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Fundamental Concepts in Variable Lebesgue Spaces Associated with Laplace-Bessel Differential Operator

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Abstract: In this study, we consider the concepts of convergence in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. In variable Lebesgue spaces, there are three types of convergence: convergences with respect to modular, norm and measure. We investigate the relationship between these convergences. Furthermore, we prove that $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ are Banach spaces.

Keywords: Cauchy sequence, completeness, variable Lebesgue spaces.

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1 Introduction

In this study, we investigate concepts of convergences in variable Lebesgue spaces connected with Laplace-Bessel differential operator

$$\Delta_B := \sum_{i=1}^{k} B_i + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad 1 \le k \le n.$$

In variable Lebesgue spaces, there are three types of convergence: modular convergence, norm convergence and measure convergence. We will examine the relationship between norm, modular, and measure convergence in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Also, we prove that $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ are Banach spaces.

Variable Lebesgue spaces which have first been considered by Orlicz [2] and have a long history, play a key role in harmonic analysis theory. Indeed, these spaces are extensions of classical Lebesgue spaces by taking the exponent function $p(\cdot)$ instead of the constant exponent p. Therefore, they have many properties similar properties with $L_{p,\gamma}(\mathbb{R}^n_{k,+})$. Of course, they also differ in many ways and for this reason there is an increasing interest on variable Lebesgue spaces.

The motivation of this paper is to study fundamental concepts of analysis such as convergence, completeness in variable Lebesgue spaces. Then, we examine the relationship between norm, modular, and measure convergence in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Also, we prove that $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ are Banach spaces. Now, we are ready to recall important definitions and notations.

Let $x = (x', x'')$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$. Denote $\mathbb{R}^n_{k,+} = \{x \in \mathbb{R}^n : x_1 > 0, \ldots, x_k > 0,\ 1 \le k \le n\}$, $\gamma = (\gamma_1, \ldots, \gamma_k)$, $\gamma_1 > 0, \ldots, \gamma_k > 0$ and $|\gamma| = \gamma_1 + \ldots + \gamma_k$.

Let $\mathcal{P}(\mathbb{R}^n_{k,+}) = \Big\{p(\cdot) : \mathbb{R}^n_{k,+} \to [1,\infty] : p(\cdot) \text{ is measurable}\Big\}$. Also let any element of $\mathcal{P}(\mathbb{R}^n_{k,+})$ is said to be a variable exponent function and also let

$$p_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n_{k,+}} p(x), \qquad p_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n_{k,+}} p(x),$$

satisfying the conditions for all $|x - y| \le \frac{1}{2}$, $x, y \in \mathbb{R}^n_{k,+}$,

$$|p(x) - p(y)| \le \frac{A_0}{-\log|x - y|},$$

and

$$|p(x) - p_\infty| \le \frac{A_\infty}{\log(e + |x|)}.$$

Here $p_\infty = \lim\limits_{x\to\infty} p(x) > 1$. If the above inequalities hold for $p(\cdot)$, then we denote it by $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n_{k,+})$, and $p(\cdot) \in \mathcal{P}^{\log}_\infty(\mathbb{R}^n_{k,+})$, respectively. Moreover, if $p(\cdot)$ provides both of inequalities, then it is denoted by $p(\cdot) \in LH(\mathbb{R}^n_{k,+})$. As in classical Lebesgue spaces, there exist three cases for $p(x)$, i.e., $p(x) = 1$, $p(x) = \infty$ or $1 < p(x) < \infty$. Therefore, three canonical subsets on $\mathbb{R}^n_{k,+}$ are introduced as follows:

$$\begin{aligned}
(\mathbb{R}^n_{k,+})_\infty &= \{x \in \mathbb{R}^n_{k,+} : p(x) = \infty\},\\
(\mathbb{R}^n_{k,+})_1 &= \{x \in \mathbb{R}^n_{k,+} : p(x) = 1\},\\
(\mathbb{R}^n_{k,+})_0 &= \{x \in \mathbb{R}^n_{k,+} : 1 < p(x) < \infty\}.
\end{aligned}$$

For $x \in \mathbb{R}^n_{k,+}$, conjugate exponent function is given by

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

Then variable Lebesgue space is defined as follows:

$$L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+}) := \left\{f : \|f\|_{L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})} = \inf\left\{\mu > 0 : \rho_{p(\cdot),\gamma}(f/\mu) \le 1\right\} < \infty\right\},$$

where f is a measurable function, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and

$$\rho_{p(\cdot),\gamma}(f) := \int_{\mathbb{R}^n_{k,+}\setminus(\mathbb{R}^n_{k,+})_\infty} |f(x)|^{p(x)}(x')^\gamma dx + \|f\|_{L_{\infty,\gamma}(\mathbb{R}^n_{k,+})_\infty} < \infty.$$

The next proposition follows easily from [1].

Proposition 1. *$f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ if and only if*

$$\rho_{p(\cdot),\gamma}(f) = \int_{\mathbb{R}^n_{k,+}} |f(x)|^{p(x)}(x')^\gamma dx < \infty,$$

where $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and $p_+ < \infty$.

Lemma 1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$. If $\|f\|_{p(\cdot),\gamma} \le 1$, then $\rho_{p(\cdot),\gamma}(f) \le \|f\|_{p(\cdot),\gamma}$ and if $\|f\|_{p(\cdot),\gamma} > 1$, then $\rho_{p(\cdot),\gamma}(f) \ge \|f\|_{p(\cdot),\gamma}$.*

Proof: If $\|f\|_{p(\cdot),\gamma} = 0$, then $f \equiv 0$ and $\rho_{p(\cdot),\gamma}(f) = 0$. If $0 < \|f\|_{p(\cdot),\gamma} \le 1$, since the modular $\rho_{p(\cdot),\gamma}$ is convex, then we have

$$\begin{aligned}
\rho_{p(\cdot),\gamma}(f) &= \rho_{p(\cdot),\gamma}\left(\|f\|_{p(\cdot),\gamma}\frac{f}{\|f\|_{p(\cdot),\gamma}}\right)\\
&\le \|f\|_{p(\cdot),\gamma}\rho_{p(\cdot),\gamma}\left(\frac{f}{\|f\|_{p(\cdot),\gamma}}\right) \le \|f\|_{p(\cdot),\gamma}.
\end{aligned}$$

If $\|f\|_{p(\cdot),\gamma} > 1$, then one can write $\rho_{p(\cdot),\gamma}(f) > 1$. Also if $\rho_{p(\cdot),\gamma}(f) \le 1$, then $\|f\|_{p(\cdot),\gamma} \le 1$. But then we have

$$\begin{aligned}
\rho_{p(\cdot),\gamma}\left(\frac{f}{\rho_{p(\cdot),\gamma}}\right) &= \int_{\mathbb{R}^n_+\setminus(\mathbb{R}^n_+)_\infty}\left(\frac{|f(x)|}{\rho_{p(\cdot),\gamma}(f)}\right)^{p(x)}(x')^\gamma dx + \rho_{p(\cdot),\gamma}(f)^{-1}\|f\|_{L_{\infty,\gamma}((\mathbb{R}^n_+)_\infty)}\\
&\le \int_{\mathbb{R}^n_+\setminus(\mathbb{R}^n_+)_\infty} |f(x)|^{p(x)}\rho_{p(\cdot),\gamma}(f)^{-1}(x')^\gamma dx + \rho_{p(\cdot),\gamma}(f)^{-1}\|f\|_{L_{\infty,\gamma}((\mathbb{R}^n_+)_\infty)} = 1.
\end{aligned}$$

Consequently, we get $\|f\|_{p(\cdot),\gamma} \le \rho_{p(\cdot),\gamma}(f)$. This completes the proof. □

2 Convergence in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$

In this section, we first give the relationship between convergences with respect to modular, norm and measure in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.

Given $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, it is said that $\{f_i\}$ converges with respect to norm to $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ if $\lim\limits_{i\to\infty}\|f - f_i\|_{p(\cdot),\gamma} = 0$. If there exists $\lambda > 0$ such that $\rho_{p(\cdot),\gamma}(\lambda(f - f_i)) \to 0$ as $i \to \infty$, then it is said that f_i converges to f with respect to modular. Finally, given $\varepsilon > 0$, if

$$\lim_{i\to\infty} \nu\left\{\left|\{x \in \mathbb{R}^n_{k,+} : |f(x) - f_i(x)| \ge \varepsilon\}\right|_\gamma < \varepsilon\right\} = 0$$

holds, then it is said that $\{f_i\}$ converges with respect to measure.

Theorem 1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and for all $i \in \mathbb{N}$, $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ nonnegative functions such that f_i increases to a function f pointwise a.e. Then $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $\|f_i\|_{p(\cdot),\gamma} \to \|f\|_{p(\cdot),\gamma}$ or $f \notin L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $\|f_i\|_{p(\cdot),\gamma} \to \infty$.*

Proof: Since $\{f_i\}$ is an increasing sequence, $\{\|f_i\|_{p(\cdot),\gamma}\}$ is also increasing and so it either converges or diverges to ∞. If $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, then $\|f_i\|_{p(\cdot),\gamma} \leq \|f\|_{p(\cdot),\gamma}$ since $f_i \leq f$. Otherwise, since $f_i \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, we have $\|f_i\|_{p(\cdot),\gamma} < \infty = \|f\|_{p(\cdot),\gamma}$. In both cases it is sufficient to show that there holds $\mu < \|f_i\|_{p(\cdot),\gamma}$ for any $\mu < \|f\|_{p(\cdot),\gamma}$, and for all sufficiently large i.

Fix $\mu > 0$. Then it is obvious that $\rho_{p(\cdot),\gamma}(f/\mu) > 1$ and from monotone convergence theorem, we have

$$\begin{aligned}
\rho_{p(\cdot),\gamma}(f/\mu) &= \int\limits_{\mathbb{R}^n_+\setminus(\mathbb{R}^n_+)_\infty} \left(\frac{|f(x)|}{\mu}\right)^{p(x)} (x')^\gamma dx + \mu^{-1}\|f\|_{L_{\infty,\gamma}((\mathbb{R}^n_+)_\infty)} \\
&= \lim_{i\to\infty}\left(\int_{\mathbb{R}^n_+\setminus(\mathbb{R}^n_+)_\infty} \left(\frac{|f_i(x)|}{\mu}\right)^{p(x)} (x')^\gamma dx + \mu^{-1}\|f_i\|_{L_{\infty,\gamma}((\mathbb{R}^n_+)_\infty)}\right) \\
&= \lim_{i\to\infty} \rho_{p(\cdot),\gamma}(f_i/\mu).
\end{aligned}$$

Thus, $\rho_{p(\cdot),\gamma}(f_i/\mu) > 1$ and $\mu < \|f_i\|_{p(\cdot),\gamma}$ for all sufficiently large i. Therefore, we complete the proof. □

Now, we will give the analog of Fatou's Lemma.

Theorem 2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$. Assume that the sequence $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $f_i \to f$ pointwise a.e. If $\liminf\limits_{i\to\infty} \|f_i\|_{p(\cdot),\gamma} < \infty$, then $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $\|f\|_{p(\cdot),\gamma} \leq \liminf\limits_{i\to\infty} \|f_i\|_{p(\cdot),\gamma}$.*

Proof: Firstly, we will define a sequence $g_i(x) = \inf_{i\leq m} |f_m(x)|$. Then $g_i(x) \leq |f_m(x)|$ for all $i \leq m$ and thus $g_i \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Furthermore, $\{g_i\}$ is an increasing sequence and

$$\lim_{i\to\infty} g_i(x) = \liminf_{m\to\infty} |f_m(x)| = |f(x)|$$

for $x \in \mathbb{R}^n_{k,+}$ a.e. by its definition. From Theorem 1, we get

$$\|f\|_{p(\cdot),\gamma} = \lim_{i\to\infty} \|g_i\|_{p(\cdot),\gamma} \leq \lim_{i\to\infty} \left(\inf_{i\leq m} \|f_m\|_{p(\cdot),\gamma}\right) = \liminf_{i\to\infty} \|f_i\|_{p(\cdot),\gamma} < \infty,$$

and $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Thus, this completes the proof. □

Notice that unlike the above theorems, to obtain the analog of dominated convergence theorem we have to suppose $p_+ < \infty$. The following theorem associated with convergence with respect to norm is required to convergence with respect to modular.

Theorem 3. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and $p_+ < \infty$. For $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, $\|f_i - f\|_{p(\cdot),\gamma} \to 0$ if and only if $\rho_{p(\cdot),\gamma}(f - f_i) \to 0$.*

Proof: Assume that $\{f_i\}$ converges with respect to norm. Then, from Lemma 1, we obtain

$$\rho_{p(\cdot),\gamma}(f - f_i) \leq \|f - f_i\|_{p(\cdot),\gamma} \leq 1,$$

for all sufficiently large i. Thus, $\rho_{p(\cdot),\gamma}(f - f_i) \to 0$.

To obtain the converse, let $\mu < 1$ be fixed. Then, we get

$$\rho_{p(\cdot),\gamma}\left(\frac{f - f_i}{\mu}\right) \leq \left(\frac{1}{\mu}\right)^{p_+} \rho_{p(\cdot),\gamma}(f - f_i).$$

This implies that $\rho_{p(\cdot),\gamma}\left(\frac{f-f_i}{\mu}\right) \leq 1$ for all i sufficiently large. Equivalently, $\|f - f_i\|_{p(\cdot),\gamma} \leq \mu$ for all sufficiently large i. Since μ is arbitrary, $f_i \to f$ with respect to norm. Therefore, we complete the proof. □

Theorem 4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and $p_+ < \infty$. If there exists a sequence $\{f_i\}$ such that $f_i \to f$ pointwise a.e., and there exists $g \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $|f_i(x)| \leq g(x)$ a.e., then $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $\lim\limits_{i\to\infty} \|f - f_i\|_{p(\cdot),\gamma} = 0$.*

Proof: From Proposition 1, we have

$$\begin{aligned}
|f(x) - f_i(x)|^{p(x)} &\leq 2^{p(x)-1}\left(|f(x)|^{p(x)} + |f_i(x)|^{p(x)}\right) \\
&\leq 2^{p_+}|g(x)|^{p(x)} \in L_{1,\gamma}(\mathbb{R}^n_+).
\end{aligned}$$

Therefore, from the dominated convergence theorem on $L_{1,\gamma}(\mathbb{R}^n_{k,+})$, $\rho_{p(\cdot),\gamma}(f - f_i) \to 0$ as $i \to 0$ and so $\|f - f_i\|_{p(\cdot),\gamma} \to 0$, by Theorem 3. Thus, we complete the proof. □

Theorem 5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$. If there exists the sequence $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $\|f - f_i\|_{p(\cdot),\gamma} \to 0$, then the sequence $\{f_i\}$ converges to f with respect to measure.*

Proof: Assume that there exists a sequence $\{f_i\}$ converges to f with respect to norm but not with respect to measure. And we also assume that there exists $0 < \varepsilon < 1$ such that,

$$E_i := |\{x \in \mathbb{R}^n_{k,+} : |f(x) - f_i(x)| \geq \varepsilon\}|_\gamma \geq \varepsilon,$$

for all i. Since there exists $|E_i \cap (\mathbb{R}^n_{k,+})_\infty|_\gamma \geq \varepsilon/2$ or $|E_i \setminus (\mathbb{R}^n_{k,+})_\infty|_\gamma \geq \varepsilon/2$ for each i, by taking another subsequence we suppose that one of these inequalities holds for all i.

If $|E_i \cap (\mathbb{R}^n_{k,+})_\infty|_\gamma \geq \varepsilon/2$ for all i, then we find

$$\|f - f_i\|_{L_{p(\cdot),\gamma}} \geq \|(f - f_i)\chi_{(\mathbb{R}^n_{k,+})_\infty}\|_{L_{p(\cdot),\gamma}} = \|f - f_i\|_{L_{\infty,\gamma}((\mathbb{R}^n_{k,+})_\infty)} \geq \varepsilon.$$

Then, this is a contradiction. If $|E_k \setminus (\mathbb{R}^n_{k,+})_\infty|_\gamma \geq \varepsilon/2$ for all k, then we get

$$\begin{aligned} \rho_{p(\cdot),\gamma}\left(\frac{f - f_k}{\varepsilon^2/2}\right) &\geq \int_{\mathbb{R}^n_{k,+}\setminus(\mathbb{R}^n_{k,+})_\infty} \left(\frac{|f(x) - f_k(x)|}{\varepsilon^2/2}\right)^{p(x)} (x')^\gamma dx \\ &\geq \int_{E_k\setminus(\mathbb{R}^n_{k,+})_\infty} \left(\frac{2}{\varepsilon}\right)^{p(x)} (x')^\gamma dx \geq \left(\frac{2}{\varepsilon}\right)^{p_-} |E_k \setminus (\mathbb{R}^n_{k,+})_\infty| \geq 1. \end{aligned}$$

Therefore, there exists $\|f - f_i\|_{L_{p(\cdot),\gamma}} \geq \varepsilon^2/2 > 0$. Again, it is a contradiction. Thus, if the sequence $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ converges to f with respect to norm, then the sequence $\{f_i\}$ converges to f in measure. □

Theorem 6. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$. Assume that the sequence $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ converges with respect to norm to $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Then there exists a subsequence $\{f_{i_j}\}$ and $g \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $f_{i_j} \to f$ pointwise a.e. and $|f_{i_j}(x)| \leq g(x)$ for a.e. $x \in \mathbb{R}^n_{k,+}$.*

Proof: From Theorem 5, we have a subsequence $\{f_{i_j}\}$ such that $f_{i_j} \to f$ pointwise a.e. Furthermore, since a convergent sequence is also Cauchy, we can fix i_j large enough for each j, $\|f_{i_{j+1}} - f_{i_j}\|_{p(\cdot),\gamma} \leq 2^{-j}$.

Define for each j,

$$h_j(x) = \sum_{m=1}^{j-1} |f_{i_{m+1}}(x) - f_{i_m}(x)|,$$

which implies $\{h_j\}$ is an increasing and pointwise convergent to h. Therefore, we have

$$\|h_j\|_{p(\cdot),\gamma} \leq \sum_{m=1}^{j-1} 2^{-m} \leq 1.$$

By monotone convergence theorem, there exists $h \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. But, we get

$$|f_{i_j}(x) - f_1(x)| \leq \sum_{m=1}^{j-1} |f_{i_{m+1}}(x) - f_{i_m}(x)| = h_j(x) \leq h(x),$$

for every j and a.e. $x \in \mathbb{R}^n_{k,+}$. Hence, if we fix $g = h + |f_1|$, we have $g \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and $|f_{i_j}(x)| \leq g(x)$ a.e. □

Now, we give the relationship between these convergence types.

Theorem 7. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$, $p_+ < \infty$, $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ and a sequence $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$. Then the followings are equivalent:*

(i) $f_i \to f$ with respect to norm,
(ii) $f_i \to f$ with respect to modular,
(iii) $f_i \to f$ with respect to measure and for some $\lambda > 0$, $\rho_{p(\cdot),\gamma}(\lambda f_i) \to \rho_{p(\cdot),\gamma}(\lambda f)$.

Proof: The equivalence of (i) and (ii) has been proved in Theorem 3. We will now prove the equivalence of (ii) and (iii).

To prove that (ii) implies (iii), by Theorem 5, notice that convergence with respect to norm implies convergence with respect to measure. To finish the proof of this argument we will obtain that convergence with respect to modular means $\rho_{p(\cdot),\gamma}(\lambda f_i) \to \rho_{p(\cdot),\gamma}(\lambda f)$ for $\lambda = 1$.

If $1 \le p < \infty$ and $u, v \ge 0$, by using the mean value theorem, then there holds

$$|u^p - v^p| \le p \max\{u^{p-1}, v^{p-1}\}|u - v| \le p(u^{p-1} + v^{p-1})|u - v|.$$

Therefore, we have

$$\begin{aligned}|\rho_{p(\cdot),\gamma}(f) - \rho_{p(\cdot),\gamma}(f_i)| &\le \int_{\mathbb{R}^n_{k,+}} \left| |f(x)|^{p(x)} - |f_i(x)|^{p(x)} \right| (x')^\gamma dx \\ &\le p_+ \int_{\mathbb{R}^n_{k,+}} \left(|f(x)|^{p(x)-1} - |f_i(x)|^{p(x)-1} \right) |f(x) - f_i(x)|(x')^\gamma dx.\end{aligned}$$

To estimate the RHS, we write $\mathbb{R}^n_{k,+} = (\mathbb{R}^n_{k,+})_1 \cup (\mathbb{R}^n_{k,+})_0$. For the integral on $(\mathbb{R}^n_{k,+})_1$, we can write

$$\begin{aligned}&p_+ \int_{(\mathbb{R}^n_{k,+})_1} \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1} \right) |f(x) - f_i(x)|(x')^\gamma dx \\ &\le 2p_+ \int_{(\mathbb{R}^n_{k,+})_1} |f(x) - f_i(x)|^{p(x)}(x')^\gamma dx \\ &\le 2p_+ \rho_{p(\cdot),\gamma}(f - f_i).\end{aligned}$$

Since convergences with respect to modular and norm are equivalent, RHS goes to 0 ($i \to \infty$).

To calculate the integral on $(\mathbb{R}^n_{k,+})_0$, let ε, $0 < \varepsilon < 1/4$ be fixed. Then by Young's inequality, we have

$$\begin{aligned}&p_+ \int_{(\mathbb{R}^n_{k,+})_0} \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1} \right) |f(x) - f_i(x)|(x')^\gamma dx \\ &\le p_+ \int_{(\mathbb{R}^n_{k,+})_0} \frac{\varepsilon^{p'(x)}}{p'(x)} \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1} \right)^{p'(x)} (x')^\gamma dx \\ &\quad + p_+ \int_{(\mathbb{R}^n_{k,+})_0} \frac{\varepsilon^{-p(x)}}{p(x)} |f(x) - f_i(x)|^{p(x)}(x')^\gamma dx \\ &= I_1 + I_2.\end{aligned}$$

First, let us calculate I_2. Since $p(x) > 1$ for all $x \in (\mathbb{R}^n_{k,+})_0$, we get

$$I_2 \le p_+ \rho_{p(\cdot),\gamma}(\varepsilon^{-1}(f - f_i)).$$

To estimate I_1, we use the following inequalities:

$$\begin{aligned}u^p + v^p &\le \max\{1, 2^{1-p}\}(u + v)^p \\ (u + v)^p &\le \max\{1, 2^{1-p}\}(u^p + v^p)\end{aligned} \qquad p > 0, \quad u, v > 0.$$

Since $1 < p'(x) < \infty$ on $(\mathbb{R}^n_{k,+})_0$, we have

$$\begin{aligned}I_1 &\le p_+ \int_{(\mathbb{R}^n_{k,+})_0} \varepsilon^{p'(x)} \max\{1, 2^{2-p(x)}\}^{p'(x)} \left(|f(x)| + |f_i(x)| \right)^{p(x)} (x')^\gamma dx \\ &\le p_+ \int_{(\mathbb{R}^n_{k,+})_0} (4\varepsilon)^{p'(x)} \left(2|f(x)| + |f(x) - f_i(x)| \right)^{p(x)} (x')^\gamma dx \\ &\le 4\varepsilon p_+ \int_{(\mathbb{R}^n_{k,+})_0} 2^{p(x)-1} \left(2^{p(x)}|f(x)|^{p(x)} + |f(x) - f_i(x)|^{p(x)} \right)^{p(x)} (x')^\gamma dx \\ &\le \varepsilon p_+ 2^{2p_+ + 1} \rho_{p(\cdot),\gamma}(f) + \varepsilon p_+ 2^{2p_+ + 1} \rho_{p(\cdot),\gamma}(f - f_i).\end{aligned}$$

Thus, we can write

$$\begin{aligned}&p_+ \int_{(\mathbb{R}^n_{k,+})_0} \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1} \right) |f(x) - f_i(x)|(x')^\gamma dx \\ &\le \varepsilon p_+ 2^{2p_+ + 1} \rho_{p(\cdot),\gamma}(f) + \varepsilon p_+ 2^{2p_+ + 1} \rho_{p(\cdot),\gamma}(f - f_i) + p_+ \rho_{p(\cdot),\gamma}(\varepsilon^{-1}(f - f_i)).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\limsup_{i \to \infty} p_+ \int_{(\mathbb{R}^n_{k,+})_0} \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1} \right) |f(x) - f_i(x)|(x')^\gamma dx \\ &\le \varepsilon p_+ 2^{2p_+ + 1} \rho_{p(\cdot),\gamma}(f).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we follow from $|\rho_{p(\cdot),\gamma}(f) - \rho_{p(\cdot),\gamma}(f_i)| \to 0$.

Now assume that $f_i \to f$ with respect to measure and $\rho_{p(\cdot),\gamma}(\lambda f_i) \to \rho_{p(\cdot),\gamma}(\lambda f)$ for $\lambda > 0$. Since $\lambda f_i \to \lambda f$ with respect to measure, we may suppose that $\lambda = 1$. Then we have, for each ε, $0 < \varepsilon < 1$,

$$\left|\left\{x \in \mathbb{R}^n_{k,+} : |f(x) - f_i(x)|^{p(x)} > \varepsilon\right\}\right|_\gamma \leq \left|\left\{x \in \mathbb{R}^n_{k,+} : |f(x) - f_i(x)|^{p(x)} > \varepsilon^{1/p_-}\right\}\right|_\gamma \leq \left|\left\{x \in \mathbb{R}^n_{k,+} : |f(x) - f_i(x)|^{p(x)} > \varepsilon\right\}\right|_\gamma \leq \varepsilon.$$

Hence, $|f(\cdot) - f_i(\cdot)|^{p(\cdot)} \to 0$ with respect to measure.

Furthermore, we get

$$\begin{aligned} &\left||f(x)|^{p(x)} - |f_i(x)|^{p(x)}\right| \\ &\leq p_+ \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1}\right)|f(x) - f_i(x)| \\ &\leq p_+|f(x)|^{p(x)-1}|f(x) - f_i(x)| + p_+ \max\{1, 2^{p(x)-2}\}\times \\ &\quad \times \left(|f(x)|^{p(x)-1} + |f_i(x)|^{p(x)-1}\right)|f(x) - f_i(x)| \\ &\leq p_+(2^{p_+ +1})|f(x)|^{p(x)-1}|f(x) - f_i(x)| + p_+2^{p_+}|f(x) - f_i(x)|^{p(x)}. \end{aligned} \tag{1}$$

Now let ε, $0 < \varepsilon < 1$ be fixed. Since $|f(\cdot)|^{p(\cdot)} \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$, there exists $K \geq 1$ such that

$$\left|\left\{x : |f(x)|^{p(x)-1} > K\right\}\right|_\gamma \leq \left|\left\{x : |f(x)|^{p(x)} > K\right\}\right|_\gamma \leq \varepsilon/2.$$

From inequality (1), since $f_i \to f$ and $|f(\cdot) - f_i(\cdot)|^{p(\cdot)} \to 0$ with respect to measure, we can write

$$\begin{aligned} &\left|\left\{x : \left||f(x)|^{p(x)} - |f_i(x)|^{p(x)}\right| > \varepsilon\right\}\right|_\gamma \\ &\leq \left|\left\{x : |f(x)|^{p(x)-1} > K\right\}\right|_\gamma + \left|\left\{x : p_+(2^{p_+ +1})K|f(x) - f_i(x)| > \varepsilon/2\right\}\right|_\gamma + \\ &\quad + \left|\left\{x : p_+2^{p_+}|f(x) - f_i(x)|^{p(x)} > \varepsilon/2\right\}\right|_\gamma \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2p_+(2^{p_+} + 1)K} + \frac{\varepsilon}{p_+2^{p_+ +1}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

for all sufficiently large i. Therefore, $|f_i(\cdot)|^{p(\cdot)} \to |f(\cdot)|^{p(\cdot)}$ with respect to measure. Define

$$h_i(x) = 2^{p_+ -1}|f_i(x)|^{p(x)} + 2^{p_+ -1}|f(x)|^{p(x)} - |f(x) - f_i(x)|^{p(x)} \geq 0,$$

then $h_i \to 2^{p_+}|f(\cdot)|^{p(\cdot)}$ with respect to measure. Hence, from Fatou's Lemma, we get

$$\begin{aligned} &2^{p_+} \int_{\mathbb{R}^n_{k,+}} |f(x)|^{p(x)}(x')^\gamma dx \\ &\quad \leq \liminf_{i\to\infty} \int_{\mathbb{R}^n_{k,+}} 2^{p_+ -1}|f_i(x)|^{p(x)} + 2^{p_+ -1}|f(x)|^{p(x)} - |f(x) - f_i(x)|^{p(x)}(x')^\gamma dx. \end{aligned}$$

Since $\rho_{p(\cdot),\gamma}(f_i) \to \rho_{p(\cdot),\gamma}(f)$, we follow from

$$\limsup_{i\to\infty} \int_{\mathbb{R}^n_{k,+}} |f(x) - f_i(x)|^{p(x)}(x')^\gamma dx \leq 0.$$

Thus, $f_i \to f$ with respect to modular. Therefore, we complete the proof. □

3 Completeness of $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$

Here, we are ready to obtain the completeness of $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, but we first need to prove the following theorem.

Theorem 8. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$ and $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ be the sequence such that $\sum_{i=1}^{\infty} \|f_i\|_{p(\cdot),\gamma} < \infty$. Then there exists $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $\sum_{i=1}^{k} f_i \to f$ in norm as $k \to \infty$ and*

$$\|f\|_{p(\cdot),\gamma} \leq \sum_{i=1}^{\infty} \|f_i\|_{p(\cdot),\gamma}.$$

Proof: Firstly, let us define F on $\mathbb{R}^n_{k,+}$ and $\{F_k\}$ as follows:

$$F(x) = \sum_{k=i}^{\infty} |f_i(x)|, \qquad F_k(x) = \sum_{i=1}^{k} |f_i(x)|.$$

Then the sequence $\{F_k\}$ is nonnegative and increasing pointwise a.e. to F. Furthermore, there exists $F_k \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$, and its norm is uniformly bounded for each k, since

$$\|F_k\|_{p(\cdot),\gamma} \leq \sum_{i=1}^{k} \|f_i\|_{p(\cdot),\gamma} \leq \sum_{i=1}^{\infty} \|f_i\|_{p(\cdot),\gamma} < \infty.$$

Therefore, from Theorem 1, we have $F \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$.

Since F is finite a.e., $\{F_k\}$ converges pointwise a.e. Therefore, if we can define $\{G_k\}$ by

$$G_k(x) = \sum_{i=1}^{k} f_i(x),$$

then it is also pointwise convergent a.e., since absolute convergence means convergence. Let us denote this limit by f, i.e, $G_k \to f$.

Now fix $G_0 = 0$. Then $G_k - G_j \to f - G_j$ pointwise a.e. for $j \geq 0$. Moreover, we have

$$\liminf_{k\to\infty} \|G_k - G_j\|_{p(\cdot),\gamma} \leq \liminf_{k\to\infty} \sum_{i=j+1}^{k} \|f_i\|_{p(\cdot),\gamma} = \sum_{i=j+1}^{\infty} \|f_i\|_{p(\cdot),\gamma} < \infty.$$

From Theorem 2, if $j = 0$, then we get

$$\|f\|_{p(\cdot),\gamma} \leq \liminf_{k\to\infty} \|G_k\|_{p(\cdot),\gamma} \leq \sum_{i=1}^{\infty} \|f_i\|_{p(\cdot),\gamma} < \infty.$$

Also, we can write, for each j,

$$\|f - G_j\|_{p(\cdot),\gamma} \leq \liminf_{k\to\infty} \|G_k - G_j\|_{p(\cdot),\gamma} \leq \sum_{i=j+1}^{\infty} \|f_i\|_{p(\cdot),\gamma},$$

since the sum on the RHS goes to 0. Hence, $G_j \to f$ with respect to norm. Therefore, we complete the proof. □

Let us state the completeness of $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ which is a corollary of Theorem 8. This result also means that $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ is Banach space for $1 < p_- \leq p(x) \leq p_+ < \infty$.

Corollary 1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n_{k,+})$. $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ is complete, i.e. every Cauchy sequence in $L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ is also convergent.*

Proof: Let $\{f_i\} \subset L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ be a Cauchy sequence. Fix i_1 such that $\|f_k - f_j\|_{p(\cdot),\gamma} < 2^{-1}$ for $k, j \geq i_1$, fix i_2 such that $\|f_k - f_j\|_{p(\cdot),\gamma} < 2^{-2}$ for $k, j \geq i_2$ and so on. This gives a subsequence $\{f_{i_j}\}$, $i_j < i_{j+1}$, such that

$$\|f_{i_{j+1}} - f_{i_j}\|_{p(\cdot),\gamma} < 2^{-j}.$$

Let $\{g_j\}$ be defined by $g_1 = f_{i_1}$ and $g_j = f_{i_j} - f_{i_{j-1}}$ for $j > 1$. Then for all j, we have the telescoping sum $\sum_{k=1}^{j} g_k = f_{i_j}$. Furthermore, we obtain

$$\sum_{j=1}^{\infty} \|g_j\|_{p(\cdot),\gamma} \leq \|f_{i_1}\|_{p(\cdot),\gamma} + \sum_{j=1}^{\infty} 2^{-j} < \infty.$$

Hence, from Theorem 8, there exists $f \in L_{p(\cdot),\gamma}(\mathbb{R}^n_{k,+})$ such that $f_{i_j} \to f$ in norm.

As a consequence of this, we have

$$\|f - f_i\|_{p(\cdot),\gamma} \leq \|f - f_{i_j}\|_{p(\cdot),\gamma} + \|f_{i_j} - f_i\|_{p(\cdot),\gamma}.$$

Since $\{f_i\}$ is a Cauchy sequence, we can get the RHS as small as desired. Therefore, $f_i \to f$ with respect to norm. This completes the proof. □

4 Conclusion

In this paper, the concepts of convergence in variable Lebesgue space has been investigated. In this space, there exists three types of convergence: convergences with respect to modular, norm, measure. The relationship between these convergences has been studied.

5 References

1 D. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue spaces, Foundations and Harmonic Analysis*, Springer Science & Business Media, 2004.
2 W. Orlicz, *Über konjugierte Exponentenfolgen*, Studia Math., **3** (1931), 200-211.