3.2565
Complete Sample $n = 50$	$\widehat{M}(t), \widehat{V}(t)$	0.5113	1.0226	2.0452	3.0678	5.1130	10.2260
		0.0056	0.0226	0.0904	0.2033	0.5648	2.2591

Table 3
Results for renewal function when F is $W(\alpha,\ \beta)$ with $\alpha = 2$ and $\beta = 1$.

Censoring Scheme	t / $M(t)$	0.5 / 0.2308	1 / 0.7537	2 / 1.8940	3 / 3.0218	5 / 5.2785	10 / 10.9204
1	$\widehat{M}(t)$	0.2375	0.8128	2.0129	3.2067	5.5939	11.5621
		0.0153	0.0637	0.2256	0.5037	1.3943	5.5717
2	$\widehat{M}(t)$	0.2324	0.8255	2.0333	3.2390	5.6503	11.6785
		0.0137	0.0615	0.2311	0.5285	1.4916	6.0483
3	$\widehat{M}(t)$	0.2303	0.8377	2.0529	3.2689	5.7012	11.7818
		0.0134	0.0640	0.2532	0.5887	1.6831	6.8910
4	$\widehat{M}(t)$	0.2304	0.7903	1.9713	3.1441	5.4895	11.3530
		0.0124	0.0481	0.1643	0.3636	0.9998	3.9762
5	$\widehat{M}(t)$	0.2289	0.7934	1.9766	3.1528	5.5049	11.3851
		0.0117	0.0463	0.1610	0.3589	0.9932	3.9687
6	$\widehat{M}(t)$	0.2241	0.8033	1.9899	3.1734	5.5405	11.4582
		0.0110	0.0441	0.1612	0.3680	1.0393	4.2187
Complete Sample $n = 16$	$\widehat{M}(t)$	0.2247	0.7625	1.9190	3.0646	5.3553	11.0820
		0.0095	0.0333	0.1061	0.2300	0.6224	2.4449
7	$\widehat{M}(t)$	0.2351	0.7838	1.9553	3.1173	5.4412	11.2509
		0.0078	0.0311	0.1068	0.2366	0.6501	2.5835
8	$\widehat{M}(t)$	0.2325	0.7927	1.9691	3.1391	5.4788	11.3281
		0.0064	0.0280	0.1038	0.2370	0.6681	2.7077
9	$\widehat{M}(t)$	0.2309	0.8003	1.9793	3.1545	5.5048	11.3807
		0.0061	0.0290	0.1143	0.2669	0.7663	3.1480
10	$\widehat{M}(t)$	0.2308	0.7700	1.9293	3.0780	5.3755	11.1190
		0.0065	0.0238	0.0786	0.1725	0.4702	1.8578
11	$\widehat{M}(t)$	0.2303	0.7718	1.9328	3.0837	5.3854	11.1398
		0.0058	0.0221	0.0752	0.1665	0.4574	1.8182
12	$\widehat{M}(t)$	0.2274	0.7763	1.9378	3.0919	5.3996	11.1690
		0.0055	0.0202	0.0721	0.1636	0.4598	1.8610
Complete Sample $n = 30$	$\widehat{M}(t)$	0.2271	0.7577	1.9064	3.0435	5.3177	11.0033
		0.0049	0.0168	0.0536	0.1164	0.3146	1.2347
13	$\widehat{M}(t)$	0.2330	0.7700	1.9276	3.0744	5.3682	11.1029
		0.0046	0.0179	0.0608	0.1343	0.3674	1.4555
14	$\widehat{M}(t)$	0.2313	0.7762	1.9380	3.0911	5.3972	11.1626
		0.0036	0.0156	0.0580	0.1325	0.3732	1.5120
15	$\widehat{M}(t)$	0.2303	0.7813	1.9447	3.1014	5.4146	11.1978
		0.0034	0.0160	0.0640	0.1499	0.4310	1.7723
16	$\widehat{M}(t)$	0.2305	0.7619	1.9124	3.0514	5.3297	11.0256
		0.0038	0.0135	0.0440	0.0961	0.2609	1.0271
17	$\widehat{M}(t)$	0.2305	0.7631	1.9146	3.0549	5.3358	11.0379
		0.0034	0.0125	0.0417	0.0918	0.2509	0.9931
18	$\widehat{M}(t)$	0.2287	0.7650	1.9166	3.0584	5.3420	11.0510
		0.0032	0.0110	0.0386	0.0870	0.2430	0.9786
Complete Sample $n = 50$	$\widehat{M}(t)$	0.2291	0.7564	1.9018	3.0351	5.3023	10.9703
		0.0031	0.0103	0.0326	0.0705	0.1898	0.7432

Table 4
Results for variance function when F is $W(\alpha, \beta)$ with $\alpha = 2$ and $\beta = 1$.

	t	0.5	1	2	3	5	10
Censoring Scheme	$V(t)$	0.1971	0.4463	0.7299	1.0418	1.6582	3.1999
1	$\hat{V}(t)$	0.1976	0.4283	0.7131	1.0142	1.6162	3.1213
		0.0089	0.0256	0.1114	0.2476	0.6805	2.7005
2	$\hat{V}(t)$	0.1927	0.4080	0.6836	0.9703	1.5441	2.9788
		0.0078	0.0260	0.1063	0.2367	0.6496	2.5749
3	$\hat{V}(t)$	0.1902	0.3935	0.6645	0.9426	1.4991	2.8906
		0.0073	0.0280	0.1080	0.2407	0.6601	2.6166
4	$\hat{V}(t)$	0.1936	0.4290	0.7078	1.0058	1.6017	3.0915
		0.0076	0.0205	0.0923	0.2051	0.5624	2.2276
5	$\hat{V}(t)$	0.1923	0.4238	0.6994	0.9931	1.5807	3.0499
		0.0071	0.0198	0.0876	0.1945	0.5329	2.1088
6	$\hat{V}(t)$	0.1886	0.4087	0.6791	0.9632	1.5321	2.9542
		0.0068	0.0224	0.0949	0.2112	0.5783	2.2885
Complete Sample $n = 16$	$\hat{V}(t)$	0.1908	0.4337	0.7092	1.0084	1.6054	3.0979
		0.0061	0.0155	0.0733	0.1623	0.4442	1.7583
7	$\hat{V}(t)$	0.1985	0.4406	0.7272	1.0348	1.6487	3.1836
		0.0047	0.0122	0.0568	0.1255	0.3436	1.3599
8	$\hat{V}(t)$	0.1963	0.4290	0.7115	1.0105	1.6089	3.1048
		0.0039	0.0120	0.0544	0.1205	0.3292	1.3004
9	$\hat{V}(t)$	0.1950	0.4197	0.7010	0.9952	1.5840	3.0559
		0.0037	0.0135	0.0585	0.1296	0.3538	1.3972
10	$\hat{V}(t)$	0.1958	0.4403	0.7231	1.0291	1.6391	3.1640
		0.0040	0.0103	0.0491	0.1082	0.2959	1.1704
11	$\hat{V}(t)$	0.1953	0.4377	0.7187	1.0222	1.6277	3.1411
		0.0036	0.0093	0.0443	0.0978	0.2673	1.0567
12	$\hat{V}(t)$	0.1934	0.4302	0.7095	1.0073	1.6033	3.0932
		0.0036	0.0107	0.0508	0.1129	0.3083	1.2176
Complete Sample $n = 30$	$\hat{V}(t)$	0.1934	0.4393	0.7179	1.0220	1.6270	3.1393
		0.0031	0.0076	0.0369	0.0809	0.2209	0.8731
13	$\hat{V}(t)$	0.1977	0.4435	0.7290	1.0380	1.6533	3.1916
		0.0028	0.0070	0.0337	0.0739	0.2020	0.7987
14	$\hat{V}(t)$	0.1962	0.4357	0.7179	1.0202	1.6241	3.1336
		0.0022	0.0064	0.0304	0.0672	0.1831	0.7223
15	$\hat{V}(t)$	0.1955	0.4299	0.7120	1.0110	1.6090	3.1040
		0.0021	0.0072	0.0338	0.0747	0.2035	0.8019
16	$\hat{V}(t)$	0.1962	0.4432	0.7262	1.0344	1.6472	3.1792
		0.0024	0.0058	0.0284	0.0623	0.1701	0.6721
17	$\hat{V}(t)$	0.1961	0.4422	0.7245	1.0317	1.6427	3.1701
		0.0021	0.0053	0.0257	0.0564	0.1539	0.6081
18	$\hat{V}(t)$	0.1949	0.4383	0.7197	1.0232	1.6288	3.1429
		0.0021	0.0059	0.0296	0.0654	0.1784	0.7040
Complete Sample $n = 50$	$\hat{V}(t)$	0.1953	0.4432	0.7250	1.0330	1.6448	3.1740
		0.0019	0.0047	0.0233	0.0509	0.1389	0.5485

$M(t) = E(N(t)),\ t \geq 0$ and $V(t) = Var(N(t)),\ t \geq 0$, respectively. Here, the functions $M(t)$ and $V(t)$ denotes mean and variance of number of renewals occurred up to time t. Note that, the former one is also called as renewal function. By considering the definitions one can easily find the well-known convolution series expressions for these functions as

$$M(t) = \sum_{k=1}^{\infty} F^{k^*}(t), \tag{2.1}$$

$$V(t) = 2\sum_{k=1}^{\infty} kF^{k^*}(t) - \sum_{k=1}^{\infty} F^{k^*}(t)\left(1 + \sum_{k=1}^{\infty} F^{k^*}(t)\right), \tag{2.2}$$

where t≥0, "*" denotes Stieltjes convolution and F^{k^*} is k-fold Stieltjes convolution of F. It is well known that the renewal function satisfies following integral equation which is called as renewal equation:

$$M(t) = F(t) + \int_0^t M(t-x)dF(x),\ t \geq 0. \tag{2.3}$$

Further, the variance function can be expressed as a function of renewal function as follows:

$$V(t) = 2M{*}M(t) + M(t)(1 - M(t)),\ t \geq 0. \tag{2.4}$$

Eqs. (2.3) and (2.4) provide simplicity for computation of the renewal and variance functions which don't have analytical expressions for many cases.

In order to explain motivation of the study, let's consider RP from warranty analysis point of view. In a standard warranty, a product is immediately replaced with an identical new one if it fails within warranty period. It is clear that, consecutive failure times of a specified product form a RP. The mean value and variance functions of RP are needed to be able to determine optimal warranty period of product for its manufacturers. To estimate these functions one way is to observe sufficient number of consecutive failure times over one or more realization of RPs. However, we don't have to observe these failure times throughout a replacement process. The failure times of a specified product can be observed over identical units simultaneously. This is probabilistically equal to observe a realization of the process since they have common distribution. This enables manufacturer to estimate renewal and variance functions to determine optimal warranty period for its product over the failure times obtained simultaneously in life tests before sale though after sale observations are thought to be more realistic.

Another example occurs in logistics problem when estimation of future demand for a product is of concern. Consider a system, having a critical component, fails with its critical component's fail. Now problem is how many spare parts of the critical component will be needed to guarantee functionality of the system in a specified period and what will be variance of renewals in this period. For a possible future demand estimation of a single product that must be replaced with a new one when it fails, we don't have to observe consecutive failure times throughout a replacement process. As mentioned above, observation of failure times can be done in life tests earlier than the product being put to market. Therefore, the renewal and variance functions can even be estimated based on the failure times obtained in lifetime test beside the ones observed throughout a renewal/replacement process. This feature allows to estimate renewal and variance functions based on different data types.

For some type of products observing failure time of a unit may take long time and be expensive. In such a situation observing all failure times of a sample of units may be costly and timely ineffective. Thus, a censoring mechanism should be preferred to decrease total test time and failure cost. For this purpose, some right censoring schemes can be applied. Recently, progressive type-II censoring (shortly progressive censoring), a generalization of type-II censoring, has gained popularity in life test applications due to its effectiveness for both time and cost reduction.

It should be considered statistically and practically to estimate renewal and variance functions when the failure times are observed as progressively censored. Evaluation of renewal and variance functions under progressively censored data may be useful to solve some reliability engineering issues such as block replacement policy, warranty cost analysis, future demand estimation for spare parts as well as other applications of these functions.

Table 5
Results for renewal function when F is LN(μ, σ) with $\mu = 0$ and $\sigma = 1$.

t	t	0.5	1	2	3	5	10
Censoring Scheme	$M(t)$	0.2599	0.6264	1.3194	1.9799	3.2573	6.3595
1	$\widehat{M}(t)$	0.2723	0.6885	1.4919	2.2693	3.7898	7.5191
		0.0189	0.0766	0.3298	0.7858	2.3449	10.1881
2	$\widehat{M}(t)$	0.2670	0.6902	1.5082	2.3013	3.8551	7.6710
		0.0168	0.0718	0.3322	0.8166	2.5030	11.1043
3	$\widehat{M}(t)$	0.2659	0.6966	1.5291	2.3375	3.9229	7.8198
		0.0160	0.0717	0.3489	0.8734	2.7156	12.1697
4	$\widehat{M}(t)$	0.2634	0.6636	1.4354	2.1804	3.6351	7.1978
		0.0151	0.0559	0.2266	0.5308	1.5688	6.8006
5	$\widehat{M}(t)$	0.2617	0.6637	1.4393	2.1884	3.6520	7.2378
		0.0143	0.0536	0.2232	0.5299	1.5851	6.9404
6	$\widehat{M}(t)$	0.2566	0.6633	1.4493	2.2097	3.6972	7.3457
		0.0132	0.0496	0.2194	0.5375	1.6547	7.4165
Complete Sample $n = 16$	$\widehat{M}(t)$	0.2635	0.6398	1.3584	2.0463	3.3810	6.6301
		0.0119	0.0376	0.1307	0.2871	0.8043	3.3658
7	$\widehat{M}(t)$	0.2686	0.6599	1.4081	2.1270	3.5262	6.9436
		0.0094	0.0350	0.1415	0.3296	0.9685	4.1904
8	$\widehat{M}(t)$	0.2648	0.6600	1.4171	2.1459	3.5664	7.0398
		0.0077	0.0304	0.1359	0.3317	1.0181	4.5663
9	$\widehat{M}(t)$	0.2624	0.6614	1.4274	2.1659	3.6070	7.1348
		0.0072	0.0300	0.1426	0.3568	1.1178	5.0895
10	$\widehat{M}(t)$	0.2620	0.6438	1.3731	2.0728	3.4328	6.7505
		0.0078	0.0266	0.1004	0.2288	0.6625	2.8464
11	$\widehat{M}(t)$	0.2611	0.6440	1.3758	2.0782	3.4441	6.7774
		0.0070	0.0247	0.0971	0.2256	0.6657	2.9062
12	$\widehat{M}(t)$	0.2581	0.6427	1.3791	2.0867	3.4639	6.8272
		0.0064	0.0221	0.0916	0.2203	0.6733	3.0345
Complete Sample $n = 30$	$\widehat{M}(t)$	0.2616	0.6338	1.3414	2.0176	3.3281	6.5157
		0.0062	0.0191	0.0651	0.1418	0.3953	1.6552
13	$\widehat{M}(t)$	0.2636	0.6433	1.3668	2.0598	3.4055	6.6854
		0.0057	0.0201	0.0769	0.1755	0.5081	2.1821
14	$\widehat{M}(t)$	0.2612	0.6442	1.3749	2.0759	3.4388	6.7645
		0.0044	0.0163	0.0710	0.1724	0.5292	2.3906
15	$\widehat{M}(t)$	0.2606	0.6456	1.3805	2.0861	3.4587	6.8091
		0.0039	0.0153	0.0718	0.1804	0.5706	2.6385
16	$\widehat{M}(t)$	0.2610	0.6363	1.3506	2.0342	3.3603	6.5903
		0.0046	0.0151	0.0552	0.1237	0.3532	1.5049
17	$\widehat{M}(t)$	0.2607	0.6368	1.3525	2.0377	3.3672	6.6062
		0.0041	0.0139	0.0528	0.1207	0.3509	1.5196
18	$\widehat{M}(t)$	0.2588	0.6357	1.3537	2.0414	3.3768	6.6317
		0.0038	0.0123	0.0485	0.1144	0.3444	1.5431
Complete Sample $n = 50$	$\widehat{M}(t)$	0.2624	0.6327	1.3347	2.0047	3.3018	6.4540
		0.0037	0.0115	0.0397	0.0868	0.2428	1.0206

3. Parametric estimation of renewal and variance functions

Firstly, the progressive censoring should be reminded. In a progressive censoring, a sample of n units is put to life test together, a prefixed number of units, say R_1, are randomly removed from the life test immediately as the first failure occurs. The second failure time is then observed over the remaining $n - R_1 - 1$ units, and again a prefixed number of units, say R_2, are randomly removed from the test when the failure occurs. This process is being continued until the mth failure occurs. In this scheme, the failure times of removed units are censored and m units are uncensored. The prefixed numbers $(R_1, R_2, \ldots, R_m)$ with $R_i \geq 0$, $i = 1, \ldots, m$, $m \leq n$ and $\sum_{i=1}^{m} R_i + m = n$ called as censoring scheme.

Assume that F is continuous (and so non-arithmetic), formally known but its parameters, say $\theta_1, \theta_2, \ldots, \theta_r$, are unknown. Assume also that we have a progressively censored sample form the distribution F with censoring scheme $(R_1, R_2, \ldots, R_m)$, $R_i \geq 0$, $i = 1, \ldots, m$, $m \leq n$ and $\sum_{i=1}^{m} R_i + m = n$. Denote the sample as $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$. Estimation of renewal and variance functions should be a two-stage procedure. Firstly, the distribution F is estimated and estimators for renewal and variance functions are constructed over it. From parametric point of view, natural estimators for the renewal and variance functions can be defined by replacing the parameters with their estimators in their functional forms. Let $\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_r$ be estimators of $\theta_1, \theta_2, \ldots, \theta_r$ based on the sample $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$. Then $\widehat{F}_{m:n} = F(\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_r)$ is the resulting estimator of F based on progressively censored

Table 6
Results for variance function when F is LN(μ, σ) with $\mu = 0$ and $\sigma = 1$.

	t	0.5	1	2	3	5	10
Censoring Scheme	$V(t)$	0.2245	0.5142	1.1255	1.7935	3.2630	7.4273
1	$\widehat{V}(t)$	0.2281	0.5206	1.1228	1.7729	3.1777	7.0501
		0.0115	0.0429	0.2083	0.5443	1.8591	9.9661
2	$\widehat{V}(t)$	0.2223	0.5064	1.0858	1.7096	3.0534	6.7440
		0.0098	0.0354	0.1755	0.4621	1.5990	8.7772
3	$\widehat{V}(t)$	0.2203	0.4999	1.0675	1.6776	2.9890	6.5816
		0.0091	0.0326	0.1639	0.4348	1.5196	8.4805
4	$\widehat{V}(t)$	0.2236	0.5144	1.1131	1.7608	3.1648	7.0521
		0.0100	0.0365	0.1765	0.4617	1.5845	8.5755
5	$\widehat{V}(t)$	0.2219	0.5104	1.1028	1.7433	3.1304	6.9661
		0.0093	0.0337	0.1641	0.4304	1.4824	8.0770
6	$\widehat{V}(t)$	0.2172	0.4998	1.0743	1.6942	3.0332	6.7249
		0.0086	0.0300	0.1487	0.3927	1.3689	7.6314
Complete Sample $n = 16$	$\widehat{V}(t)$	0.2287	0.5237	1.1373	1.8034	3.2570	7.3315
		0.0089	0.0321	0.1490	0.3849	1.3113	7.1219
7	$\widehat{V}(t)$	0.2294	0.5247	1.1416	1.8117	3.2714	7.3400
		0.0062	0.0232	0.1104	0.2875	0.9808	5.2587
8	$\widehat{V}(t)$	0.2254	0.5155	1.1180	1.7710	3.1897	7.1295
		0.0050	0.0181	0.0867	0.2269	0.7829	4.3085
9	$\widehat{V}(t)$	0.2227	0.5094	1.1023	1.7439	3.1350	6.9869
		0.0046	0.0165	0.0799	0.2104	0.7325	4.1064
10	$\widehat{V}(t)$	0.2253	0.5172	1.1267	1.7893	3.2356	7.2810
		0.0054	0.0199	0.0938	0.2439	0.8351	4.5381
11	$\widehat{V}(t)$	0.2241	0.5147	1.1205	1.7789	3.2151	7.2277
		0.0049	0.0177	0.0835	0.2176	0.7470	4.0796
12	$\widehat{V}(t)$	0.2215	0.5091	1.1055	1.7526	3.1617	7.0897
		0.0045	0.0159	0.0754	0.1968	0.6810	3.8074
Complete Sample $n = 30$	$\widehat{V}(t)$	0.2267	0.5193	1.1321	1.7994	3.2605	7.3732
		0.0046	0.0166	0.0771	0.1994	0.6814	3.7265
13	$\widehat{V}(t)$	0.2266	0.5194	1.1333	1.8018	3.2636	7.3625
		0.0039	0.0146	0.0689	0.1791	0.6122	3.3060
14	$\widehat{V}(t)$	0.2240	0.5135	1.1182	1.7757	3.2104	7.2204
		0.0030	0.0107	0.0507	0.1322	0.4553	2.5155
15	$\widehat{V}(t)$	0.2233	0.5113	1.1114	1.7632	3.1836	7.1467
		0.0027	0.0096	0.0455	0.1191	0.4137	2.3341
16	$\widehat{V}(t)$	0.2249	0.5162	1.1275	1.7935	3.2521	7.3519
		0.0032	0.0117	0.0547	0.1420	0.4857	2.6364
17	$\widehat{V}(t)$	0.2244	0.5152	1.1250	1.7893	3.2438	7.3294
		0.0029	0.0104	0.0489	0.1273	0.4358	2.3694
18	$\widehat{V}(t)$	0.2229	0.5119	1.1164	1.7743	3.2133	7.2493
		0.0027	0.0095	0.0444	0.1154	0.3968	2.2015
Complete Sample $n = 50$	$\widehat{V}(t)$	0.2271	0.5195	1.1343	1.8046	3.2751	7.4250
		0.0027	0.0097	0.0449	0.1160	0.3957	2.1562

sample $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$. Therefore, we can define the following plug-in estimators for renewal and variance functions for each fixed $t \geq 0$ as

$$\widehat{M}(t) = \sum_{k=1}^{\infty} \widehat{F}_{m:n}^{k*}(t), \tag{3.1}$$

$$\widehat{V}(t) = 2\sum_{k=1}^{\infty} k\widehat{F}_{m:n}^{k*}(t) - \sum_{k=1}^{\infty} \widehat{F}_{m:n}^{k*}(t)\left(1 + \sum_{k=1}^{\infty} \widehat{F}_{m:n}^{k*}(t)\right). \tag{3.2}$$

If the functions $M(t)$ and $V(t)$ have analytical expressions as a function of parameters of F, then the estimators defined on Eqs. (3.1) and (3.2) are obtained by replacing the parameters with their estimators in their analytical forms. If so, investigating statistical properties of the estimators $\widehat{M}(t)$ and $\widehat{V}(t)$ may not be a challenging problem. However, this is not always the case since these functions don't have analytical expressions except some specific distributions. Therefore, statistical properties of these estimators should be derived in a general way. Note that, these estimators are natural generalizations of the parametric estimators proposed by [17] for complete sample.

4. Statistical properties and possible applications of the estimators

4.1. Statistical properties

In order to make the plug-in estimators $\widehat{M}(t)$ and $\widehat{V}(t)$ consistent for the renewal function $M(t)$ and variance function $V(t)$, parameters of the distribution function F have to be estimated consistently from a progressively censored data. Then it is expected that the estimators $\widehat{M}(t)$ and $\widehat{V}(t)$ defined in (3.1) and (3.2) become consistent. Further, the estimators are likely to be unbiased asymptotically when we consider consistency property though they may not be unbiased generally. The consistency and asymptotic unbiasedness of the estimators are established as the following.

Property 4.1. (Strong Consistency) Let $F = F(\theta_1, \theta_2, \ldots, \theta_r)$ be a absolutely continuous distribution function with probability density function $f = f(\theta_1, \theta_2, \ldots, \theta_r)$, $\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_r$ be the estimators of parameters $\theta_1, \theta_2, \ldots, \theta_r$ based on a progressively censored sample $X_{1:m:n}$, $X_{2:m:n}, \ldots, X_{m:m:n}$ with censoring scheme $(R_1, R_2, \ldots, R_m)$, $R_i \geq 0$, $i = 1$,

Table 8
Estimates of renewal and variance functions based on aircraft windshields data when F is IW.

t	5	10	15	20	25
$\widehat{M}(t)$ for complete sample	1.3947	2.7658	4.0438	5.2713	6.4655
$\widehat{M}(t)$ for progressively censored sample	1.3151	2.5840	3.7552	4.8727	5.9549
$\widehat{V}(t)$ for complete sample	0.9975	2.5280	4.4228	6.6161	9.0694
$\widehat{V}(t)$ for progressively censored sample	0.9815	2.4636	4.2859	6.3881	8.7347

Table 7
Failure times of aircraft windshields given in [28].

0.301	0.309	0.557	0.943	1.070	1.124	1.248	1.281	1.281	1.303
1.432	1.480	1.505	1.506	1.568	1.615	1.619	1.652	1.652	1.757
1.795	1.866	1.876	1.899	1.911	1.912	1.914	1.981	2.010	2.038
2.085	2.089	2.097	2.135	2.154	2.190	2.194	2.223	2.224	2.229
2.300	2.324	2.349	2.385	2.481	2.610	2.625	2.632	2.646	2.661
2.688	2.823	2.890	2.902	2.934	2.962	2.964	3.000	3.103	3.114
3.117	3.166	3.344	3.376	3.385	3.443	3.467	3.478	3.578	3.595
3.699	3.779	3.924	4.035	4.121	4.167	4.240	4.255	4.278	4.305
4.376	4.449	4.485	4.570	4.602	4.663	4.694			

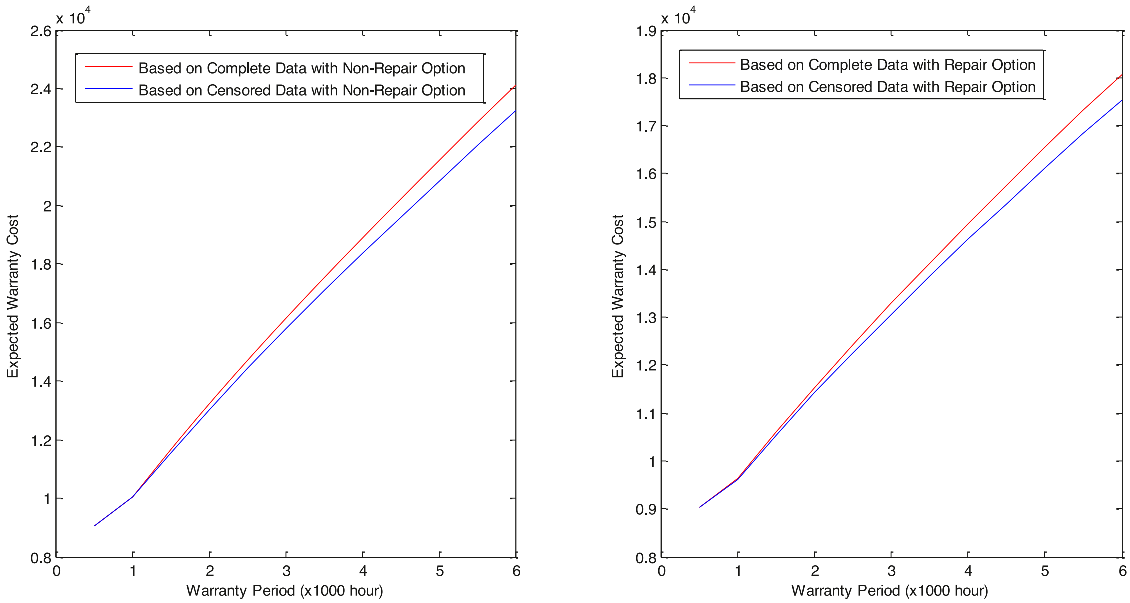

Fig. 1. Expected warranty costs for per windshield under FRW with repair and non-repair option based on both complete and censored samples.

Table 9
Failure times of 50 devices given in [1].

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	67
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

$\ldots,m$, $m \le n$ and $\sum_{i=1}^{m} R_i + m = n$ and $\widehat{F}_{m:n} = F(\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_r)$. Assume f is continuous in each parameter θ_i, $i = 1, 2, \ldots, r$ and $m = m(n)$ such that $m \to \infty$ as $n \to \infty$. If the estimators $\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_r$ are strongly consistent for $\theta_1, \theta_2, \ldots, \theta_r$ as $n \to \infty$, then for fixed t, $\widehat{M}(t)$ and $\widehat{V}(t)$ are strongly consistent for $M(t)$ and $V(t)$, respectively, as $n \to \infty$.

Property 4.2. (Asymptotic Unbiasedness) Assume that the conditions of Property 4.1 hold. If $F(t_0) < 1$ for any $t_0 \ge 0$, then for all $t \le t_0$, $\widehat{M}(t)$ and $\widehat{V}(t)$ are asymptotically unbiased as $n \to \infty$, that is $\lim_{n\to\infty} E(\widehat{M}(t)) = M(t)$ and $\lim_{n\to\infty} E(\widehat{V}(t)) = V(t)$.

Proof of the properties are given in Appendix.

To construct consistent estimators for $M(t)$ and $V(t)$ for a fixed t under a progressively censored sample, the parameters of F may be estimated by the well-known maximum likelihood method due to its consistency feature under regularity conditions which hold for most of the practical distributions. For asymptotic unbiasedness, the condition $F(t_0) < 1$ is not restrictive in practice since for most of distributions encountered in applications such as exponential, Weibull, lognormal and gamma, it holds for any finite values of t_0.

4.2. Possible applications

In maintenance analysis, deteriorating products is decided to be replaced or repaired according to a predetermined policy. A classical replacement policy is block replacement for which the deteriorating products are replaced by new one periodically and at failure. If replacement period is chosen as T, expected cost per unit of time for block replacement policy is given by $ER_c(T) = (c_f M(T) + c_p)/T$, where c_f and c_p are failure and preventive replacement costs, respectively, and M is the renewal function associated with failure distribution F, see [42].

Manufacturers generally offers warranty contracts to customers for customer satisfaction. Analyzing the warranty cost is vital to maintain profitability. Although there are much warranty types offered specifically, a basic and attractive warranty type for customers is free replacement warranty. It is generally offered for non-repairable or expensively repaired products and it guarantees replacement of a failed product with a new one within warranty period. Assume the warranty period is T, then expected value and variance of the warranty cost are $E_c(T) = c_r M(T)$ and $V_c(T) = c_r^2 V(T)$, respectively, where c_r is replace-

Table 11
Estimates of renewal and variance functions when F is MWD.

t	100	200	300	400	500
$\widehat{M}(t)$ for complete sample	1.9191	4.0978	6.2861	8.4749	10.6637
$\widehat{M}(t)$ for prog. cens. sample	1.8646	3.9643	6.0607	8.1570	10.2532
$\widehat{V}(t)$ for complete sample	1.2208	2.2038	3.1687	4.1305	5.0922
$\widehat{V}(t)$ for prog. cens. sample	1.4991	2.6243	3.7635	4.9042	6.0451

Table 10
Progressively censored sample generated from Table 7 by [29].

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$x_{i:m:n}$	0.1	0.2	1	1	1	1	1	2	3	6	7	11	18	18	18	18	21	32
R_i	0	0	0	3	0	0	0	0	0	0	3	0	0	0	0	0	0	3
i	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	
$x_{i:m:n}$	36	45	47	50	55	60	63	63	67	67	75	79	82	84	84	85	86	
R_i	0	0	0	0	0	0	3	0	0	0	0	0	0	3	0	0	0	

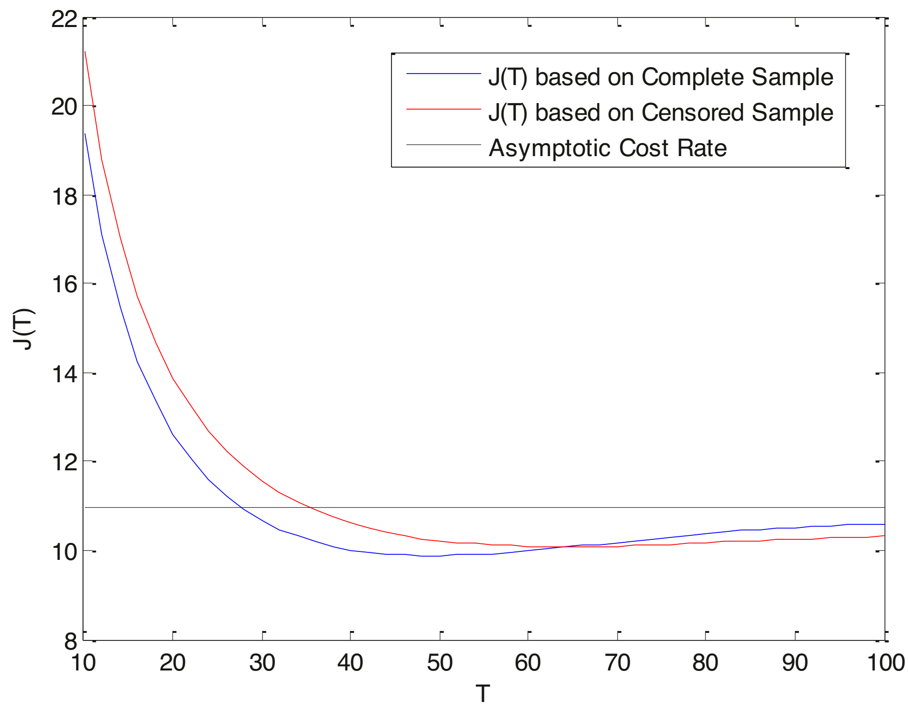

Fig. 2. Expected cost rate curves based on complete and censored samples.

ment cost, M and V are renewal and variance functions associated with failure distribution F. If the product is repairable and cost of the repairment is inexpensive, then free repair warranty may be offered instead of free replacement warranty. If the repair is done "as good as new", the repair process forms a RP and therefore expected value and variance of cost of the free repair warranty is obtained with renewal and variance functions, see for the details [42].

In maintenance and warranty analyzes, some processes such as non-homogeneous poisson process, generalized-renewal process, quasi-renewal process besides the RP have been used to model repair or replacement processes as a solution of some specified problems. Also, there are a group of study in maintenance and warranty literatures modelling the number of replacements or renewals with certain probability distributions. For such recent studies, see [26,32,33]. Besides these different approaches, the RP model and its arguments are still being utilized in maintenance and warranty studies. For recent examples, see [31,34,41].

In logistics and inventory management, one of the main objectives is to control of stock of spare parts for critical systems or components. In order to keep a system functioning uninterruptedly for specified period of time, it should be estimated how many spare parts will be needed in that period. For the components that will be replaced with a new identical one after its fail, mean and variance of number of spare parts that will be needed in the specified period, say T, is estimated by $M(T)$ and $V(T)$ as the replacement process of failed products can be modelled with a RP. Although this is the basic approach for future demand estimation of spare parts first studied by [40], the RP model and its arguments are still being used for spare part analysis, see [11,12,23,37] for recent studies.

It is seen that the renewal and variance functions are main tools to determine related policies in above issues. If the probabilistic model of failure distribution is not known completely, statistical inference based on failure times must be done. The failure data is obtained either from field, i.e. the product is on market, or from early life tests before the product have not been put on market. In early life tests, manufacturers may prefer an accelerating technique to reduce total test time. Recently, [44] have used accelerated life test to predict warranty cost with imperfect repair. As a similar approach, the progressive censoring mechanism may be preferred to accelerate life test experiment since it reduces both test time and cost. Then, the estimators proposed for renewal and variance functions under progressively censored data may be utilized to estimate future renewal/replacement processes and related functions given above as well as to solve other reliability problems where the RP is used as the model.

5. Computational remarks

In this section computation of renewal and variance functions and their plug-in estimators are considered. We know that the functions $M(t)$ and $V(t)$ do not have analytical forms except some specific cases for F, so do their plug-in estimators $\widehat{M}(t)$ and $\widehat{V}(t)$. In order to compute these functions some numerical computational procedures must be used. Numerical computation of these functions will be quite difficult if we consider the Eqs. (2.1) and (2.2) as basis due to series of convolution integrals. Instead of that, utilizing the Eqs. (2.3) and (2.4) may be easier. A numerical integration procedure for computation of renewal function was presented by [43]. This method solves the integral Eq. (2.3) iteratively using the Riemann-Stieltjes integration. It is well known that an integral of the form $\int_a^b g(t)dh(t)$ can be approximated numerically as

$$\int_a^b g(x)dh(x) \cong \sum_{i=1}^{p} g\left(\frac{x_i + x_{i-1}}{2}\right)(h(x_i) - h(x_{i-1})) \tag{5.1}$$

where $\{x_0, x_1, \ldots, x_p\}$ is a partition of the interval $[a, b]$ such that $a = x_0 < x_1 < \ldots < x_p = b$. Let t is given and fixed and $\{t_0, t_1, \ldots, t_p\}$ be a partition of the interval $[0, t]$ such that $0 = t_0 < t_1 < \ldots < t_p = t$. Then, considering the Eqs. (2.3) and (5.1), we have

$$M(t_i) \cong F(t_i) + \sum_{j=1}^{i} F\left(\frac{t_i - t_{i-1}}{2}\right)\left(M(t_j) - M(t_{j-1})\right) \tag{5.2}$$

for $i = 1,\ldots,p$. Note that the Eq. (2.3) can be equally rewritten as $M(t) =$

$F(t) + \int_0^t F(t-x)dM(x)$. Thus, we can approximately compute $M(t_i)$ by the following recursive formulae:

$$M(t_i) \cong \frac{F(t_i) + T_i - F\left(t_i - \frac{t_i+t_{i-1}}{2}\right)M(t_{i-1})}{1 - F\left(t_i - \frac{t_i+t_{i-1}}{2}\right)}, \tag{5.3}$$

where $T_i = \sum_{j=1}^{i-1} F((t_i - t_{i-1})/2)(M(t_j) - M(t_{j-1}))$ and $M(t_0) := 0$ [43]. If the distribution function F has not a closed form expression, it can be computed with its density as $F(t_i) = F(t_{i-1}) + f((t_i + t_{i-1})/2)t/n$, $i = 1, 2, \ldots, n$ and $F(t_0) := 0$. By utilizing this approach, we can also compute the variance function approximately by considering the Eq. (2.4). Using the same partition, we have for $i = 1, 2, \ldots, p$

$$M^*M(t_i) \cong \sum_{j=1}^{i} M\left(t_i - \frac{t_j + t_{j-1}}{2}\right)\left(M(t_j) - M(t_{j-1})\right) \tag{5.4}$$

Hence, for $i = 1, 2, \ldots, p$ we have

$$V(t_i) \cong 2M^*M(t_i) + M(t_i)(1 - M(t_i)). \tag{5.5}$$

For the computational simplicity, a uniform partition of the interval $[0, t]$ should be preferred. To be able to use same computation points for both renewal and variance functions, the partitions $\{0, t/p, 2t/p, \ldots, t\}$ and $\{0, 2t/p, 4t/p, \ldots, t\}$ have to be used, respectively. Further, if we restate $F((i+0.5)t/p)$, $M(t_i)$, $F(t_i)$ and $V(t_i)$ by $F(i)$, $M(i)$, $G(i)$ and $V(i)$, respectively, we can write for $i = 1, 2, \ldots, p$

$$M(i) = \frac{G(i) + \sum_{j=1}^{i} F(i-j)(M(j) - M(j-1)) - F(0)M(i-1)}{1 - F(0)}, \tag{5.6}$$

and

$$V(i) = 2\sum_{j=1}^{i} M(2(i-j)+1)(M(2j) - M(2(j-1))). \tag{5.7}$$

These formulas are quite understandable and easy to program.

The plug-in estimators $\widehat{M}(t)$ and $\widehat{V}(t)$ can also be computed by means of this approach. To compute $\widehat{M}(t)$ and $\widehat{V}(t)$ for a fixed t, we only need to replace the parameters of F with their estimates in the related functions.

6. Simulation

In this section, a simulation study is carried out to observe performance of the plug-in estimators $\widehat{M}(t)$ and $\widehat{V}(t)$ under progressive censoring. For this purpose, the following distributions are considered as possible candidates for F because of their common usage:

1 Exponential distribution $\text{Exp}(\lambda)$ having the probability density function as $f(x) = (1/\lambda)\exp\{-x/\lambda\}$, $x \geq 0$; $\lambda > 0$.
2 Weibull distribution $W(\alpha, \beta)$ with shape parameter α and scale parameter β having the probability density function as $f(x) = (\alpha/\beta)(x/\beta)^{\alpha-1}\exp\{-(x/\beta)^{\alpha}\}$, $x \geq 0$; $\alpha > 0, \beta > 0$.
3 3. Lognormal distribution $\text{LN}(\mu, \sigma)$ with shape parameter σ^2 and scale parameter e^{μ} having the probability density function as $f(x) = (x\sigma\sqrt{2\pi})^{-1}\exp\{-(\ln x - \mu)^2/(2\sigma^2)\}$, $x \geq 0$; $-\infty < \mu < \infty$, $\sigma > 0$.

Let $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ be a progressively censored sample from the distribution F with censoring scheme $(R_1, R_2, \ldots, R_m)$, $R_i \geq 0$, $i = 1, \ldots, m$, $m \leq n$ and $\sum_{i=1}^{m} R_i + m = n$. In order to estimate the renewal and variance functions, first step is to estimate the parameters of F based on that sample. When F is $\text{Exp}(\lambda)$, it is well known that the renewal and variance functions are $M(t) = t/\lambda$ and $V(t) = t/\lambda$ and their plug-in estimators are $\widehat{M}(t) = t/\widehat{\lambda}$ and $\widehat{V}(t) = t/\widehat{\lambda}$, respectively. To estimate the parameter λ, we can use maximum likelihood (ML) method due to its applicability and theoretical properties. ML estimator of the parameter λ based on the sample $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ can be easily found as $\widehat{\lambda} = \frac{1}{m}\sum_{i=1}^{m}(R_i + 1)X_{i:m:n}$. Hence, the renewal and variance functions can be easily estimated for a fixed t. Although the ML estimator $\widehat{\lambda}$ is unbiased for λ, the plug-in estimators $\widehat{M}(t)$ and $\widehat{V}(t)$ are not unbiased but unbiased asymptotically. Note that the ML estimator $\widehat{\lambda}$ is strongly consistent for λ as $m \to \infty$. By considering Property 4.1, the plug-in estimators $\widehat{M}(t)$ and $\widehat{V}(t)$ almost surely converge to $M(t)$ and $V(t)$, for a fixed t, as $m \to \infty$. Here the condition "$m \to \infty$" is written instead of "$m = m(n)$ such that $m \to \infty$ as $n \to \infty$".

As for the Weibull and lognormal cases, neither the renewal and variance functions nor the ML estimators of the parameters do not have analytical expressions. Maximum likelihood estimates (MLEs) of parameters under progressively censored data for Weibull and lognormal cases must be computed numerically. [30], well studied ML estimation of parameters of Weibull and lognormal models under progressively censored data. They utilized EM algorithm to compute MLEs of parameters for both Weibull and lognormal distributions. We refer to this study to obtain MLEs for Weibull and lognormal models and omit details. For the estimation of model parameters of Weibull and lognormal distributions we choose tolerance level as 0.00005 when utilizing the EM algorithm. We use approximate maximum likelihood estimators (AMLEs) for starting points in the EM algorithm for both Weibull and lognormal distributions since they have analytical forms and are easy to compute. For the AMLEs we refer to [4] and [3] for Weibull and lognormal cases, respectively. The simulation is conducted on MATLAB. The number of replications is chosen as 10^4. Sample sizes are taken as 16 (small), 30 (moderate) and 50 (large). Two censoring proportions, approximately 1/3 and 1/2, are considered for each sample sizes. That is, the cases considered are; $n = 16$, $m = 10$; $n = 16$, $m = 8$; $n = 30$, $m = 20$; $n = 30$, $m = 15$; $n = 50$, $m = 35$; $n = 50$, $m = 25$. There different censoring schemes are applied for each sample. The first one censors units in early stages. The second one censors units after each failure, as possible. It may be considered as all stages censoring. And the third one censors units in final stages. These censoring schemes are tabulated in Table 1. Besides these censoring schemes, complete samples are also used for estimation of renewal and variance functions to compare. ML estimators of parameters of exponential and lognormal distributions under complete sample are well-known and they have analytical expressions. *wblfit* function on MATLAB is used for numerical computation of MLEs of parameters of Weibull distribution under complete sample. For the sake of simplicity, only one setting of parameters is evaluated in the simulation. These are; $\text{Exp}(\lambda = 1)$ with mean 1, $W(\alpha = 2, \beta = 1)$ with mean 0.8862 and $\text{LN}(\mu = 0, \sigma = 1)$ with mean 1.6487. The renewal and variance functions and their related plug-in estimators are computed at points $t = 0.5, 1, 2, 3, 5, 10$. When computing these functions numerically via RS method described in Section 5, partition intervals are chosen equally as 0.01 for renewal function and its estimator and 0.02 for variance function and its estimator. All results are presented in Tables 2–6. Since renewal and variance functions are equal for exponential case, they are given together in Table 2. Note that, first row of the estimators denotes estimation values and second row denotes mean square errors (MSEs) of estimators.

The results show that the plug-in estimators $\widehat{M}(t)$ and $\widehat{V}(t)$ based on MLEs under progressive censoring estimate the values of renewal and variance functions quite effectively when t is small or moderate for all distribution cases. For the larger values of t, biases and MSEs get larger as compared to the case where t is smaller. This is an expected situation since more observations are needed to achieve effective estimation for the larger values of t. Further, there is no remarkable and prominent difference among the censoring schemes since all MSE values of the estimators are close to each other for three different schemes. The plug-in estimators $\widehat{M}(t)$ and $\widehat{V}(t)$ based on complete sample have less bias and MSE values compared to those based on progressive censoring. However, the plug-in estimators $\widehat{M}(t)$ and $\widehat{V}(t)$ under progressive censoring seem to be effective sufficiently since the MSE values of estimators based on complete and progressively censored samples haven't big differences even for large values of t. All the biases and MSEs decrease as the sample size n and number of uncensored observations m increase for all cases, as expected. This is numerical illustration of convergence and asymptotic unbiasedness of the plug-in estimators $\widehat{M}(t)$ and $\widehat{V}(t)$.

7. Illustrative examples

In this section, estimation of renewal and variance functions under progressive censoring is exemplified by some reliability problems. For each example, renewal and variance functions are estimated based on both complete and progressively censored samples. The renewal function is further estimated based on complete and progressively censored samples to observe how progressive censoring has an impact on solution of some replacement problems.

Example 1. [10] studied reliability analysis of aircraft windshields data comprehensively. This data consists of failure and service times of commercial windshields used in many aircrafts. In their case study, [10] analyzed this data set to predict warranty costs and to determine optimal warranty policy. For this purpose, they considered two warranty policies: free replacement warranty (FRW) and pro-rata warranty (PRW). To compute warranty cost under FRW renewal function must be estimated based on the data. Later, [28] analyzed the failure times of aircraft windshields under inverse Weibull (IW) model and progressive censoring. The failure times of windshields given by [28] are presented in Table 7. Note that, unit of measurement for failure times is 1000 hour. To analyze the failure times under IW distribution [28] generated some progressively censored data. One of them censors last ten observations which corresponds to classical type-II censoring. This censoring scheme is given as $(0_{1\times 76}, 10)$. Let $F(x) = \exp\{-(x\beta)^{-\alpha}\},\ x \geq 0;\ \alpha > 0,\ \beta > 0$ be distribution function of IW distribution. MLEs of α and β were found by [28] as $(\widehat{\alpha},\ \widehat{\beta}) = (1.3918,\ 0.5755)$ for complete sample, as $(\widehat{\alpha},\ \widehat{\beta}) = (1.3131,\ 0.5549)$ for progressively censored sample. Based on these MLEs, renewal and variance functions are computed for $t = 5:5:25$. Partition intervals for numerical computation of renewal and variance functions are chosen as 0.01 and 0.02, respectively. Estimated values of renewal and variance functions based on the MLEs for both complete and censored samples are given in Table 8.

Estimates of both renewal and variance functions are close to each other for complete and progressively censored samples for small values of t. Estimated values of renewal and variance functions based on complete sample become a little greater than for those based on progressively censored sample as t increases. But these differences seem to negligible even for large values of t.

Now, let's consider average per unit cost to the manufacturer if the windshields are sold under non-renewing FRW policy. Here non-renewing means for; the warranty period is kept fixed, that is, it is not renewed after a replacement of a failed item occurred within the warranty period. Note that, these windshields are repairable products and the repaired ones are accepted as good as new. So, the warranty cost can be computed for both repair and non-repair option. For the non-repair option, the expected cost for the non-renewing FRW with warranty period T is given as $E[C_m(T)] = c_s[1 + M(T)]$, where c_s is the manufacturer's average per unit cost of providing an item to the costumer and $M(T)$ is the renewal function associated with the failure distribution of windshields. For the repair option, this cost becomes $E[C_m(T)] = c_s + c_r M(T)$, where c_r is the average per repair cost. For the details, see [10]. To compare the behavior of expected warranty costs under non-renewing FRW with repair and non-repair option, the expected warranty costs are computed based on both complete and censored samples given above. Renewal functions are computed based on the MLEs of IW model. The expected warranty costs are computed for warranty period of $T = 0.5:0.5:6$ ($\times$ 1000 hour), and manufacturing and repair costs are chosen as $c_s = \$9000$ and $c_r = \$5400$ (%60 of a new item). These values are plotted in the Fig. 1.

As it is seen, the estimated expected warranty costs based on complete and censored samples are quite close to each other even for varying period of warranty. Thus, the expected warranty cost can be estimated reliable under progressively censored sample as we do with complete sample.

Example 2. [1] studied failure times of 50 devices, to explain a method he proposed to determine bathtub-shaped failure rate. [29] generated a progressively censored sample from this data set to model a modified Weibull distribution (MWD), which has a bathtub-shaped hazard rate, under progressively censored sample. Complete and progressively censored samples are given in Tables 9 and 10, respectively. Although there is not any information about what the devices [1] used are, let's consider them as critical component of a system that must run for a specified period of time. For such a situation, how many spare parts from these devices will be needed in that period can be estimated by means of renewal function. Also, the variance function can be used to estimate the variance of renewals occurred in that period. The MWD firstly introduced by [25] has distribution function as $F(x) = 1 - \exp\{-ax^b e^{\lambda x}\}$, $x \geq 0$; $a > 0$, $b \geq 0$, $\lambda > 0$. [29] computed MLEs of the parameters for complete sample as $(\widehat{a},\ \widehat{b},\ \widehat{\lambda}) = (0.0624,\ 0.355,\ 0.02332)$, for progressively censored sample as $(\widehat{a},\ \widehat{b},\ \widehat{\lambda}) = (0.0714,\ 0.398,\ 0.01702)$. Mean of the MWD is computed as 45.6871 for complete sample and 47.7051 for progressively censored sample. We computed the renewal and variance functions for $t = 100:100:500$. Note that, the partition intervals for numerical computation of renewal and variance functions are chosen as 0.01 and 0.02, respectively. Estimated values of renewal and variance functions based on the MLEs for both complete and censored samples are given in Table 11.

Estimates of the variance function based on the censored sample are a little greater from the values of obtained for complete sample case. However, difference between the estimated values of renewal function for complete and censored cases are quite small even for large values of t. Also note that, the estimated values of renewal function for large values of t are consistent with the well-known asymptotic approximation of renewal function t/μ. Here μ denotes the expectation of failure times.

Now, let's consider a system consisting of several identical devices analyzed above. Suppose that one will implement a block replacement policy for these devices inside the system. For this purpose, the optimal preventive replacement interval has to be determined which is achieved by minimizing the expected cost rate given as $J(T) = [c_p + c_f M(T)]/T$, see [22]. Here, c_p and c_f are preventive and failure replacement costs, respectively and M is the renewal function associated with the failure distribution of identical devices. The expected cost rate is estimated based on both complete and censored failure times for comparative purposes. Thus, renewal functions are computed based on the MLEs of MWD for both complete and censored failure times. Fig. 2 shows cost rate curves with $T = 10:10:100$, $c_p = 100$ and $c_f = 500$ and asymptotic cost rate c_f/μ.

Behaviors of the curves seem very similar since the values of renewal functions based on complete and censored samples are close to each other for the period of time considered. As a result, the optimal preventive replacement interval can be determined based on the progressively censored sample as well as the complete sample.

8. Conclusions

Renewal and variance functions are two important characteristics of a RP and they are required in some reliability problems especially for warranty cost analysis and future demand estimation of a product. When the failure times of a product are observed after being put on market, the sample available commonly occurs as complete or random right censored. Therefore, renewal and variance functions are estimated based on these samples. Early studies focus on this situation. However, the failure times can be obtained in different data forms since observing a realization of RP is probabilistically equal to observe failure times simultaneously. This feature allows the researcher to gather information about possible future renewal/replacement process with early failure times rather than the failure times obtained throughout the renewal/ replacement process itself. If the researcher prefers experimental observation in a lifetime test, then a sample can be obtained as progressively censored in order to reduce time and cost of the test. In this situation, parametric estimation of the renewal and variance functions can be done with the plug-in estimators when there is enough evidence to assume a parametric form for distribution function of lifetime of the product.

In order to construct plug-in estimators, we firstly need consistent estimators of parameters of distribution function. ML estimators are a good option owing to their desirable properties such as consistency under regularity conditions. Also, they can be easily computed with the well-known EM algorithm. Therefore, the plug-in estimators for renewal and variance functions become consistent and asymptotically unbiased under general conditions.

This study provides some theoretical properties and computational approaches for the estimators of renewal and variance functions under progressive censoring and illustrates their usage in practical examples such as optimization of block replacement policy and warranty cost analysis. As seen from the simulation and examples, the estimators perform well under progressive censoring as much as the estimators based on complete sample. Therefore, the renewal and variance functions associated with some reliability problems can be estimated under progressive censoring to gather information about possible future renewal/replacement process associated with the problem.

CRediT authorship contribution statement

Ömer Altındağ: Conceptualization, Methodology, Formal analysis, Software, Writing – original draft. **Halil Aydoğdu:** Supervision, Validation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

Proof of Property 4.1: The proof is given by tracking and extending the steps given in Appendix A of [17]. Let $\widehat{f}_{m:n} = f(\widehat{\theta}_1, \widehat{\theta}_2, \ldots, \widehat{\theta}_r)$. For the sake of simplicity, we will use $m\to\infty$ instead of $m = m(n)$ such that $m\to\infty$ as $n\to\infty$. For each fixed $t \geq 0$,

$$\widehat{f}_{m:n}(t) \xrightarrow{a.s.} f(t)$$

as $m\to\infty$ by the continuity of f. Since f is a probability density function it is obtained that for each fixed $t \geq 0$

$$\widehat{F}_{m:n}(t) \xrightarrow{a.s.} F(t)$$

as a result of Scheffe's theorem as $m\to\infty$. Assume that $\widehat{F}^{k^*}_{m:n}(t) \xrightarrow{a.s.} F^{k^*}(t)$ as $m\to\infty$ for a $k \in \mathbb{N}$ and any fixed $t \geq 0$. From the definition of convolution, it is written that $F^{(k+1)^*}(t) = \int_0^t F^{k^*}(t-x)f(x)dx$. Then $\widehat{F}^{k^*}_{m:n}(t-x)\widehat{f}_{m:n}(x) \leq \left|f(x) - \widehat{f}_{m:n}(x)\right| + f(x)$ since F^{k^*} is a distribution function and $F^{k^*}(t) \leq 1$ for each $t \geq 0$. Denote right hand side of the above inequality by $g_{m:n}(x)$. Then $g_{m:n}(x) \xrightarrow{a.s.} f(x)$ as $m\to\infty$. Since f is integrable, we have $\widehat{F}^{(k+1)^*}_{m:n}(t) \xrightarrow{a.s.} F^{(k+1)^*}(t)$ for each fixed $t \geq 0$ as a result of generalized Lebesgue dominated convergence theorem as $m\to\infty$. Consequently, it is clear that

$$\widehat{F}^{k^*}_{m:n}(t) \xrightarrow{a.s.} F^{k^*}(t)$$

for each $k \geq 1$ and fixed $t \geq 0$ by induction. Let (Ω, U, P) be the probability space on which random variable corresponding to the distribution function F defined. There is a set $A \in U$ such that $\widehat{F}^{k^*}_{m:n}(t)$ converges pointwise to $F^{k^*}(t)$ for all $\omega \in A$, each $k \geq 1$ and fixed $t \geq 0$. It is obvious that $P(A) = 1$. There is at least a $c > 0$ such that $F(c) < 1$ since $F(0) < 1$ and F is a distribution function. Consider a $s \in \mathbb{N}$ such that $t \leq sc$. It is obtained that $F^{s^*}(t) \leq 1 - [1 - F(t/s)]^s$, $[1 - F(t/s)]^s > 0$. Then $F(t/s) < 1$ since $F(c) < 1$ and $t/s \leq c$. Therefore, there is at least an integer $s \geq 1$ such that $F^{s^*}(t) < 1$ for $t \geq 0$. Let $\delta = 1 - F^{s^*}(t)$ satisfying $F^{s^*}(t) < 1$, then there is a $m_0(\omega)$ for all $\omega \in A$ such that $\widehat{F}^{s^*}_{m:n}(t) < 1 - \delta/2$ for all $m \geq m_0(\omega)$. It is clear that $1 - \delta/2 < 1$. For positive integers s, i and j, we have

$$\widehat{F}^{(is+j)^*}_{m:n}(t) = \int_0^t \widehat{F}^{((i-1)s+j)^*}_{m:n}(t-x)d\widehat{F}^{s^*}_{m:n}(x) \leq \widehat{F}^{((i-1)s+j)^*}_{m:n}(t)\widehat{F}^{s^*}_{m:n}(t).$$

Direct iteration yields to the relation $\widehat{F}^{(is+j)^*}_{m:n}(t) \leq [\widehat{F}^{s^*}_{m:n}(t)]^i \widehat{F}^{j^*}_{m:n}$. Then, it is obtained that $\widehat{F}^{k^*}_{m:n}(t) \leq [\widehat{F}^{s^*}_{m:n}(t)]^{\lfloor k/s \rfloor}$ for $k \geq s$ where $\lfloor \cdot \rfloor$ denotes the floor function. Define $h(m, m_0(\omega)) = (1 - \delta/2)I(m \geq m_0(\omega)) + I(m < m_0(\omega))$ where $I(\cdot)$ denotes indicator function. It can be written that $k^a\widehat{F}^{k^*}_{m:n}(t) \leq$

$k^a[h(m, m_0(\omega))]^{\lfloor k/s \rfloor}$ for $k \geq s$. Hence, we have

$$\sum_{k=1}^{\infty} k^a \widehat{F}_{m:n}^{k^*}(t) = \sum_{k=1}^{s} k^a \widehat{F}_{m:n}^{k^*}(t) + \sum_{k=s+1}^{\infty} k^a \widehat{F}_{m:n}^{k^*}(t)$$

$$\leq \sum_{k=1}^{s} k^a \left[\widehat{F}_{m:n}^{s^*}(t)\right]^{\lfloor k/s \rfloor} + \sum_{k=s+1}^{\infty} k^a [h(m, m_0(\omega))]^{\lfloor k/s \rfloor} < \infty$$

for all $m \geq m_0(\omega)$ and fixed $t \geq 0$. Therefore,

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} k^a \widehat{F}_{m:n}^{k^*}(t) = \sum_{k=1}^{\infty} k^a F^{k^*}(t)$$

for all $\omega \in A$ and fixed $t \geq 0$, as a result of Lebesgue dominated convergence theorem. We have for fixed $t \geq 0$

$$\sum_{k=1}^{\infty} k^a \widehat{F}_{m:n}^{k^*}(t) \xrightarrow{a.s.} \sum_{k=1}^{\infty} k^a F^{k^*}(t)$$

as $m\to\infty$ since $P(A) = 1$. By taking $a = 0$ and $a = 1$, we obtain for fixed t, $\widehat{M}(t) \xrightarrow{a.s.} M(t)$ and $\widehat{V}(t) \xrightarrow{a.s.} V(t)$ as $m\to\infty$.

Proof of Property 4.2: There is always a constant $c > 0$ such that $F(t_0) \leq c/(1+c)$ since $F(t_0) < 1$. Then, we have $M(t_0) = \sum_{k=1}^{\infty} F^{k^*}(t) \leq \sum_{k=1}^{\infty} [F(t_0)]^k \leq \sum_{k=1}^{\infty} [c/(1+c)]^k = c$. As a result $\widehat{M}(t_0) \leq c$ for all $t \leq t_0$. By considering the bounded convergence theorem, it is clear that $\lim_{n\to\infty} E(\widehat{M}(t)) = M(t)$ for all $t \leq t_0$. Similarly, we have $M^* M(t) \leq M^2(t)$ and $V(t) \leq M(t)(1+M(t))$ from Eq. (2.4). Then it is obvious that $\widehat{V}(t_0) \leq c(1+c)$ for all $t \leq t_0$. Again, by considering the bounded convergence theorem $\lim_{n\to\infty} E(\widehat{V}(t)) = V(t)$ for all $t \leq t_0$ and the proof is concluded.

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